UDC 517.9

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JACOBI OPERATORS AND ORTHONORMAL MATRIX-VALUED POLYNOMIALS. II

ОПЕРАТОРИ ЯКОБІ ТА ОРТОГОНАЛЬНІ ОПЕРАТОРНОЗНАЧНІ ПОЛІНОМИ. II

We use a system of operator-valued orthogonal polynomials to construct analogs of L. de Branges spaces and establish their relationship with the theory of nonself-adjoint operators.

Із використанням системи операторнозначних ортогональних поліномів побудовано аналоги просторів Л. де Бранжа та встановлено їх зв'язок з теорією несамоспряжених операторів.

1. Introduction. This work is based on the main constructions of the first part of the study (Jacobi operators and orthonormal matrix-valued polynomials. I). Establishment of links between the constructions of the first part (Jacobi operators and orthogonal polynomials) with the theory of nonself-adjoint operators and further realization of these constructions in analogues of vector-valued L. de Branges spaces is the aim of the present study. The works [14, 15] are dedicated to the connection of Jacobi matrices with L. de Branges spaces. Method of reconstruction of the system of orthogonal polynomials by the functions specifying the L. de Branges space is one of results of this paper. In Section 1 using a system of orthonormal matrix-valued polynomials a Hilbert L. de Branges space is constructed which is an analogue of the well-known L. de Branges space [10, 12] in the discrete case. In Section 2 connection of these L. de Branges spaces with triangular models of nonself-adjoint nilpotent operators [10] is established. In Section 3 resolvent of the truncated Jacobi operator is calculated, and in the case of the finite dimensionality of E, an orthonormal basis of generating kernels is found.

2. L. de Branges spaces. I. In this section we remind main definitions and facts from the first part of the work. Denote by $L^2_{\mathbb{R}}(E, dF(x))$ the Hilbert space of E-valued vector-functions on $\mathbb{R}$ $(\dim E = r < \infty)$,

$$L^2_{\mathbb{R}}(E, dF(x))) \stackrel{\text{df}}{=} \left\{ f(x) \colon \int\limits_{\mathbb{R}} \langle dF(x)f(x), f(x)\rangle_E < \infty \right\}. \tag{1}$$

Let the measure $dF(x)$ satisfy the nd-condition

$$\int\limits_{\mathbb{R}} \langle dF(x)P_n(x), P_n(x)\rangle > \delta_n \sum_{k=0}^{n} \|g_k\|^2 ,$$

for all $P_n(x) = \sum_{k=0}^{n} x^k g_k$ $(g_k \in E,\ 1 \leq k \leq n,\ n \in \mathbb{Z}_+)$. Then (see Part I) there exists the family of matrix-valued polynomials $\{P_n(x)\}_0^\infty$, such that

$$\int\limits_{\mathbb{R}} P_k^*(x) dF(x) P_n(x) = \delta_{k,n} I_E, \quad k, n \in \mathbb{Z}_+. \tag{2}$$

The polynomials $\{P_n(x)\}_0^\infty$ are the solutions of the finite-difference equations

$$xP_n(x) = P_{n+1}(x)B_n + P_n(x)C_n + P_{n-1}(x)B_{n-1}^*, \quad n \in \mathbb{Z}_+.$$

Let $Q_n(x)$ be polynomials of the second kind,

$$Q_n(x) \stackrel{\text{df}}{=} \int\limits_{\mathbb{R}} dF(\xi)\frac{P_n(\xi) - P_n(x)}{\xi - x}, \quad n \in \mathbb{Z}_+.$$

Construct the Jacobi operator

$$J_E \stackrel{\text{df}}{=} \begin{bmatrix} C_0 & B_0^* & 0 & 0 & \cdots \\ B_0 & C_1 & B_1^* & 0 & \cdots \\ 0 & B_1 & C_2 & B_2^* & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{3}$$

by $\{B_n, C_n\}_0^\infty$. Define the operator-function

$$W_n(\lambda) \stackrel{\text{df}}{=} \begin{bmatrix} P_n(\lambda) & P_{n+1}(\lambda)B_n \\ Q_n(\lambda) & Q_{n+1}(\lambda)B_n \end{bmatrix}$$

and the involution

$$J \stackrel{\text{df}}{=} \begin{bmatrix} 0 & iI_E \\ -iI_E & 0 \end{bmatrix}. \tag{4}$$

Specify the operator-function

$$S_n(\lambda) \stackrel{\text{df}}{=} W_n(\lambda)W_n^{-1}(0) = \begin{bmatrix} A_n(\lambda) & B_n(\lambda) \\ C_n(\lambda) & D_n(\lambda) \end{bmatrix}, \tag{5}$$

besides,

$$A_n(\lambda) = I - \lambda\sum_{k=0}^{n} P_k(\lambda)Q_k^*(0),$$

$$B_n(\lambda) = \lambda\sum_{k=0}^{n} P_k(\lambda)P_k^*(0),$$

$$C_n(\lambda) = -\lambda\sum_{k=0}^{n} Q_k(\lambda)Q_k^*(0),$$

$$D_n(\lambda) = 1 + \lambda\sum_{k=0}^{n} Q_k^*(\lambda)P_k^*(0).$$

Then

$$S_n(\lambda) = S_{n-1}(\lambda)a_n(\lambda), \quad n \in \mathbb{Z}_+, \tag{6}$$

where

$$a_n(\lambda) \stackrel{\text{df}}{=} I - i\lambda m_n J, \quad m_n \stackrel{\text{df}}{=} \begin{bmatrix} P_n(0)P_n^*(0) & P_n(0)Q_n^*(0) \\ Q_n(0)P_n^*(0) & Q_n(0)Q_n^*(0) \end{bmatrix} \geq 0, \quad n \in \mathbb{Z}_+. \tag{7}$$

II. Consider the subspaces

$$\mathcal{L}_n \stackrel{\text{df}}{=} \text{span}\left\{\sum_{k=0}^{n} P_k(x)f_k : f_k \in E(1 \leq k \leq n)\right\}, \quad n \in \mathbb{Z}_+, \tag{8}$$

in $L^2_{\mathbb{R}}(E, dF(x))$ (1). The kernel

$$K_n(\lambda, w) \stackrel{\text{df}}{=} \sum_{k=1}^{n} P_k(\lambda)P_k^*(w), \quad \lambda, w \in \mathbb{C}, \tag{9}$$

is Hermitian positive [2, 3] and is generating in $\mathcal{L}_n$,

$$f(\lambda) = \int_{\mathbb{R}} K_n(\lambda, x)dF(x)f(x)$$

for all $f(\lambda) \in \mathcal{L}_n$ ($\forall \lambda \in \mathbb{C}$). The orthoprojector in $L^2_{\mathbb{R}}(E, dF(x))$ on $\mathcal{L}_n$ is given by

$$(P_{\mathcal{L}_n} f)(x) \stackrel{\text{df}}{=} \int_{\mathbb{R}} K_n(x, y)dF(y)f(y).$$

Following [2, 3], we every vector $f = \text{col}\,[f_0, f_1, \ldots]$, $f_k \in E$, $k \in \mathbb{Z}_+$, from $l^2_{\mathbb{Z}_+}(E)$ juxtapose with function $f(x) \in L^2_{\mathbb{R}}(E, dF(x))$,

$$f(x) = Vf, \quad f(x) \stackrel{\text{df}}{=} \sum_{k=0}^{\infty} P_k(x)f_k, \tag{10}$$

where $\{P_k(x)\}_0^\infty$ is a family of orthonormal (2) polynomials. Series (10) converges in the topology of $L^2_{\mathbb{R}}(E, dF(x))$ for all $f \in l^2_{\mathbb{Z}_+}(E)$. Operator V (10) isometrically maps $l^2_{\mathbb{Z}_+}(E)$ on the subspace in $L^2_{\mathbb{R}}(E, dF(x))$,

$$\mathcal{L}_\infty \stackrel{\text{df}}{=} \text{span}\left\{f(x) : f(x) \in \mathcal{L}_n, n \in \mathbb{Z}_+\right\},$$

which is the closure of the linear span of the subspaces $\mathcal{L}_n$ (8). To calculate the inverse of V (10) is to calculate the Fourier coefficients

$$f_k = \int_{\mathbb{R}} P_k^*(x)dF(x)f(x), \quad k \in \mathbb{Z}_+.$$

The operator V (10) is a unitary operator from $l^2_{\mathbb{Z}_+(n)}(E)$ on $\mathcal{L}_n$ (8),

$$l^2_{\mathbb{Z}_+(n)}(E) \stackrel{\text{df}}{=} \text{span}\left\{f = \text{col}\,[f_0, f_1, \ldots, f_n] : f_k \in E; \sum_{k=1}^{n} \|f_k\|^2 < \infty\right\}, \tag{11}$$

besides, $\mathbb{Z}_+(n) = \{k \in \mathbb{Z}_+; 0 \le k \le n\}$, $n \in \mathbb{Z}_+$.

Let $L_n(\lambda) = [I_E, 0]\, S_n(\lambda)$,

$$L_n(\lambda) = [A_n(\lambda), B_n(\lambda)], \quad n \in \mathbb{Z}_+,$$

in view of (5). It is obvious that

$$L_n(\lambda) J L_n^*(w) = \frac{\lambda - \bar{w}}{i} K_n(\lambda, w),$$

besides, $K_n(\lambda, w)$ is given by formula (9). Consider the orthoprojectors $P_\pm = \dfrac{1}{2}(I_{E\oplus E} \pm J)$ in $E \oplus E$, then

$$L_n(\lambda) P_+ = \frac{1}{2} E_n(\lambda) [I_E, iI_E], \quad L_n(\lambda) P_- = \frac{1}{2} \widetilde{E}_n(\lambda) [I_E, -iI_E],$$

where the operator-functions $E_n(\lambda)$ and $\widetilde{E}_n(\lambda)$ in E equal

$$E_n(\lambda) \stackrel{\text{df}}{=} A_n(\lambda) - iB_n(\lambda), \quad \widetilde{E}_n(\lambda) \stackrel{\text{df}}{=} A_n(\lambda) + iB_n(\lambda), \quad \lambda \in \mathbb{C}, \quad n \in \mathbb{Z}_+. \tag{12}$$

Using $L_n(\lambda) J L_n^*(w) = L_n(\lambda) P_+ L_n^*(w) - L_n(\lambda) P_- L_n^*(w)$, we obtain

$$K_n(\lambda, w) = i \frac{E_n(\lambda) E_n^*(w) - \widetilde{E}_n(\lambda) \widetilde{E}_n^*(w)}{\lambda - \bar{w}}, \quad n \in \mathbb{Z}_+. \tag{13}$$

Theorem 1. *For all $n \in \mathbb{Z}_+$ the operator-functions $E_n(\lambda)$, $\widetilde{E}_n(\lambda)$ (12) satisfy the relations*

$$E_n(\lambda) E_n^*(\lambda) - \widetilde{E}_n(\lambda) \widetilde{E}_n(\lambda) \begin{cases} > 0, & \lambda \in \mathbb{C}_+, \\ = 0, & \lambda \in \mathbb{R}, \\ < 0, & \lambda \in \mathbb{C}_-, \end{cases} \tag{14}$$

besides, the function $E_n(\lambda)$ $(\widetilde{E}_n(\lambda))$ is invertible and $\overline{\mathbb{C}}_+$ (correspondingly, in $\overline{\mathbb{C}}_-$).

Proof. The relations (13) follow from the Hermitian positiveness of the kernel $K_n(\lambda, w)$ (12). The inequality $E_n(\lambda) E_n^*(\lambda) - \widetilde{E}_n(\lambda) \widetilde{E}_n^*(\lambda) > 0$ for all $\lambda \in \mathbb{C}_+$ follows from

$$E_n(\lambda) E_n^*(\lambda) - \widetilde{E}_n(\lambda) \widetilde{E}_n^*(\lambda) = \frac{\lambda - \bar{\lambda}}{i} \sum_{k=0}^{n} P_k(\lambda) P_k^*(\lambda) \ge \frac{\lambda - \bar{\lambda}}{i} P_0(\lambda) P_0^*(\lambda) =$$

$$= \frac{\lambda - \bar{\lambda}}{i} D_0^2 > 0,$$

in view of the invertibility of D. This implies the invertibility of $E_n^*(\lambda)$ (and so the invertibility of $E_n(\lambda)$ also) for all $\lambda \in \mathbb{C}_+$.

If for some $x \in \mathbb{R}$ there is such a subsequence $\{f_s\}_0^\infty$ ($\|f_s\| = 1$ $\forall s \in \mathbb{Z}_+$) that $E_n^*(x) f_s \to 0$, $s \to \infty$, then $K_n(\lambda, x) f_s \to 0$, $s \to \infty$, in virtue of (12), for all $\lambda \in \mathbb{C}$. Thus $\sum_{k=0}^{n} P_k(\lambda) P_k^*(x) f_s \to 0$, $s \to \infty$, therefore $\sum_{k=0}^{n} \|P_k^*(x) f_s\|^2 \to 0$, $s \to \infty$, which is impossible, since $\|P_0^*(x) f_s\|^2 = \|D_0 f_s\|^2 > 0$.

In the case of the self-adjointness of $A_n(\lambda)$ and $B_n(\lambda)$ as $\lambda \in \mathbb{R}$, the functions $\widetilde{E}_n(\lambda) = E_n^*\left(\bar{\lambda}\right)$ and $E_n(\lambda)$ belong to the Hermite – Biehler class [14, 15].

The operator-function $L_k(\lambda)$ satisfies the recurrent relation

$$L_k(\lambda) = L_{k-1}(\lambda) a_k(\lambda), \quad k \in \mathbb{Z}_+, \tag{15}$$

in view of (6). Taking into account the form of $a_k(\lambda)$ (7), we obtain that $L_k(\lambda)$ is the solution of the system of equations

$$L_k(\lambda) + i\lambda \sum_{s=0}^{k} L_{s-1}(\lambda) m_s J = \left[I_E, 0\right], \quad 0 \le k \le n,\ n \in \mathbb{Z}_+,$$

where m_k are given by (7) and $L_{-1}(\lambda) \stackrel{\mathrm{df}}{=} \left[I_E, 0\right]$. The J-properties of $a_k(\lambda)$ (7) imply

$$S_n(\lambda) J S_n^*(w) - J = \frac{\lambda - \bar{w}}{i} \sum_{k=0}^{n} S_{k-1}(\lambda) m_k S_{k-1}^*(w),$$

and thus

$$L_n(\lambda) J L_n^*(w) = \frac{\lambda - \bar{w}}{i} \sum_{k=0}^{n} L_{k-1}(\lambda) m_k L_{k-1}^*(w),$$

so,

$$K_n(\lambda, w) = \sum_{k=0}^{n} L_{k-1}(\lambda) m_k L_{k-1}^*(w). \tag{16}$$

Define the weight spaces

$$l^2_{\mathbb{Z}_+(n)}(E \oplus E, m) \stackrel{\mathrm{df}}{=} \operatorname{span}\left\{ f = [f_0, \ldots, f_n] : \sum_{k=0}^{n} \langle m_k f_k, f_k \rangle < \infty \right\}, \tag{17}$$

where m_k are given by (7); $f_k = \operatorname{col}\left[u_k, v_k\right]$; $u_k,\ v_k \in E,\ 0 \le k \le n$. Factorization by the metric kernel is executed in this space (17). The spaces (17) are ordered by inclusion $l^2_{\mathbb{Z}_+(n)}(E \oplus \oplus E, m) \subseteq l^2_{\mathbb{Z}_+(p)}(E \oplus E, m)$ as $n < p$. The inner product in $l_{\mathbb{Z}_+(n)}(E \oplus E, m)$ is given by $\langle f, g \rangle = \sum_{k=0}^{n} \langle m_k f_k, g_k \rangle$, where $f = [f_0, \ldots, f_n]$, $g = [g_0, \ldots, g_n]$ are from $l_{\mathbb{Z}_+(n)}(E \oplus E, m)$.

III. Every $f \in l^2_{\mathbb{Z}_+(n)}(E \oplus E, m)$ we juxtapose with the E-valued vector-function $F(\lambda)$,

$$F(\lambda) = \mathcal{B} f \stackrel{\mathrm{df}}{=} \sum_{k=0}^{n} L_{k-1}(\lambda) m_k f_k, \tag{18}$$

assuming that $L_k(\lambda)$ is the solution of (15). The map $\mathfrak{B}$ is said to be the **L. de Branges transform** [10, 12]. Obviously, $\deg F(\lambda) \le n$. Describe the class of functions $F(\lambda)$ (18).

Lemma 1. *For all* $n \in \mathbb{Z}_+$ *the operator-functions* $E_n(\lambda)$, $\widetilde{E}_n(\lambda)$ (12) *are polynomials of degree* $n+1$, *leading coefficients of which are invertible operators.*

Consider the expression $E_n^{-1}(\lambda)F(\lambda)$ where $F(\lambda)$ is given by (18) and $\lambda \in \mathbb{C}_+$. Theorem 1 yields that this function is holomorphic in $\mathbb{C}_+$ and $E_n^{-1}(\lambda)F(\lambda) \to 0$ as $\lambda \to \infty$ (since $\deg F(\lambda) \le n$). In connection with the fact that the function $E_n^{-1}(x)F(x)$, $x \in \mathbb{R}$, behaves at infinity as $x^{-p}N$ ($\alpha \in \mathbb{N}$, N is a linear bounded operator in E), the integral

$$\int\limits_{\mathbb{R}} \left\| E_n^{-1}(x)F(x) \right\|_E^2 dx < \infty$$

converges. Similar fact takes place for the function $\widetilde{E}^{-1}(\lambda)F(\lambda)$ in $\mathbb{C}$, besides,

$$\int\limits_{\mathbb{R}} \left\| E_n^{-1}(x)F(x) \right\|_E^2 dx = \int\limits_{\mathbb{R}} \left\| \widetilde{E}_n^{-1}(x)F(x) \right\|_E^2 dx,$$

since $\left(E_n^*(x)\right)^{-1} E_n^{-1}(x) = \left(\widetilde{E}_n^*(x)\right)^{-1} \widetilde{E}_n^{-1}(x)$ as $x \in \mathbb{R}$ in virtue of (14). As a result, we obtain the inner description of the space of the function $F(\lambda)$ (17).

Definition 1. *Let $E_n(\lambda)$ and $\widetilde{E}_n(\lambda)$ be operator-valued polynomials in E of degree $n+1$, the leading coefficients of which are invertible operators, $E_n(0) = \widetilde{E}_n(0) = I$, and* (14) *take place, besides, $E_n(\lambda)$ and $\widetilde{E}_n(\lambda)$ are invertible in the semiplanes $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$ correspondingly.*

The linear span of the entire E-valued ($\dim E < \infty$) *functions $F(\lambda)$ is said to be the L. de Branges space $\mathcal{B}(A_n, B_n)$* (*here $2A_n = E_n + \widetilde{E}_n$, $2iB_n = \widetilde{E}_n - E_n$*) *if*

a) $E_n^{-1}(\lambda)F(\lambda)$ ($\widetilde{E}_n^{-1}(\lambda)F(\lambda)$) *is holomorphic in* $\mathbb{C}_+$ (*in* $\mathbb{C}_-$) *function, besides,* $E_n^{-1}(\lambda)F(\lambda) \to$ $\to 0$ ($\widetilde{E}_n^{-1}(\lambda)F(\lambda) \to 0$) *as* $\lambda \to \infty$ *and* $\lambda \in \mathbb{C}_+$ (*correspondingly,* $\lambda \in \mathbb{C}_-$);

b) *the integral*

$$\int\limits_{\mathbb{R}} \left\| E_n^{-1}(x)F(x) \right\|_E^2 dx = \int\limits_{\mathbb{R}} \left\| \widetilde{E}_n^{-1}(x)F(x) \right\|_E^2 dx < \infty \tag{19}$$

is finite.

The inner product in $\mathcal{B}(A_n, B_n)$ is given by

$$\langle F(\lambda), G(\lambda) \rangle = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} \left\langle E_n^{-1}(x)F(x), E_n^{-1}(x)G(x) \right\rangle_E dx.$$

Factorization by the metric kernel is executed in the space $\mathcal{B}(A_n, B_n)$.

This definition of the L. de Branges space is the generalization of the classical L. de Branges space [10, 12] and for vector-valued functions is new.

The function $K_n(\lambda, w)$ (13) is the **reproducing kernel** in $\mathcal{B}(A_n, B_n)$,

$$\langle F(\lambda), K_n(\lambda, w)g \rangle_{\mathcal{B}} = \langle F(w), g \rangle_E \tag{20}$$

for all $F(\lambda) \in \mathcal{B}(A_n, B_n)$, all $w \in \mathbb{C}$, and all $g \in E$.

Theorem 2. *The operator $\mathcal{B}$* (18) *specifies the one-to-one correspondence between the spaces $l^2_{\mathbb{Z}_+(n)}(E \oplus E, m)$* (17) *and $\mathcal{B}(A_n, B_n)$, besides, the Parseval equality*

$$\langle F(\lambda), G(\lambda) \rangle_{\mathcal{B}} = \langle f, g \rangle_{l^2},$$

takes place, where $F(\lambda) = \mathcal{B}f$, $G(\lambda) = \mathcal{B}g$, and f, $g \in l^2_{\mathbb{Z}_+(n)}(E \oplus E, m)$.

Proof. Since

$$\langle K_n(\lambda,w)f,g\rangle_E = \langle K_n(x,w)f, K_n(x,\lambda)g\rangle_{\mathcal{B}},$$

where $f,\ g \in E$, then using (16) we obtain

$$\langle K_n(\lambda,w)f,g\rangle_E = \langle f(w), g(\lambda)\rangle_{l^2},$$

besides, $f(w)$ and $g(\lambda)$ are given by $f(w) = [f_0(w),\ldots,f_n(w)]$, $g(\lambda) = [g_0(\lambda),\ldots,g_n(\lambda)]$ $(f_k(w) = \operatorname{col}\left[A^*_{k-1}(w)f, B^*_{k-1}(w)f\right],\ g_k(\lambda) = \operatorname{col}\left[A^*_{k-1}(\lambda)g, B^*_{k-1}(\lambda)g\right],\ 0 \le k \le n)$ and belong to space (17), for all $\lambda,\ w \in \mathbb{C}$ and all $f,\ g \in E$. Since $\mathcal{B}f(w) = K_n(\lambda,w)f$ and $\mathcal{B}g(w) = K_n(\lambda,w)g$, we obtain the Parseval equality (20).

To conclude the proof, we need to ascertain that $L_n \overset{\text{df}}{=} \operatorname{span}\{f(w) \colon f \in E, w \in \mathbb{C}\}$ coincides with $L^2_{\mathbb{Z}_+(n)}(E\oplus E, m)$, and, secondly, to show that the space $\mathcal{B}_n \overset{\text{df}}{=} \operatorname{span}\{K_n(\lambda,w)f \colon f\in E, w\in\mathbb{C}\}$ coincides with $\mathcal{B}(A_n,B_n)$. The equality $\mathcal{B}_n = \mathcal{B}(A_n,B_n)$ follows from (19); since if there is the function $F(\lambda) \in \mathcal{B}(A_n,B_n)$ orthogonal to $\mathcal{B}_n$, then, taking into account (20), we obtain $\langle F(w), f\rangle_E = 0$ for all $w\in\mathbb{C}$ and all $f \in E$, and thus $F(\lambda)\equiv 0$.

Let $L_n \ne l^2_{\mathbb{Z}_+(n)}(E\oplus E,m)$, then there is such a vector $g \in L^2_{\mathbb{Z}_+(n)}(E\oplus E,m)$ that $g\perp L_n$,

$$\langle f(w), g\rangle_{l^2} = \left\langle f, \sum_{k=0}^{n} L_{k-1}(w) m_k g_k \right\rangle_E \qquad (\forall f\in E),$$

therefore

$$\sum_{k=0}^{n} L_{k-1}(w) m_k g_k = 0,$$

$$L_{k-1}(w)m_k = \left[P_k(w)P_k^*(0), P_k(w)Q_k^*(0)\right], \quad k\in\mathbb{Z}_+,$$

therefore

$$0 = \sum_{k=0}^{n} L_{k-1}(w)m_k g_k = \sum_{k=0}^{n} P_k(w)\left\{P_k^*(0)u_k + Q_k^*(0)v_k\right\},$$

where $g_k = \operatorname{col}[u_k, v_k]$, $1 \le k \le n$. Taking into account the orthonormality (2) of the polynomials $\{P_k(\lambda)\}_0^n$, we obtain

$$P_k^*(0)u_k + Q_k^*(0)v_k = 0, \quad 0\le k\le n.$$

This signifies that the norm of the vector g in $L^2_{\mathbb{Z}_+(n)}(E\oplus E,m)$ is equal to zero. So, $g=0$.

3. Nonself-adjoint operators. IV. Specify the nonself-adjoint operator

$$(A_n f)_k \overset{\text{df}}{=} -i\sum_{s=k+1}^{n} J m_s f_s, \quad 0\le k\le n, \tag{21}$$

in $l^2_{\mathbb{Z}_+(n)}(E\oplus E,m)$ (17), where J and m_k are given by (4) and (7) $(\dim E<\infty)$. It is obvious that A_n (21) is nilpotent. Specifying the operator $\varphi\colon l^2_{\mathbb{Z}_+(n)}(E\oplus E,m)\to E\oplus E$ by the formula

$$\varphi_n f \overset{\text{df}}{=} \sum_{s=0}^{n} m_s f_s$$

we obtain the colligation [10]

$$\Delta_n = \left(A_n, l^2_{\mathbb{Z}_+(n)}(E\oplus E,m), \varphi_n, E\oplus E, -J\right). \tag{22}$$

Theorem 3. *The characteristic Livsic – Brodskii function* $S_{\Delta_n}(\lambda) = I - i\varphi\left(A_n - \lambda I\right)^{-1}\varphi_n^*(-J)$ *of the colligation* Δ_n (22) *coincides with* $S_n\left(\lambda^{-1}\right)$ (5), $S_{\Delta_n}(\lambda) = S_n\left(\lambda^{-1}\right)$.

The L. de Branges transform $\mathcal{B}$ (18) transforms the operator A_n (21) into the shift operator $\widetilde{A}_n$,

$$\left(\widetilde{A}_n F\right)(\lambda) \stackrel{\mathrm{df}}{=} \frac{1}{\lambda}(F(\lambda) - F(0)), \quad F(\lambda) \in \mathcal{B}\left(A_n, B_n\right). \tag{23}$$

The operator $\widetilde{\varphi}_n^* = \mathcal{B}\varphi_n^*$ equals

$$\widetilde{\varphi}_n^* g \stackrel{\mathrm{df}}{=} e_1(\lambda)u + e_2(\lambda)v, \quad g = \mathrm{col}[u, v],$$

where

$$e_1(\lambda) = \lambda^{-1}B_n(\lambda), \quad e_2(\lambda) = \lambda^{-1}\left(I - A_n(\lambda)\right). \tag{24}$$

So, we have the colligation

$$\widetilde{\Delta}_n = \left(\widetilde{A}_n, \mathcal{B}\left(A_n, B_n\right), \widetilde{\varphi}_n, E \oplus E, -J\right), \tag{25}$$

which is unitary equivalent to Δ_n (22).

V. Let us turn our attention to the finding of the weight operators $\{m_k\}_0^n$ by the pair of functions $E_n(\lambda)$, $\widetilde{E}_n(\lambda)$, for which Theorem 1 is true and $E_n(0) = \widetilde{E}_n(0) = I$. To do this, we by $E_n(\lambda)$, $\widetilde{E}_n(\lambda)$ construct the L. de Branges space $\mathcal{B}\left(A_n, B_n\right)$, in which we define the operator $\widetilde{A}_n$ (23), and then we construct the colligation $\widetilde{\Delta}_n$ and (25). Let $S_n(w) = S_{\widetilde{\Delta}_n}\left(w^{-1}\right)$, where $S_{\widetilde{\Delta}_n}(\lambda)$ is the characteristic function of $\widetilde{\Delta}_n$ (25). It is obvious that $S_n^{1,1}(w) = A_n(w)\ \left(= 2^{-1}\left(E_n(w) + \widetilde{E}_n(w)\right)\right)$ and $S_n^{1,2}(w) = B_n(w)\ \left(= (2i)^{-1}\left(\widetilde{E}_n(w) - E_n(w)\right)\right)$, where $\left\{S_n^{k,s}(w)\right\}_1^2$ are blocks $S_n(w)$ corresponding to the decomposition $E \oplus E$. The remaining blocks $C_n(w) = S_n^{2,1}(w)$ and $D_n(w) = S_n^{2,2}(w)$ equal

$$\begin{aligned} C_n(w) &= \frac{w}{2\pi}\int\limits_{\mathbb{R}} \frac{I - A_n^*(\lambda)}{\lambda}\left(E_n^*(\lambda)\right)^{-1}E_n^{-1}(\lambda)\frac{A_n(\lambda) - A_n(w)}{\lambda - w}d\lambda, \\ D_n(w) &= I + \frac{w}{2\pi}\int\limits_{\mathbb{R}} \frac{I - A_n^*(\lambda)}{\lambda}\left(E_n^*(\lambda)\right)^{-1}E_n^{-1}(\lambda)\frac{B_n(\lambda) - B_n(w)}{\lambda - w}d\lambda. \end{aligned} \tag{26}$$

Theorem 4. *Let there be determined two such matrix-functions* $E_n(\lambda)$ *and* $\widetilde{E}_n(\lambda)$ *that* a) $E_n(\lambda)$, $\widetilde{E}_n(\lambda)$ *are polynomials of degree* $n+1$ *with invertible leading coefficients*; b) $E_n(0) = \widetilde{E}_n(0) = I$; c) (12) *take place*; d) $E_n(\lambda)$ *and* $\widetilde{E}_n(\lambda)$ *are invertible in* $\overline{\mathbb{C}}_+$ *and in* $\overline{\mathbb{C}}_-$ *correspondingly. Then by the pair of functions* $A_n(\lambda) = \dfrac{1}{2}(\widetilde{E}_n(\lambda) + E_n(\lambda))$, $B_n(\lambda) = \dfrac{1}{2i}(\widetilde{E}_n(\lambda) - E_n(\lambda))$ *we can construct the functions* $C_n(\lambda)$, $D_n(\lambda)$ (26) *such that the operator-function* $S_n(\lambda)$ (5) *has the* J*-properties.*

Using the J-theory of V. P. Potapov [9, 10], we expand $S_n(\lambda)$ (5) into factors, then the simplest factors $a_k(\lambda)$ are the polynomials of the first degree and are given by (7), where $m_k \geq 0$ and $m_k J m_k = 0$, $0 \leq k \leq n$.

Lemma 2. *Let* $m \geq 0$ *be an operator in* E $(\dim E < \infty)$ *such that* $mJm = 0$, *where* J *is given by* (24), *then*

$$m = \left[\begin{array}{cc} PP^* & PQ^* \\ QP^* & QQ^* \end{array} \right],$$

besides, P, Q *are linear bounded operators in* E *satisfying the condition* $P^*Q = Q^*P$.

Proof. The lemma on block-matrix [7] yields that nonnegativity of the operator

$$m = \left[\begin{array}{cc} A & B \\ B* & C \end{array} \right] \geq 0$$

is equivalent to the conditions: 1) $A \geq 0$; 2) solution X of the equation $AX = B$ exists; 3) $C - X^*AX \geq 0$, besides, the expression X^*AX does not depend on X. Nonnegativity $A \geq 0$ signifies that $A = PP^*$ (we can take $P = \sqrt{A}$ as P). Condition 2 yields that $PP^*X = B$, therefore, specifying $Q^* = P^*X$, we obtain $B = PQ^*$.

The requirement $mJm = 0$ is equivalent to the equalities

$$AB^* = BA, \quad AC = B^2, \quad B^*C = CB.$$

The first relation implies

$$0 = AB^* - BA = PP^*QP^* - PQ^*PP^* = P\left\{P^*Q - Q^*P\right\}P^*,$$

therefore $\{P^*Q - QP^*\}|_{P^*E} = 0$ since the images P^* and Q^* belong to $\overline{P^*E}$. Taking into account that $i\left\{P^*Q - Q^*P\right\}$ is self-adjoint and equals zero on its image, we obtain the desired condition $P^*Q = Q^*P$.

To conclude the proof of lemma, it is left to show that $C = QQ^*$. Condition 3 $C - X^*AX \geq 0$ implies

$$0 \leq C - X^*AX = C - X^*B = C - X^*PQ^* = C - QQ^*.$$

Existence of the solution of $AX = B$ signifies that A is invertible on $\overline{BE}$, therefore the equality $AC = B^2$ yields

$$C = XB = XPQ^* = X^*PQ^* + (X - X^*)\,PQ^* = QQ^* + (X - X^*)\,PP^*X,$$

and thus $(X - X^*)\,PP^*X \geq 0$. Since $X^*\colon PE \to QE$ and $X\colon QE \to PE$, the self-adjoint operator $(X - X^*)\,PP^*X$ maps the subspace $\mathrm{span}\{PE + QE\}$ onto itself. Note that $0 = P^*Q - Q^*P = P^*(X - X^*)P$ and thus the restriction of $X - X^*$ on PE equals zero. Therefore to prove that $(X - X^*)\,PP^*X = 0$, it is necessary to ascertain that the image of the operator $(X - X^*)\,PP^*X$ belongs to PE. Let $f \in \mathrm{span}\{PE + QE\}$ and $f \perp PE$, then $X\colon QE \to PE$ implies

$$0 \leq \langle (X - X^*)\,PP^*Xf, f\rangle = -\langle X^*PP^*Xf, f\rangle = -\left\|P^*Xf\right\|^2,$$

and thus $P^*Xf = 0$ and $(X - X^*)\,PP^*Xf = 0$. Thus $C = QQ^*$.

The condition $B^*C = CB$ holds automatically:

$$B^*C - CB = QP^*QQ^* - QQ^*PQ^* = Q\left\{P^*Q - Q^*P\right\}Q^* = 0,$$

in virtue of $P^*Q = Q^*P$.

Thus the operators m_k in $a_k(\lambda)$ (7) are always given by

$$m_k = \begin{bmatrix} P_k P_k^* & P_k Q_k^* \\ Q_k P_k^* & Q_k Q_k \end{bmatrix}, \quad 1 \le k \le n. \tag{27}$$

The "first line" $L_n(\lambda) = [A_n(\lambda), B_n(\lambda)]$ of the operator-function $S_n(\lambda)$ (5) satisfies the recurrent relation $L_n(\lambda) = L_{n-1}(\lambda) a_n(\lambda)$. Since $a_n^{-1}(\lambda) = I + i\lambda m_n J$, then $L_{n-1}(\lambda) = L_n(\lambda)\,(I + i\lambda m_n J)$, and thus

$$A_{n-1}(\lambda) = A_n(\lambda) + \lambda \mathcal{P}_n(\lambda) Q_n^*,$$

$$B_{n-1}(\lambda) = B_n(\lambda) - \lambda \mathcal{P}_n(\lambda) P_n^*,$$

where $\mathcal{P}_n(\lambda)$ is given by

$$\mathcal{P}_n(\lambda) \overset{\text{df}}{=} A_n(\lambda) P_n + B_n(\lambda) Q_n. \tag{28}$$

Repeating this step-by-step procedure, we find all the polynomials $\{\mathcal{P}_k(\lambda)\}_0^n$, besides,

$$A_n(\lambda) = I - \lambda \sum_{k=0}^{n} \mathcal{P}_k(\lambda) Q_k^*, \quad B_n(\lambda) = \lambda \sum_{k=0}^{n} \mathcal{P}_k(\lambda) P_k^*. \tag{29}$$

Theorem 5. *Let operator-functions $E_n(\lambda)$ and $\widetilde{E}_n(\lambda)$ satisfying the suppositions* a) – d) *of Theorem* 4 *be given in a Hilbert space E. Then there exists such set of nonnegative operators $\{m_k\}_0^n$* (27) *in $E \oplus E$ that $P_k^* Q_k = Q_k^* P_k$, $0 \le k \le n$, besides,*

1) *using the recurrent relation $L_{k-1}(\lambda) = L_k(\lambda)\,(I + i\lambda m_k J)$ we can construct the whole $L_k(\lambda) = [A_k(\lambda), B_k(\lambda)]$, $0 \le k \le n$, by the line $L_n(\lambda) = [A_n(\lambda), B_n(\lambda)]$ $(2A_n(\lambda) = \widetilde{E}_n(\lambda) + E_n(\lambda)$, $2iB_n(\lambda) = \widetilde{E}_n(\lambda) - E_n(\lambda))$, besides,* $\deg A_k(\lambda) = \deg B_k(\lambda) = k + 1$;

2) *for the functions $E_k(\lambda)$, $\widetilde{E}_k(\lambda)$ corresponding to $L_k(\lambda)$, the suppositions* a) – d) *of Theorem* 4 *are true*;

3) *the equalities* (28) *specify the system of polynomials $\{\mathcal{P}_k(\lambda)\}_0^n$ such that* $\deg \mathcal{P}_k(\lambda) = k$, *the leading coefficient of $\mathcal{P}_k(\lambda)$ is invertible, $\mathcal{P}_k(0) = P_k$, and the formulas* (29) *are true.*

4. Jacobi operator. VI. Consider the "nth cut" of the Jacobi operator J_E (3),

$$J_{E,n} \overset{\text{df}}{=} \begin{bmatrix} C_0 & B_0^* & 0 & \cdots & \cdots & \cdots \\ B_0 & C_1 & B_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & B_{n-2} & C_{n-1} & B_{n-1}^* \\ \cdots & \cdots & \cdots & 0 & B_{n-1} & C_n \end{bmatrix}, \tag{30}$$

which is self-adjoint in $l^2_{\mathbb{Z}_+(n)}(E)$ (11). Using the form of operator V (10), we obtain

$$V J_{E,n} f = x f(x) + R_n(x) f_n, \tag{31}$$

where $f(x) = Vf = \sum_{k=1}^{n} P_k(x) f_k \in \mathcal{L}_n$ (10) ($f = \operatorname{col}[f_0, \ldots, f_n] \in l^2_{\mathbb{Z}_+(n)}(E)$), and $R_n(x)$ is a polynomial of degree $n + 1$,

$$R_n(x) \overset{\text{df}}{=} P_n(x)\,[C_n - xI] + P_{n-1}(x) B_{n-1}^*, \quad n \in \mathbb{Z}_+. \tag{32}$$

Taking into account (15), we write $R_n(x)$ in the form $R_n(x) = -P_{n+1}(x)B_n$.

The operator $J_{E,n}$ (30), after the transform V, turns into the restriction of the operator of multiplication by the independent variable in $\mathcal{L}_n$,

$$\left(\widetilde{J}_{E,n}f\right)(x) \stackrel{\mathrm{df}}{=} P_{\mathcal{L}_n} x f(x) \quad (VJ_{E,n} = \widetilde{J}_{E,n}V),$$

where $P_{\mathcal{L}_n}$ is the orthoprojector on $\mathcal{L}_n$.

Calculate the resolvent $(J_{E,n} - wI)^{-1}$, and let $g(w) = (J_{E,n} - wI)^{-1} f$, then $J_{E,n}g(w) - wg(w) = f$. Using (31), we obtain

$$(x - w)g(x, w) + R_n(x)g_n(w) = f(x),$$

where $f(x) = Vf = \sum_{k=0}^{n} P_k(x) f_k$ and $g(x, w) = Vg(w) = \sum_{k=0}^{n} P_k(x)g_k(w)$. Supposing in this equality that $x = w$, we find $R_n(w)g_n(w) = f(w)$. If $w \in \mathbb{C}$ is such that $R_n(w)$ is invertible (which is possible in view of the invertibility of the leading coefficient of $R_n(x)$), then from the last equality we obtain

$$g(x, w) = \left(\widetilde{J}_{E,n} - wI\right)^{-1} f(x) = \frac{f(x) - R_n(x)R_n^{-1}(w)f(w)}{x - w}. \tag{33}$$

Therefore the "kth" component $g_k(w)$ of the vector $g(w) = (J_{E,n} - wI)^{-1} f$ equals

$$g_k(w) = \int\limits_{\mathbb{R}} P_k^*(x)dF(x) \left\{ \frac{f(x) - R_n(x)R_n^{-1}(w)f(w)}{x - w} \right\}, \quad 0 \le k \le n. \tag{34}$$

Theorem 6. *The resolvent of the operator* $\widetilde{J}_{E,n}$ (33) *is given by* (34), *and the resolvent of the operator* $J_{E,n}$ (30) *is given by formula* (47), *where* $f(x) = Vf$ (10), $g_k(w)$ *is the "kth" component of the vector* $g(w)$, *besides,* $R_n(w)$ *equals* (32).

We can write the L. de Branges map $\mathcal{B}$ (18) from $l^2_{\mathbb{Z}_+(n)}(E \oplus E, m)$ in $\mathcal{B}(A_n, B_n)$ in the form

$$F(\lambda) = \mathcal{B}f = \sum_{k=0}^{n} P_k(\lambda) \left[P_k^*(0)u_k + Q_k^*(0)v_k\right]$$

since $A_{k-1}(\lambda)P_k(0) + B_{k-1}(\lambda)Q_k(0) = P_k(\lambda)$, $1 \le k \le n$, where $f = [f_0, \ldots, f_n] \in l^2_{\mathbb{Z}_+(n)}(E \oplus E, m)$ ($f_k = \mathrm{col}\,[u_k, v_k]$, $0 \le k \le n$). After the transform $\mathcal{B}$ (18), the Jacobi operator $J_{E,n}$ (30) equals

$$\left(\widehat{J}_{E,n}F\right)(\lambda) = P_{\mathcal{B}(A_n,B_n)}\lambda F(\lambda),$$

where $F(\lambda) \in \mathcal{B}(A_n, B_n)$ and $P_{\mathcal{B}(A_n,B_n)}$ is the orthoprojector on $\mathcal{B}(A_n, B_n)$. Similarly to (34), the formula

$$\left(\widehat{J}_{E,n} - wI\right)^{-1} F(\lambda) = \frac{F(\lambda) - R_n(\lambda)R_n^{-1}(w)F(w)}{\lambda - w}$$

is true, besides, $R_n(\lambda)$ is given by (32).

Let $\dim E = r < \infty$. Let us find the eigenvectors $J_{E,n}f = \lambda f$ of the operator $J_{E,n}$ (30), where $f = \mathrm{col}\,[f_0, \ldots, f_n] \in l^2_{\mathbb{Z}_+(n)}(E)$ (11). Then

$$f_0 = P_0^*(\lambda)h, \ldots, f_n = P_n^*(\lambda)h, \quad R_n^*(\lambda)h = 0,$$

where the function $R_n(\lambda)$ is given by (32). The last of the equations $R_n^*(\lambda)h = 0$ defines the eigenvalues $\{\lambda_k\}_1^{r(n+1)}$, $\lambda_k \in \mathbb{R}$, of the operator $J_{E,n}$ which are the zeros of the polynomial $\det R_n^*(\lambda) = 0$ and $h_k \in \operatorname{Ker} R_n^*(\lambda_k)$. Therefore $f_{\lambda_k} = \operatorname{col}[P_0^*(\lambda_k)h_k, \ldots, P_n^*(\lambda_k)h_k]$ are the eigenvectors of $J_{E,n}$, besides, $f_{\lambda_k} \perp f_{\lambda_s}$ as $\lambda_k \neq \lambda_s$. When $\dim \operatorname{Ker} R_n^*(\lambda_k) = l_k > 1$, it is easy to construct such a basis $\{h_k^s\}_1^{l_k}$ in $\operatorname{Ker} R_n^*(\lambda_k)$ that the vectors $f_{\lambda_k}^s = \operatorname{col}[P_0(\lambda_k)h_k^s, \ldots, P_n(\lambda_k)h_k^s]$, $1 \leq s \leq l_k$, are orthogonal. To do this one should use the Hilbert–Schmidt orthogonalization. Let the vectors h_k and $\hat{h}_k$ from $\operatorname{Ker} R_n^*(\lambda_k)$ correspond to f_{λ_k} and $\hat{f}_{\lambda_k}$ from $\operatorname{Ker}(J_{E,n} - \lambda_k I)$, then $h_k' = \hat{h}_k - \mu_k h_k$ corresponds to the linear span $f_{\lambda_k}' = \hat{f}_{\lambda_k} - \mu_k f_{\lambda_k}$, $\mu_k \in \mathbb{C}$. Then from $f_{\lambda_k}' \perp f_{\lambda_k}$ we find $\mu_k = \left\langle K_n(\lambda_k, \lambda_k)\hat{h}_k, h_k \right\rangle \{\langle K_n(\lambda_k, \lambda_k)h_k, h_k\rangle\}^{-1}$. Using this technique the required number of times, we obtain an orthogonal set of vectors in $\{f_{\lambda_k}^s\}_1^l$ in $\operatorname{Ker}\{J_{E,n} - \lambda_k I\}$. So, the orthogonal basis $\{f_{\lambda_k}^s\}$ of the eigenvectors of $J_{E,n}$ in $l^2_{\mathbb{Z}_+(n)}(E)$ (11) exists.

Theorem 7. *Let* $\dim E = r < \infty$ *and* $\{\lambda_k\}$ *be zeros of the polynomial* $\det R_n^*(\lambda) = 0$, *where* $R_n(\lambda)$ *is given by* (32), *besides,* $\operatorname{Ker} R_n^*(\lambda_k) = L_k$ *and* $\dim L_k = l_k > 1$. *Suppose that* $\{h_k^s\}_1^{l_k}$ *from* L_k *are such that* $\langle K_n(\lambda_k, \lambda_k)h_k^s, h_k^p\rangle = 0$ *as* $s \neq p$. *Then the reproducing kernel* $K_n(\lambda, w)$ (9) *generates the orthogonal basis*

$$\{K_n(x, \lambda_k)h_k^s\}, \quad 1 \leq s \leq l_k, \quad 1 \leq k \leq (n+1)r),$$

in the space $\mathcal{L}_n$ (8), *besides, every function* (34) *is an eigenfunction for* $\widetilde{J}_{E,n}$ (33) *and corresponds to the eigenvalue* λ_k.

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Received 06.10.15,
after revision — 10.03.16