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CO-COATOMICALLY SUPPLEMENTED MODULES

КО-КОАТОМНО ПОПОВНЕНІ МОДУЛІ

It is shown that if a submodule N of M is co-coatomically supplemented and M/N has no maximal submodule, then M is a co-coatomically supplemented module. If a module M is co-coatomically supplemented, then every finitely M-generated module is a co-coatomically supplemented module. Every left R-module is co-coatomically supplemented if and only if the ring R is left perfect. Over a discrete valuation ring, a module M is co-coatomically supplemented if and only if the basic submodule of M is coatomic. Over a nonlocal Dedekind domain, if the torsion part $T(M)$ of a reduced module M has a weak supplement in M, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. Over a nonlocal Dedekind domain, if a reduced module M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P, then M is a co-coatomically supplemented module.

Показано, що у випадку, коли субмодуль N модуля M є ко-коатомно поповненим, а M/N не має максимального субмодуля, модуль M є ко-коатомно поповненим. Якщо модуль M є ко-коатомно поповненим, то кожен скінченно M-породжений модуль є ко-коатомно поповненим. Кожний лівий R-модуль є ко-коатомно поповненим тоді і тільки тоді, коли кільце R є лівим досконалим. Поза дискретним метризаційним кільцем модуль M є ко-коатомно поповненим тоді і тільки тоді, коли базовий субмодуль M є коатомним. Поза нелокальною дедекіндовою областю у випадку, коли торсіонна частина $T(M)$ зведеного модуля M має слабке поповнення в M, модуль M є ко-коатомно поповненим тоді і тільки тоді, коли $M/T(M)$ є подільним, а $T_P(M)$ — обмеженим для кожного максимального ідеалу P. Поза нелокальною дедекіндовою областю у випадку, коли зведений модуль M є ко-коатомно широко поповненим, $M/T(M)$ є подільним, а $T_P(M)$ — обмеженим для кожного максимального ідеалу P. Навпаки, якщо $M/T(M)$ є подільним, а $T_P(M)$ — обмеженим для кожного максимального ідеалу P, то модуль M є ко-коатомно поповненим.

1. Introduction. Throughout this paper R denotes an associative ring with identity and all modules are *left* unitary R-modules ($_RM$) unless otherwise stated. Let U be a submodule of M. A submodule V of M is called a *supplement* of U in M if V is minimal element in the set of submodules $L \leq M$ with $U+L = M$. V is a supplement of U in M if and only if $U+V = M$ and $U \cap V \ll V$. A module M is called *supplemented* if every submodule of M has a supplement in M (see [9], Section 41, or [5], Chapter 4). Semisimple, artinian and hollow (in particular local) modules are supplemented. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule (see [12]). Let N be a submodule of a module M. We say that N is a *co-coatomic* submodule in M if M/N is coatomic. Semisimple, finitely generated and local modules are coatomic modules. Since every factor module of a coatomic module is coatomic, every submodule of semisimple, finitely generated and local modules is co-coatomic. A module M is said to be *co-coatomically supplemented* module if every co-coatomic submodule of M has a supplement in M. A submodule N of M is called *cofinite* if M/N is finitely generated. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M (see [1]). Clearly a co-coatomically supplemented module is cofinitely supplemented and a coatomic module is co-coatomically supplemented if and only if it is a supplemented module. A module M is called *co-coatomically weak supplemented* if every co-coatomic submodule N of M has a weak supplement in M, i.e., $N + K = M$ and $N \cap K \ll M$ for some submodule K of M. It is clear that a co-coatomically supplemented module is co-coatomically weak supplemented. A submodule U of an R-module M has *ample supplements* in M if, for every submodule V of M with $U + V = M$, there exists a supplement V' of U with $V' \leq V$ (see [5, p. 237]). A module M is called *co-coatomically amply supplemented* if every co-coatomic submodule of M has ample supplements in M. Clearly a co-coatomically amply supplemented module is co-coatomically supplemented.

In Section 2, we show that if a submodule N of M is co-coatomically supplemented and M/N has no maximal submodule, then M is co-coatomically supplemented. Every left R-module is co-coatomically supplemented if and only if the ring R is left perfect.

In Section 3, we study on co-coatomically supplemented modules over a discrete valuation ring. We show that a module M is co-coatomically supplemented if and only if the basic submodule of M is coatomic if and only if $M = T(M) \oplus X$, where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.

In Section 4, we study on co-coatomically supplemented modules over nonlocal Dedekind domains. A torsion module M is co-coatomically weak supplemented if and only if it is co-coatomically supplemented. We show that for a reduced module M, if the torsion part $T(M)$ of M has a weak supplement in M, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P. For a reduced module M, if M is co-coatomically amply supplemented, then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R. Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R, then M is a co-coatomically supplemented module.

2. Co-coatomically supplemented modules. For any module M, $\operatorname{Soc}(M)$ denotes the socle of M and $\operatorname{Rad}(M)$ denotes the radical of M. The Jacobson radical of ${}_RR$ is denoted by $\operatorname{Jac}(R)$.

Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be the family of simple submodules of M that are direct summands of M. $\operatorname{Soc}^{\oplus}(M)$ will denote the sum of M_λs for all $\lambda \in \Lambda$. That is $\operatorname{Soc}^{\oplus}(M) = \sum_{\lambda\in\Lambda} M_\lambda$. Clearly $\operatorname{Soc}^{\oplus}(M) \leq \operatorname{Soc}(M)$.

Theorem 2.1. *Let R be a ring. The following are equivalent for an R-module M:*

1. *Every co-coatomic submodule of M is a direct summand of M.*
2. *Every cofinite submodule of M is a direct summand of M.*
3. *Every maximal submodule of M is a direct summand of M.*
4. *$M/\operatorname{Soc}^{\oplus}(M)$ does not contain a maximal submodule.*
5. *$M/\operatorname{Soc}(M)$ does not contain a maximal submodule.*

Proof. (1) $\Rightarrow$ (2) is clear since every cofinite submodule is co-coatomic.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). Suppose $M/\operatorname{Soc}^{\oplus}(M)$ contains a maximal submodule $K/\operatorname{Soc}^{\oplus}(M)$. It follows that K is a maximal submodule of M. By hypothesis, $M = K \oplus K'$ and K' is simple. $K' \leq \operatorname{Soc}^{\oplus}(M) \leq$ $\leq K$. Contradiction.

(4) $\Rightarrow$ (5). Clear because $\operatorname{Soc}^{\oplus}(M) \leq \operatorname{Soc}(M)$.

(5) $\Rightarrow$ (1). Let N be a co-coatomic submodule of M. Since

$$M/(N + \operatorname{Soc}(M)) \cong (M/N)/\big((N + \operatorname{Soc}(M))/N\big)$$

and M/N is coatomic, $M/(N + \operatorname{Soc}(M))$ is also coatomic. Since $M/\operatorname{Soc}(M)$ has no maximal submodule, $M/(N + \operatorname{Soc}(M))$ also has no maximal submodule, therefore $M = N + \operatorname{Soc}(M)$. It follows that $M = N \oplus N'$ for any submodule N' such that $\operatorname{Soc}(M) = \big(N \cap \operatorname{Soc}(M)\big) \oplus N'$.

A supplemented module is co-coatomically supplemented but a co-coatomically supplemented modules need not be supplemented as it is shown in the following example.

***Example* 2.1.** The $\mathbb{Z}$-module $\mathbb{Q}$ is co-coatomically supplemented since the only co-coatomic submodule is $\mathbb{Q}$ itself. But the $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented since $\mathbb{Q}$ is not torsion (see [10], Theorem 3.1).

Proposition 2.1. *Let M be a semilocal module with small radical* $\operatorname{Rad}(M)$. *Then M is co-coatomically supplemented if and only if M is supplemented.*

Proof. Let N be a submodule of M. Since M is semilocal, $M/\operatorname{Rad}(M)$ is semisimple, i.e., coatomic. Consider the following statement:

$$M/\big(N + \operatorname{Rad}(M)\big) \cong \big(M/\operatorname{Rad}(M)\big)/\big((N + \operatorname{Rad}(M))/\operatorname{Rad}(M)\big).$$

Since $M/\operatorname{Rad}(M)$ is coatomic, $M/(N + \operatorname{Rad}(M))$ is coatomic. Therefore $N + \operatorname{Rad}(M)$ has a supplement in M, say K. Then $M = N + \operatorname{Rad}(M) + K$ and $(N + \operatorname{Rad}(M)) \cap K \ll K$. Since $\operatorname{Rad}(M) \ll M$, it follows that $M = N + K$ and $N \cap K \leq (N + \operatorname{Rad}(M)) \cap K \ll K$. Thus M is supplemented.

A co-coatomically supplemented module is cofinitely supplemented but the example below show that a cofinitely supplemented module need not be co-coatomically supplemented.

A ring R is said to be a semiperfect if $R/\operatorname{Jac}(R)$ is semisimple and idempotents in $R/\operatorname{Jac}(R)$ can be lifted to R (see [9], 42.6).

A ring is called left perfect if $R/\operatorname{Jac}(R)$ is left semisimple and $\operatorname{Jac}(R)$ is right t-nilpotent (see [9], 43.9).

${}_RR^{(\mathbb{N})}$ denotes the direct sum of R-module R by index set $\mathbb{N}$. Note that $\mathbb{N}$ denotes the set of all positive integers.

Any direct sum of cofinitely supplemented modules is cofinitely supplemented [1] (Corollary 2.4).

***Example* 2.2.** Let p be a prime integer and consider the following ring:

$$R = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a,\, b \in \mathbb{Z},\, b \neq 0,\, (b, p) = 1 \right\}$$

which is the localization of $\mathbb{Z}$ at (p). In this case, the R-module R is supplemented. Then the R-module $R^{(\mathbb{N})}$ is cofinitely supplemented by [1] (Corollary 2.4). Furthermore, R is a semiperfect ring and therefore $R/\operatorname{Jac}(R)$ is semisimple (see [9], 42.6). Hence R is semilocal. However, R is not a perfect ring since its Jacobson radical is not t-nilpotent by [9] (43.9). $\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big)$ is a co-coatomic submodule of ${}_RR^{(\mathbb{N})}$ but $\operatorname{Rad}\big({}_RR^{(\mathbb{N})}\big)$ does not have supplement in ${}_RR^{(\mathbb{N})}$ since R is not a perfect ring (see [3], Theorem 1). Hence ${}_RR^{(\mathbb{N})}$ is not co-coatomically supplemented.

Example 2.2 shows that over semiperfect rings and discrete valuation rings, cofinitely supplemented modules and co-coatomically supplemented modules need not coincide.

Proposition 2.2. *A factor module of a co-coatomically supplemented module is co-coatomically supplemented.*

Proof. Let M be a co-coatomically supplemented module and N be a submodule of M. Then any co-coatomic submodule of M/N is a submodule of the form L/N where L is co-coatomic submodule of M. By hypothesis, L has a supplement in M, say K. It follows that $(K + N)/N$ is a supplement of L/N in M/N by [9] (41.1(7)).

Proposition 2.3. *Let M be a co-coatomically supplemented module. Then every co-coatomic submodule of the module $M/\operatorname{Rad}(M)$ is a direct summand.*

Proof. Any co-coatomic submodule of $M/\operatorname{Rad}(M)$ has the form $N/\operatorname{Rad}(M)$ where N is a co-coatomic submodule of M. Since M is co-coatomically supplemented, there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll K$. It follows that $N \cap K \leq \operatorname{Rad}(M)$. Thus

$$M/\operatorname{Rad}(M) = \big(N/\operatorname{Rad}(M)\big) + \big((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)\big),$$

$$\big(N/\operatorname{Rad}(M)\big) \cap \big((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)\big) = \big(N \cap K + \operatorname{Rad}(M)\big)/\operatorname{Rad}(M) = 0.$$

Hence

$$M/\operatorname{Rad}(M) = \big(N/\operatorname{Rad}(M)\big) \oplus \big((K + \operatorname{Rad}(M))/\operatorname{Rad}(M)\big).$$

To prove that a finite sum of co-coatomically supplemented modules is a co-coatomically supplemented module, we use the following standard lemma (see [9], 41.2).

Lemma 2.1. *Let N and L be submodules of an R-module M such that N is co-coatomic, L is co-coatomically supplemented and $N + L$ has a supplement in M. Then N has a supplement in M.*

Proof. Let K be a supplement of $N + L$ in M. Note that

$$L/\big(L \cap (N + K)\big) \cong (N + K + L)/(N + K) = M/(N + K).$$

The last module is coatomic, therefore there is a supplement H of $L \cap (N + K)$ in L, i.e.,

$$L = H + L \cap (N + K) \qquad \text{and} \qquad H \cap L \cap (N + K) \ll H.$$

Now

$$M = N + L + K = N + K + H + L \cap (N + K) = N + K + H,$$

$$N \cap (H + K) \leq H \cap (N + K) + K \cap (N + H) \leq$$

$$\leq H \cap (N + K) + K \cap (N + L) \ll H + K.$$

Therefore $H + K$ is a supplement of N in M.

A (direct) sum of infinitely many co-coatomically supplemented modules need not be co-coatomically supplemented by Example 2.2 but a finite sum of co-coatomically supplemented modules is always co-coatomically supplemented.

Theorem 2.2. *A finite sum of co-coatomically supplemented modules is co-coatomically supplemented.*

Proof. Clearly it is sufficient to prove that the sum $M = M_1 + M_2$ of two co-coatomically supplemented modules M_1 and M_2 is co-coatomically supplemented. Let U be a co-coatomic submodule of M. Then $M = M_1 + M_2 + U$. Since $M_2 + U$ is co-coatomic submodule of M and M_1 is co-coatomically supplemented, $M_2 + U$ has a supplement in M by Lemma 2.1. Since M_2 is co-coatomically supplemented and U is co-coatomic, then again by Lemma 2.1, U has a supplement in M. Thus M is co-coatomically supplemented.

Let M and N be R-modules. If there is an epimorphism $f \colon M^{(\Lambda)} \to N$ for some finite set Λ, then N is called a *finitely M-generated* module.

The following corollary follows from Proposition 2.2 and Theorem 2.2.

***Corollary* 2.1.** *If M is co-coatomically supplemented module, then any finitely M-generated module is a co-coatomically supplemented module.*

A ring R is called a left *V-ring* if every simple R-module is injective (see [9, p. 192]). A commutative ring R is a V-ring if and only if R is a von Neumann regular ring (see [9], 23.5).

Proposition 2.4. *A module M over a V-ring R is co-coatomically supplemented if and only if M is semisimple.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Since M is a co-coatomically supplemented module, $M/\operatorname{Soc}(M)$ has no maximal submodule by Theorem 2.1. It follows from [9] (23.1) that $M/\operatorname{Soc}(M) = \operatorname{Rad}(M/\operatorname{Soc}(M)) = 0$ since R is a V-ring. Thus M is semisimple.

***Corollary* 2.2.** *Over a left V-ring, any direct sum co-coatomically supplemented modules is co-coatomically supplemented.*

Proof. By Proposition 2.4, co-coatomically supplemented and semisimple modules coincide over left V-rings.

Theorem 2.3. *Let N be a co-coatomically supplemented submodule of an R-module M such that M/N has no maximal submodule. Then M is a co-coatomically supplemented module.*

Proof. Let L be a submodule of M such that M/L is coatomic. Clearly $M/(N+L)$ is also coatomic. Since M/N has no maximal submodule, $M/(N+L)$ also has no maximal submodule, therefore $M = N + L$. By Lemma 2.1, L has a supplement in M. Thus M is a co-coatomically supplemented module.

The following corollary is a direct result of Theorem 2.3.

***Corollary* 2.3.** *Let M be a module and $M/\operatorname{Soc}(M)$ have no maximal. Then M is co-coatomically supplemented.*

Proposition 2.5. *Let M be a co-coatomically supplemented R-module. If M contains a maximal submodule, then M contains a local submodule.*

Proof. Let L be a maximal submodule of M. Then L is a co-coatomic submodule of M. Since M is a co-coatomically supplemented module, there exists a submodule K of M such that K is a supplement of L in M, i.e., $M = K + L$ and $K \cap L \ll K$. It follows from [9] (41.1(3)) that K is local.

A module M is called *linearly compact* if for every family of cosets $\{x_i + M_i\}_{\triangle}$, $x_i \in M$, and submodules $M_i \leq M$ (with M/M_i finitely cogenerated), the intersection of any finitely many of these cosets is not empty, then the intersection is also not empty (see [9], 29.7(c)).

The following proposition gives a characterization of a co-coatomically supplemented module by a linearly compact submodule.

Proposition 2.6. *Let K be a linearly compact submodule of an R-module M. Then M is co-coatomically supplemented if and only if M/K is co-coatomically supplemented.*

Proof. ($\Rightarrow$) By Proposition 2.2.

($\Leftarrow$) Let N be a co-coatomic submodule of M. Then $(N+K)/K$ is co-coatomic submodule of M/K since $N+K$ is co-coatomic submodule of M. Since M/K is co-coatomically supplemented, $(N+K)/K$ has a supplement in M/K. K has a supplement in every submodule L of M with $K \leq L$ since K is linearly compact (see [8], Lemma 2.3). K is supplemented by [9] (29.8(2)) and [8] (Lemma 2.3). Therefore N has a supplement in M by [8] (Corollary 2.7). Thus M is co-coatomically supplemented.

***Remark* 2.1.** A module M is called Σ-*selfprojective* if for each index set I, the module $M^{(I)}$ is selfprojective. For an R-module M, if M is Σ-selfprojective and $U \leq \operatorname{Rad}(M)$, then the following holds: U has a supplement in M, so U is small in M [11] (Satz 4.1). Clearly ${}_RR^{(\mathbb{N})}$ is Σ-selfprojective and $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \leq \operatorname{Rad}({}_RR^{(\mathbb{N})})$, therefore if $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$, then $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$.

Theorem 2.4. *Every left R-module is co-coatomically supplemented if and only if the ring R is left perfect.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ By hypothesis, every left R-module is co-coatomically supplemented, so every left R-module is cofinitely supplemented. Then R is semiperfect by [1] (Theorem 2.13). Therefore $R/\operatorname{Jac}(R)$ is semisimple by [9] (42.6). It follows that ${}_RR^{(\mathbb{N})}/\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is semisimple. Thus $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ is a co-coatomic in ${}_RR^{(\mathbb{N})}$. By hypothesis, $\operatorname{Rad}({}_RR^{(\mathbb{N})})$ has a supplement in ${}_RR^{(\mathbb{N})}$. By Remark 2.1, $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$. Since $R/\operatorname{Jac}(R)$ is semisimple and $\operatorname{Rad}({}_RR^{(\mathbb{N})}) \ll {}_RR^{(\mathbb{N})}$, ${}_RR$ is perfect by [9] (43.9). Thus the ring R is left perfect.

3. Co-coatomically supplemented modules over discrete valuation rings. Throughout this section R will be a discrete valuation ring. An R-module M is called radical-supplemented if $\operatorname{Rad}(M)$ has a supplement in M (see [11]). A module M is radical supplemented if and only if the basic submodule of M is coatomic (see [11], Satz 3.1). A module M is coatomic if and only if M is reduced and supplemented (see [10], Lemma 2.1).

Proposition 3.1. *Let M be an R-module. Then M is co-coatomically supplemented module if and only if the basic submodule of M is coatomic.*

Proof. $(\Rightarrow)$ $M/\operatorname{Rad}(M) = M/pM$ is semisimple and therefore coatomic. Since M is co-coatomically supplemented module, pM has a supplement. Thus M is a radical-supplemented module. Then the basic submodule of M is coatomic by [11] (Satz 3.1).

$(\Leftarrow)$ Let X be a submodule of M such that M/X is coatomic and B be the basic submodule of M. Then $M/(X+B)$ is also coatomic. Furthermore, $M/(X+B)$ is reduced by [10] (Lemma 2.1). On the other hand, $M/(X+B)$ is divisible since M/B is divisible. Therefore $M/(X+B) = 0$, that is $M = X+B$. By hypothesis, B is coatomic, so supplemented by [10] (Lemma 2.1). Therefore X has a supplement in M by Lemma 2.1. Hence M is a co-coatomically supplemented module.

***Corollary* 3.1.** *Co-coatomically supplemented modules and radical supplemented modules coincide.*

The following corollary follows from [11] (Satz 3.1) and Corollary 3.1.

***Corollary* 3.2.** *A module M is co-coatomically supplemented if and only if $M = T(M) \oplus X$ where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.*

The following properties are given in [11] (Lemma 3.2) for radical-supplemented modules over a discrete valuation ring. Since co-coatomically supplemented modules and radical-supplemented modules coincide, clearly they hold for co-coatomically supplemented modules.

***Corollary* 3.3.** *For an R-module M the following hold*:

1. *The class of co-coatomically supplemented modules is closed under pure submodules and extensions.*

2. *If M is co-coatomically supplemented and M/U is reduced*, *then U is also co-coatomically supplemented.*

3. *Every submodule of M is co-coatomically supplemented if and only if $T(M)$ is supplemented and $M/T(M)$ has a finite rank.*

4. Co-coatomically supplemented modules over nonlocal Dedekind domains. Throughout this section R will be a nonlocal Dedekind domain unless otherwise stated.

Theorem 4.1. *Let R be a Dedekind domain and M be an R-module. M is a module whose co-coatomic submodules are direct summand if and only if*

1) $T(M) = M_1 \oplus M_2$ *where M_1 is semisimple and M_2 is divisible*,

2) $M/T(M)$ *is divisible.*

Proof. By Theorem 2.1 and [4] (Theorem 6.11).

A submodule N of a module M has(is) a weak supplement in M if $M = N+K$ and $N \cap K \ll M$ for some submodule K of M. Clearly every supplement is a weak supplement.

Recall that over an arbitrary ring R, a module M is called co-coatomically weak supplemented if every co-coatomic submodule has a weak supplement in M.

Proposition 4.1. *Over an arbitrary ring, a small cover of a co-coatomically weak supplemented module is co-coatomically weak supplemented.*

Proof. Let M be a small cover of a co-coatomically weak supplemented module N. Then $N \cong M/K$ for some $K \ll M$. Take a co-coatomic submodule L of M. $(L+K)/K$ is co-coatomic submodule of M/K since $L+K$ is co-coatomic submodule of M. By hypothesis, M/K is co-coatomically weak supplemented so $(L+K)/K$ has a weak supplement in M/K, say X/K. Since $K \ll M$, $(X \cap L) + K = X \cap (L+K) \ll M$ (see [5], 2.2(3)). Therefore $M = L + X$ and $L \cap X \ll M$, i.e., X is a weak supplement of L in M. Thus M is co-coatomically weak supplemented.

Proposition 4.2. *Over an arbitrary ring, a factor module of a co-coatomically weak supplemented module is co-coatomically weak supplemented.*

Proof. Let M be a co-coatomically weak supplemented module and N be a submodule of M. Then any co-coatomic submodule of M/N is a submodule of the form L/N where L is co-coatomic submodule of M. By hypothesis, L has a weak supplement in M, say K. It follows that $(K+N)/N$ is a weak supplement of L/N in M/N by [5] (2.2(5)).

Let M be a module and K be a submodule of M. A submodule L of M is called complement of K in M if it is maximal in the set of all submodules N of M with $K \cap N = 0$. A submodule L of M is called a complement submodule if it is a complement of some submodule of M (see [5], 1.9). A submodule of M is a complement if and only if it is closed (see [5], 1.10). A submodule L of M is called coclosed in M if L has no proper submodule K for which $L/K \ll M/K$ (see [5], 3.6). Over a Dedekind domain, a submodule N of M is closed if and only if N is coclosed (see [10], Lemma 3.3). Over a domain R, torsion submodule $T(M)$ of a module M is a closed submodule of M (see [7], Example 6.34). Therefore over a Dedekind domain, torsion submodule $T(M)$ of a module M is a coclosed submodule of M.

Proposition 4.3. *Let M be a torsion R-module. Then M is co-coatomically weak supplemented if and only if it is co-coatomically supplemented.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ Let K be a submodule of M such that M/K is coatomic. Since M is co-coatomically weak supplemented K has a weak supplement in M, say N. Then $M = K + N$ and $K \cap N \ll M$. Since M is torsion, N is also torsion so it is coclosed. Therefore $K \cap N \ll N$ by [5] (3.7(3)). Hence M is co-coatomically supplemented.

Let R be a Dedekind domain and $\mathcal{P}$ be the set of all maximal ideals of R. For some $P \in \mathcal{P}$, the submodule $\{m \in M \mid P^n m = 0 \text{ for some integer } n \geq 1\}$ is said to be the P-primary component of M. This submodule is denoted by $T_P(M)$.

Over a discrete valuation ring, if a module M is torsion, reduced and radical of M has a supplement in M, then M is bounded (see [10, p. 48], 2nd Folgerung).

Theorem 4.2. *Let M be a reduced R-module. If $T(M)$ has a weak supplement in M, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P.*

Proof. $(\Rightarrow)$ Let M be a co-coatomically supplemented reduced R-module. Then the module $M/T(M)$ is radical: Suppose K is a maximal submodule of M with $T(M) \subseteq K$. Since M is co-coatomically supplemented, K has a supplement, say V. Since K is maximal, V is local, therefore V is cyclic, i.e., $V \cong R/I$ (see [9], 41.1(3)). On the other hand, R is nonlocal so $I \neq 0$, i.e., V is torsion so $V \subseteq T(M)$, contradiction. Hence $M/T(M)$ has no maximal so $M/T(M)$ is divisible (see [1], Lemma 4.4). $T(M)$ is closed by [7] (Example 6.34), i.e., it is coclosed by [10] (Lemma 3.3). Since $T(M)$ has a weak supplement, it is a supplement by [5] (20.2). Therefore there is a submodule N in M such that $T(M) + N = M$ and $T(M) \cap N \ll T(M)$. Then

$$T(M)/T(M) \cap N \cong (T(M) + N)/N = M/N.$$

Since M is co-coatomically supplemented, it is co-coatomically weak supplemented so $T(M)/T(M) \cap N$ is co-coatomically weak supplemented. By Proposition 4.1, $T(M)$ is co-coatomically weak supplemented. By Proposition 4.2, $T_P(M)$ is also co-coatomically weak supplemented for each P as it is direct summand of $T(M)$. $T_P(M)$ is co-coatomically supplemented module by Proposition 4.3. Thus $T_P(M)$ is bounded for each maximal ideal P (see [10, p. 48], 2nd Folgerung).

$(\Leftarrow)$ Each $T_P(M)$ is bounded so it is supplemented by [10] (Lemma 2.1). Therefore $T(M)$ is supplemented by [10] (Theorem 3.1). Now let K be a submodule of M such that M/K is coatomic. Then $M/(K+T(M))$ is also coatomic. By hypothesis, $M/T(M)$ is divisible, i.e., it has no maximal submodule (see [1], Lemma 4.4). Therefore $M = K + T(M)$. By Lemma 2.1, K has a supplement in M. Hence M is co-coatomically supplemented.

***Remark* 4.1.** We see that "if" part of the theorem is true without the condition that "T(M) has a weak supplement in M". We do not know if this condition is necessary for the "only if" part.

***Corollary* 4.1.** *Let R be a nonlocal Dedekind domain and M be a reduced R-module. If $\operatorname{Rad}(T(M)) \ll T(M)$, then M is co-coatomically supplemented if and only if $M/T(M)$ is divisible.*

Proof. $(\Rightarrow)$ Clear by the proof of Theorem 4.2.

$(\Leftarrow)$ By [2] (Corollary 4.1.2.), $T(M)/\operatorname{Rad}(T(M))$ is semisimple, so it is co-coatomically weak supplemented. Then $T(M)$ is co-coatomically weak supplemented since $\operatorname{Rad}(T(M)) \ll T(M)$ by Proposition 4.1. Therefore $T(M)$ is co-coatomically supplemented by Proposition 4.3. Since $M/T(M)$ is divisible, $M/T(M)$ has no maximal submodule. Therefore M is co-coatomically supplemented by Theorem 2.3.

Theorem 4.3. *Let R be a nonlocal Dedekind domain and M be a reduced R-module. If M is co-coatomically amply supplemented then $M/T(M)$ is divisible and $T_P(M)$ is bounded for each $P \in \mathcal{P}$.*

Conversely, if $M/T(M)$ is divisible and $T_P(M)$ is bounded for each maximal ideal P of R then M is co-coatomically supplemented.

Proof. Let R be a nonlocal Dedekind domain and M be a co-coatomically amply supplemented reduced R-module. Then by the proof of Theorem 4.2, $M/T(M)$ is divisible. Now suppose that $T_P(M)$ is not bounded for some $P \in \mathcal{P}$. If basic submodule $B_p(M)$ is bounded then by [6] (Theorem 5), $T_P(M) = B_P(M) \oplus D$ where D is divisible. Therefore M is not reduced, a contradiction.

Therefore $B_p(M)$ is not bounded. We will prove that $B_P(M)$ is co-coatomically supplemented. Let K be a co-coatomic submodule of $B_P(M)$, i.e., $B_P(M)/K$ is coatomic. Therefore $B_P(M)/K$ is bounded by [10, p. 48] (2nd Folgerung). We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & K & & & & & & \\ & & \downarrow i & & & & & & \\ E: 0 & \longrightarrow & B_P(M) & \xrightarrow{\text{pure}} & M & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow \sigma & & \downarrow & & \| & & \\ E': 0 & \longrightarrow & B_P(M)/K & \xrightarrow{\text{pure}} & M' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Since E is pure E' is also pure. Hence E' is splitting since $B_P(M)/K$ is bounded (see [6], Theorem 5). By applying Ext, we obtain exact sequence

$$\to \mathrm{Ext}_R(X, K) \xrightarrow{i_*} \mathrm{Ext}_R(X, B_P(M)) \xrightarrow{\sigma_*} \mathrm{Ext}_R(X, B_P(M)/K) \to .$$

Since $\mathrm{Ext}(X, B_P(M)/K) = 0$, $\sigma_*(E) = 0$ and therefore $E \in \mathrm{Ker}\,\sigma_* = \mathrm{Im}\, i_*$. Thus there is a short exact sequence

$$E'': 0 \to K \to N \to X \to 0$$

such that $i_*(E'') = E$. Therefore we obtain the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B_P(M) & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & B_P(M)/K & = & B_P(M)/K & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Without loss of generality, we can assume that K, $B_P(M)$ and N are submodules of M. In this diagram $B_P(M) \cap N = K$, $B_P(M) + N = M$ (see [9], Noether isomorphism theorem). Moreover M/N is coatomic. Since M is co-coatomically amply supplemented there exists a submodule L of $B_P(M)$ such that $N + L = M$ and $N \cap L \ll L$. Therefore $B_P(M) = B_P(M) \cap (N + L) = L + (B_P(M) \cap N) = L + K$ and $L \cap K \leq L \cap N \ll L$. Thus K has a supplement in $B_P(M)$ and so $B_P(M)$ is co-coatomically supplemented. Therefore $B_P(M)$ is bounded by [10, p. 48] (2nd Folgerung). This is a contradiction. Thus $T_P(M)$ is bounded for each $P \in \mathcal{P}$.

The converse is clear by Theorem 4.2.

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