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TRI-ADDITIVE MAPS AND LOCAL GENERALIZED (α,β)-DERIVATIONS

ТРИАДИТИВНІ ВІДОБРАЖЕННЯ ТА ЛОКАЛЬНІ УЗАГАЛЬНЕНІ (α,β)-ПОХІДНІ

Let R be a prime ring with nontrivial idempotents. We characterize a tri-additive map $f\colon R^3\to R$ such that $f(x,y,z)=0$ for all $x,y,z\in R$ with $xy=yz=0$. As an application, we show that, in a prime ring with nontrivial idempotents, any local generalized (α,β)-derivation (or a generalized Jordan triple (α,β)-derivation) is a generalized (α,β)-derivation.

Нехай R — просте кільце з нетривіальними ідемпотентами. Охарактеризовано триадитивне відображення $f\colon R^3\to R$ таке, що $f(x,y,z)=0$ для всіх $x,y,z\in R$ таких, що $xy=yz=0$. Як застосування показано, що у простому кільці з нетривіальними ідемпотентами довільна локальна узагальнена (α,β)-похідна (або узагальнена жорданова потрійна (α,β)-похідна) є узагальненою (α,β)-похідною.

1. Introduction. Throughout this paper, R denotes a prime ring with center $Z(R)$, right (resp. left) Martindale quotient ring Q_r (resp. Q_ℓ), and symmetric Martindale quotient ring Q_s. Let Q_{mr} (resp. Q_{ml}) denote the maximal right (resp. left) ring of quotients of R. We refer the reader to the book [1] for details.

In [5], Chebotar, Ke and Lee characterized some maps preserving zero products: assume that the ring R possesses nontrivial idempotents. If $\phi\colon R\to R$ is a bijective additive map such that $\phi(x)\phi(y)=0$ whenever $xy=0$, then $\phi(xy)\phi(z)=\phi(x)\phi(yz)$ for any $x,y,z\in R$. Moreover, if $1\in R$, then $\phi(xy)=\lambda\phi(x)\phi(y)$ for any $x,y\in R$, where $\lambda=\phi(1)^{-1}\in C$ [5] (Theorem 3). In [2], Brešar also discussed additive maps preserving zero products. In [6], Chuang and Lee considered a general case, namely, a bi-additive map $\phi\colon R\times R\to R$ such that $\phi(x,y)=0$ whenever $xy=0$ (see Theorem 2.1). In this paper, we will generalize this result to a tri-additive map $f\colon R^3\to R$ such that $f(x,y,z)=0$ whenever $xy=yz=0$.

Let M be a R-bimodule. An additive mapping $g\colon R\to M$ is called a generalized derivation with associated derivation $d\colon R\to M$ if $g(xy)=g(x)y+xd(y)$ for all $x,y\in R$. In [11], Lee gave a characterization of generalized derivations: every generalized derivation g on a dense right ideal of R can be extended to Q_{mr} and can be written in the form $g(x)=ax+d(x)$ for some $a\in Q_{mr}$ and some derivation d on Q_{mr}. Let $\alpha,\beta\colon R\to R$ be automorphisms of R. An additive map $\delta\colon R\to M$ is called a skew derivation, or an (α,β)-derivation, if $\delta(xy)=\delta(x)\alpha(y)+\beta(x)\delta(y)$ for any $x,y\in R$. An additive map $g\colon R\to M$ is called a generalized (α,β)-derivation if there is an associated (α,β)-derivation $d\colon R\to M$ such that $g(xy)=g(x)\alpha(y)+\beta(x)d(y)$ for any $x,y\in R$. See [4] and [12] for a discussion of some of its properties.

An additive map $d\colon R\to R$ is called a local derivation if for every $x\in R$ there exists a derivation $d_x\colon R\to R$ such that $d(x)=d_x(x)$. Kadison [8] and Larson and Sourour [9] asked under what conditions a local derivation is a derivation. In [2], Brešar proved that a local derivation is a derivation if R has nontrivial idempotents.

Recently, Wang generalized Brešar's result to the case of generalized derivations. An additive map $g\colon R \to R$ is called a local generalized derivation if for every $x \in R$, there exists a generalized derivation $g_x\colon R \to R$ such that $g(x) = g_x(x)$. Wang proved that a local generalized derivation is actually a generalized derivation if R has nontrivial idempotents [14]. In Section 3, we will prove an analogous result for generalized (α, β)-derivations. Precisely, we will prove that a local generalized (α, β)-derivation on a prime ring with nontrivial idempotents is a generalized (α, β)-derivation. We will also prove that a generalized Jordan triple (α, β)-derivation on a prime ring with nontrivial idempotents is a generalized (α, β)-derivation, which is a special case of [13] (Theorem 3).

2. Tri-additive maps preserving zero products. Let E be the additive subgroup generated by all idempotents of R, and $\overline{E}$ denote the subring generated by E. Recall that in [5] Chebotar, Ke and Lee proved that if $\phi\colon R \to R$ is a bijective additive map such that $\phi(x)\phi(y) = 0$ whenever $xy = 0$, then $\phi(xy)\phi(z) = \phi(x)\phi(yz)$ for any $x, y, z \in R$. In [6], Chuang and Lee considered bi-additive maps preserving zero products. We write their theorem in the following form.

Theorem 2.1 ([6], Theorem 2.3)**.** *Let R be a prime ring with nontrivial idempotents. Assume $\phi\colon R \times R \to R$ is a bi-additive map preserving zero products. Then there exists a nonzero ideal I such that $\phi(xa, y) = \phi(x, ay)$ for any $x, y \in R$ and $a \in I$.*

Note that because R has nontrivial idempotents, $[E, E] \neq 0$, and by examining the proof of Theorem 2.1, we see that the nonzero ideal I can be chosen to be $R[E, E]R$. Moreover, $R[E, E]R \subseteq$ $\subseteq \overline{E}$ by Herstein's arguments in [7, p. 4].

Now we consider a more general case. Let $f\colon R^3 \to R$ be a tri-additive map, that is, a map $f(x, y, z)$ that is is additive in each argument. In view of Theorem 2.1 and the proof in [6], we can prove the following theorem.

Theorem 2.2. *Let R be a prime ring with nontrivial idempotents. Let $f(x, y, z)$ be a tri-additive map with $f(x, y, z) = 0$ whenever $xy = yz = 0$. Then*

$$f(xa, yb, z) - f(x, ayb, z) = f(xa, y, bz) - f(x, ay, bz) \tag{2.1}$$

for all $x, y, z \in R$ and $a, b \in I$, where I is some nonzero ideal of R.

Proof. For $z \in R$ and e idempotent, define $F(x, y) \overset{\text{df}}{=} f(x, ye, (1 - e)z)$, then $F(x, y) = 0$ for $xy = 0$. By Theorem 2.1 there exists a nonzero ideal I such that $F(xa, y) = F(x, ay)$ for any $a \in I$. That is,

$$f(xa, ye, (1 - e)z) = f(x, aye, (1 - e)z). \tag{2.2}$$

Note that by the remark after Theorem 2.1, the choice of I is independent of e and z. In fact, we can choose $I = R[E, E]R$. Thus, (2.2) holds for any $x, y, z \in R$, any $a \in I$ and any idempotent e. Analogously,

$$f(xa, y(1 - e), ez) = f(x, ay(1 - e), ez). \tag{2.3}$$

Comparing (2.2) and (2.3), we see that

$$f(xa, ye, z) - f(x, aye, z) = f(xa, y, ez) - f(x, ay, ez).$$

It can be easily checked that

$$f(xa, y\bar{e}, z) - f(x, ay\bar{e}, z) = f(xa, y, \bar{e}z) - f(x, ay, \bar{e}z)$$

for any $x, y, z \in R$, any $a \in I$, and any $\bar{e} \in \overline{E}$. Because $I = R[E, E]R \subseteq \overline{E}$, we get

$$f(xa, yb, z) - f(x, ayb, z) = f(xa, y, bz) - f(x, ay, bz)$$

for any $x, y, z \in R$, any $a, b \in I$, as asserted.

3. Generalized (α,β)-derivations. Let α, β be automorphisms of R, and let M be an R-bimodule. Recall that an additive map $g\colon R\to M$ is a generalized (α,β)-derivation if $g(xy) = g(x)\alpha(y)+\beta(x)d(y)$ for some (α,β)-derivation $d\colon R\to M$.

Here we need a property on extensions of (α,β)-derivations. It is well known that the automorphisms of R and (α,β)-derivations of R can be uniquely extended to $Q_{m\ell}$. We want to show that an (α,β)-derivation from a nonzero ideal to $Q_{m\ell}$ can be also extended to an (α,β)-derivation of $Q_{m\ell}$. The proof simply follows the standard arguments in [10] (Lemma 2) and [11] (Theorem 2) for the case of derivations. For brevity, we only sketch it here.

Proposition 3.1. *Let R be a prime ring, I be a nonzero ideal of R, and α, β be automorphisms of R. Then every (α,β)-derivation $\delta\colon R\to Q_{m\ell}$ can be uniquely extended to an (α,β)-derivation $\tilde{\delta}\colon Q_{m\ell}\to Q_{m\ell}$. Moreover, every (α,β)-derivation $\delta\colon I\to Q_{\ell}$ can be uniquely extended to an (α,β)-derivation $\tilde{\tilde{\delta}}\colon Q_{m\ell}\to Q_{m\ell}$.*

Proof (Sketch of Proof). Let $\delta\colon R\to Q_{m\ell}$ be an (α,β)-derivation. For any $q\in Q_{m\ell}$ choose a dense left ideal λ of R such that $\lambda q\subseteq R$. Define $\phi\colon Q_{m\ell}\lambda\to Q_{m\ell}$ by $\phi\left(\sum u_ia_i\right) = \sum u_i\beta^{-1}\big((\delta(a_iq)-\delta(a_i)\alpha(q))\big)$, where $u_i\in Q_{m\ell}$ and $a_i\in\lambda$. Then ϕ is a right multiplier induced by an element $\hat{q}$ in the maximal left quotient ring of $Q_{m\ell}$, which is just $Q_{m\ell}$ itself (see Proposition 2.1.7 and Theorem 2.1.11 in [1]). In this sense, δ can be extended to a map $\tilde{\delta}\colon Q_{m\ell}\to Q_{m\ell}$ by defining $\tilde{\delta}(q)\overset{\mathrm{df}}{=}\beta(\hat{q})$. It can be checked that $\tilde{\delta}$ is an (α,β)-derivation of $Q_{m\ell}$ and that this extension is unique. The second part of the proof simply follows the arguments in [11] (Theorem 2).

Now we can prove the following theorem.

Theorem 3.1. *Let R be a prime ring with nontrivial idempotents. If $g\colon R\to R$ is an additive map such that $\beta(x)g(y)\alpha(z)=0$ for any $x,y,z\in R$ with $xy=yz=0$, then g is a generalized (α,β)-derivation.*

Proof. Because R possesses nontrivial idempotents, by Theorem 2.2 we know that

$$\beta(xa)g(yb)\alpha(z)-\beta(x)g(ayb)\alpha(z)=\beta(xa)g(y)\alpha(bz)-\beta(x)g(ay)\alpha(bz) \tag{3.1}$$

for any $x,y,z\in R$ and $a,b\in I$, where I is a nonzero ideal of R. Because R is prime and α, β are automorphisms and rearranging the terms, the equation (3.1) can be reduced to

$$\beta(a)(g(yb)-g(y)\alpha(b))=g(ayb)-g(ay)\alpha(b). \tag{3.2}$$

Now, define $F_b(y)=g(yb)-g(y)\alpha(b)$; then (3.2) becomes $\beta^{-1}(F_b(ay))=a\beta^{-1}(F_b(y))$. That is, $\beta^{-1}F_b$ is a left I-module map and hence a left R-module map. Therefore, $\beta^{-1}F_b$ is a right multiplier induced by an element in Q_ℓ (see [1], Proposition 2.2.1). This implies

$$g(yb)-g(y)\alpha(b)=\beta(y)d(b), \tag{3.3}$$

for any $y\in R$ and any $b\in I$, where $d\colon I\to Q_\ell$ is an additive map. For $y\in R$ and $b,c\in I$, by (3.3)

$$g(ybc)-g(y)\alpha(bc)=\beta(y)d(bc). \tag{3.4}$$

Expanding otherwise and simplifying, the equation (3.4) reduces to

$$g(ybc)-(g(y)\alpha(b)+\beta(y)d(b))\alpha(c)=\beta(yb)d(c). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain $d(bc) = d(b)\alpha(c) + \beta(b)d(c)$, so $d\colon I \to Q_\ell$ is an (α, β)-derivation. Up to now we have

$$g(xa) = g(x)\alpha(a) + \beta(x)d(a) \tag{3.6}$$

for any $x \in R$ and any $a \in I$, where $d\colon I \to Q_\ell$ is an (α, β)-derivation. By Proposition 3.1, d can be uniquely extended to an (α, β)-derivation of Q_ℓ, which we still denote by d. For any $x, y \in R$ and any $a \in I$, by (3.6)

$$\begin{aligned} g(x(ya)) &= g(x)\alpha(ya) + \beta(x)d(ya) = \\ &= g(x)\alpha(y)\alpha(a) + \beta(x)d(y)\alpha(a) + \beta(x)\beta(y)d(a). \end{aligned} \tag{3.7}$$

On the other hand,

$$g((xy)a) = g(xy)\alpha(a) + \beta(xy)d(a). \tag{3.8}$$

Comparing (3.7), (3.8) and using the primeness, we get $g(xy) = g(x)\alpha(y) + \beta(x)d(y)$ for any $x, y \in R$. This means that g is a generalized (α, β)-derivation of R.

By analogy with the local derivations and local generalized derivations mentioned in Section 1, we introduce the following notion.

Definition 3.1. *An additive map $g\colon R \to R$ is called a local generalized (α, β)-derivation if for every $x \in R$, there exists a generalized (α, β)-derivation g_x, which depends on x, such that $g(x) = g_x(x)$.*

The following theorem shows that a local generalized (α, β)-derivation is a generalized (α, β)-derivation. This generalizes the derivation case in [2] and the generalized derivation case in [14].

Theorem 3.2. *Let R be a prime ring with nontrivial idempotents, and let α, β be automorphisms of R. Then a local generalized (α, β)-derivation is a generalized (α, β)-derivation.*

Proof. Let g be a local generalized (α, β)-derivation of R. For every $y \in R$, there is a generalized (α, β)-derivation g_y with associated (α, β)-derivation d_y such that $g(y) = g_y(y)$. Hence for any $x, y, z \in R$ with $xy = yz = 0$, we have

$$\beta(x)g(y)\alpha(z) = \beta(x)g_y(y)\alpha(z) = \beta(x)g_y(yz) - \beta(xy)d_y(z) = 0.$$

By Theorem 3.1, g is actually a generalized (α, β)-derivation.

Recall that an additive map $\delta\colon R \to R$ is called a Jordan triple (α, β)-derivation, if

$$\delta(xyx) = \delta(x)\alpha(y)\alpha(x) + \beta(x)\delta(y)\alpha(x) + \beta(x)\beta(y)\delta(x) \tag{3.9}$$

for any $x, y \in R$. An additive map $g\colon R \to R$ is called a generalized Jordan triple (α, β)-derivation if there exists a Jordan triple (α, β)-derivation δ of R such that

$$g(xyx) = g(x)\alpha(y)\alpha(x) + \beta(x)\delta(y)\alpha(x) + \beta(x)\beta(y)\delta(x) \tag{3.10}$$

for any $x, y \in R$.

In [13], Liu and Shiue proved that a generalized Jordan triple (α, β)-derivation on a 2-torsion free semiprime ring must be a generalized (α, β)-derivation [13] (Theorem 3). Now we want to prove an analogous theorem for the special case of prime rings with nontrivial idempotents, but where the associated map δ in (3.10) is any map.

In order to prove the theorem, we need a result in functional identities.

Lemma 3.1. *Let R be a prime ring and $\alpha, \beta\colon R \to R$ be automorphisms of R. If $F, G\colon R \to R$ are two additive maps such that $F(x)\alpha(y) = \beta(x)G(y)$ for any $x, y \in R$, then there exists an element $q \in Q_s$ such that $F(x) = \beta(x)q$ and $G(y) = q\alpha(y)$.*

Proof. It is well known that any automorphism of R can be uniquely extended to an automorphism of Q_s, Q_ℓ, or Q_r. A direct computation shows that $F(rx)\alpha(y) - \beta(r)F(x)\alpha(y) = 0$ for any $r, x, y \in R$, so because R is prime, we see that $F(rx) = \beta(r)F(x)$. That is, $\beta^{-1}F$ is a left R-module map of R. Therefore, there exists an element $s \in Q_\ell$ such that $\beta^{-1}F(x) = xs$. Hence, $F(x) = \beta(x)q$, where $q = \beta(s) \in Q_\ell$. By assumption we have $\beta(x)q\alpha(y) = \beta(x)G(y)$, which implies that $G(y) = q\alpha(y)$ because R is a prime ring. Moreover, q is an element of Q_s because $qR \subseteq R$.

Theorem 3.3. *Let R be a prime ring with nontrivial idempotents, and let α, β be automorphisms of R. If $g\colon R \to R$ is an additive map and $d\colon R \to R$ is any map such that*

$$g(xyx) = g(x)\alpha(y)\alpha(x) + \beta(x)d(y)\alpha(x) + \beta(x)\beta(y)d(x) \tag{3.11}$$

for any $x, y \in R$, then g is a generalized (α, β)-derivation with the associated derivation δ, and one of the following holds:

(1) $d = \delta$, *is exactly the associated (α, β)-derivation of g*;

(2) char $R = 2$ *and there exists an invertible element $q \in Q_s$, such that $d(x) = \delta(x) + \beta(x)q = \delta(x) - q\alpha(x)$ and $\beta(x) = q\alpha(x)q^{-1}$.*

Proof. For any $s \in R$ and $x, y, z \in R$ with $xy = yz = 0$, it follows from (3.11) that

$$0 = \beta(x)g(yzsy) = \beta(x)g(y)\alpha(z)\alpha(s)\alpha(y).$$

Because α, β are automorphisms and R is prime, we have $\beta(x)g(y)\alpha(z) = 0$ or $\alpha(y) = 0$. Take $I_1 = \{y \in R \mid \beta(x)g(y)\alpha(z) = 0\}$ for all $x, z \in R$ and $I_2 = \{y \in R \mid \alpha(y) = 0\}$. Clearly, I_1 and I_2 both are additive subgroups of R, whose union is R. But, a group can not be union of two of its proper subgroups. Hence, either $I_1 = R$ and $I_2 = R$. But, if $I_2 = R$ gives $\alpha = 0$, a contradiction. Hence, $\beta(x)g(y)\alpha(z) = 0$ for all $x, y, z \in R$ with $xy = yz = 0$. Hence g is a generalized (α, β)-derivation with associated (α, β)-derivation δ by Theorem 3.1.

Now we claim that d is additive. Substituting y by $y + z$ in (3.11), and because g, α and β are all additive, we get

$$\beta(x)\big(d(y+z) - d(y) - d(z)\big)\alpha(x) = 0. \tag{3.12}$$

Linearizing on x, it follows that

$$\beta(u)\big(d(y+z) - d(y) - d(z)\big)\alpha(x) + \beta(x)\big(d(y+z) - d(y) - d(z)\big)\alpha(u) = 0. \tag{3.13}$$

Substituting u by ux in (3.13) and using (3.12), we see that

$$\beta(x)\big(d(y+z) - d(y) - d(z)\big)\alpha(ux) = 0$$

for all $u, x, y, z \in R$. Again, because α is an automorphism and R is prime, $\beta(x)\big(d(y+z) - d(y) - d(z)\big) = 0$ or $\alpha(x) = 0$ for all $x, y, z \in R$. As discuss in the beginning of the theorem, we have $\beta(x)\big(d(y+z) - d(y) - d(z)\big) = 0$ for all $x, y, z \in R$. This implies that $d(y+z) = d(y) + d(z)$ for all $y, z \in R$. That is, d is additive.

Now g is a generalized (α, β)-derivation with associated (α, β)-derivation δ. From (3.9) and (3.11) we get

$$\beta(x)d(y)\alpha(x) + \beta(x)\beta(y)d(x) = \beta(x)\delta(y)\alpha(x) + \beta(x)\beta(y)\delta(x)$$

for all $x, y \in R$, and hence $d(y)\alpha(x) + \beta(y)d(x) = \delta(y)\alpha(x) + \beta(y)\delta(x)$. That is, $(d-\delta)(y)\alpha(x) + \beta(y)(d-\delta)(x) = 0$. Because $d-\delta$ is additive, it follows by Lemma 3.1 that $(d-\delta)(x) = \beta(x)q = -q\alpha(x)$ for some $q \in Q_s$, which means that $d(x) = \delta(x) + \beta(x)q = \delta(x) - q\alpha(x)$. For any $x, y \in R$, we have

$$\beta(xy)q = \beta(x)\beta(y)q = -\beta(x)q\alpha(y) = q\alpha(x)\alpha(y).$$

Therefore, $2qR^2 = 0$, and this implies that $2q = 0$. If char $R \neq 2$, then $q = 0$ and $d = \delta$, as asserted. In case char $R = 2$ and $q \neq 0$, by $\beta(x)q = -q\alpha(x) = q\alpha(x)$ we can conclude that q is invertible in Q_s and hence $\beta(x) = q\alpha(x)q^{-1}$.

The following is a special case of [3] (Theorem 1).

***Corollary* 3.1.** *Let R be a prime ring with nontrivial idempotents and α, β be automorphisms of R. If* char $(R) \neq 2$ *and $d\colon R \to R$ is a Jordan triple (α, β)-derivation, then d is an (α, β)-derivation.*

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