

J. L. Shen (College Comput. and Inform. Technology, Henan Normal Univ., China),

A. Chen (School Math. Sci., Inner Mongolia Univ., China)

A NOTE ON PROPERTY (gaR) AND PERTURBATIONS*

ПРО ВЛАСТИВІСТЬ (gaR) ТА ЗБУРЕННЯ

We introduce a new property (gaR) extending the property (R) considered by Aiena. We study the property (gaR) in connection with Weyl type theorems and establish sufficient and necessary conditions under which the property (gaR) holds. In addition, we also study the stability of the property (gaR) under perturbations by finite-dimensional operators, by nilpotent operators, by quasinilpotent operators, and by algebraic operators commuting with T . The classes of operators are considered as illustrating examples.

Введено нову властивість (gaR) , що узагальнює властивість (R) , яку розглядав Айєна. Властивість (gaR) вивчається у зв'язку з теоремами типу Вейля. Встановлено необхідні та достатні умови для того, щоб властивість (gaR) виконувалася. Крім того, вивчається стабільність властивості (gaR) при збуреннях скінченновимірними, нільпотентними, квазінілпотентними та алгебраїчними операторами, що комутують з T . Ці класи операторів розглянуто як ілюстративні приклади.

1. Introduction. Let X be an infinite-dimensional complex Banach space and $L(X)$ be the algebra of all bounded linear operators on X . For $T \in L(X)$, we denote the null space, the range, the spectrum, the approximate point spectrum, the surjective spectrum, the isolated points of the spectrum and the isolated points of the approximate point spectrum of T by $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\text{iso } \sigma(T)$ and $\text{iso } \sigma_a(T)$, respectively.

If $R(T)$ is closed and $\alpha(T) = \dim N(T) < \infty$ (resp. $\beta(T) = \dim X/R(T) < \infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. In the sequel $\Phi_+(X)$ (resp. $\Phi_-(X)$) is written for the set of all upper (resp. lower) semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $\Phi_{\pm}(X) = \Phi_+(X) \cup \Phi_-(X)$, in this case the index of T is given by $i(T) = \alpha(T) - \beta(T)$. Denote $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ the set of all Fredholm operators. Define $W_+(X) = \{T \in \Phi_+(X) : i(T) \leq 0\}$ the set of all upper semi-Weyl operators, while $W_-(X) = \{T \in \Phi_-(X) : i(T) \geq 0\}$ the set of all lower semi-Weyl operators. The set of all Weyl operators is defined by $W(X) = W_+(X) \cap W_-(X) = \{T \in \Phi(X) : i(T) = 0\}$. The classes of operators defined above generate the following spectrum: the Weyl spectrum of T is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W(X)\}$, while the upper semi-Weyl spectrum and the lower semi-Weyl spectrum of T are defined by $\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_+(X)\}$ and $\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_-(X)\}$, respectively.

Let $p = p(T)$ be the ascent of T , i.e., the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T , i.e., the smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$, and if such integer does not exist we put $q(T) = \infty$ [15] (Proposition 38.3). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T , see Proposition 50.2 of Heuser [15]. The class of all upper semi-Browder operators is defined by $B_+(X) = \{T \in \Phi_+(X) : p(T) < \infty\}$ and the class of all Browder operators is defined by $B(X) = \{T \in \Phi(X) : p(T) = q(T) < \infty\}$. The Browder spectrum

* This work is supported by the National Natural Science Foundation of China (№ 11601130) and the Natural Science Foundation of the Department of Education of Henan Province (16A110033, 17A110005).

of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}$ and the upper semi-Browder spectrum is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\}$, respectively. Let $\pi_{00}^a(T) = \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}$ and $p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$.

For $T \in L(X)$, an operator T is called B -Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \rightarrow Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B -Fredholm operator T is called B -Weyl if $i(T_{[n]}) = 0$.

The B -Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda)\}$.

For $T \in L(X)$, an operator T is called an upper semi- B -Weyl operator if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator $T_{[n]} : R(T^n) \ni x \rightarrow Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $i(T_{[n]}) \leq 0$ [10]. We define

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-}B\text{-Weyl}\}.$$

We say that generalized a -Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T),$$

where $E_a(T) = \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(T - \lambda)\}$.

If $p(T) < \infty$ and $R(T^{p(T)+1})$ is closed, then T is called left Drazin invertible. If $p(T) = q(T) < \infty$, then T is called Drazin invertible. The Drazin spectrum $\sigma_D(T)$ and left Drazin spectrum $\sigma_{LD}(T)$ of an operator T are defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$$

and

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}.$$

Let

$$\Pi(T) = \{\lambda \in \text{iso } \sigma(T) : T - \lambda \text{ is Drazin invertible}\}$$

denote the set of all poles of T and

$$p_{00}(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda) < \infty\}$$

be the set of all poles of T of finite rank,

$$\Pi_a(T) = \{\lambda \in \text{iso } \sigma_a(T) : T - \lambda \text{ is left Drazin invertible}\}$$

denotes the set of all left poles of T and

$$p_{00}^a(T) = \{\lambda \in \Pi_a(T) : \alpha(T - \lambda) < \infty\}$$

denotes the set of all left poles of T of finite rank. According to [15], the space $R((T - \lambda I)^{p(T - \lambda I) + 1})$ is closed for each $\lambda \in \Pi(T)$. Hence we have $\Pi(T) \subseteq \Pi_a(T)$ and $p_{00}(T) \subseteq p_{00}^a(T)$. We say that generalized a -Browder's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T).$$

According to [9] we say that generalized Browder's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T).$$

Recall [12] that property (gaw) is said to hold for T if $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$. Recall [5] that property (R) holds for T if $p_{00}^a(T) = \pi_{00}(T)$, where $\pi_{00}(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$. According to [18], an operator T is said to satisfy property (aR) if $p_{00}(T) = \pi_{00}^a(T)$.

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [16] and Aiena [1]. In this article we shall consider the following local version of this property.

Let $T \in L(X)$. The operator T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbrev. SVEP at λ_0), the only analytic function $f: D \rightarrow X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

It is known that both Browder's (equivalently, generalized Browder's) theorem and a -Browder's (equivalently, generalized a -Browder's) theorem hold for T if T or T^* has SVEP. Precisely, we have that a -Browder's (equivalently, generalized a -Browder's) theorem holds for T if and only if T has SVEP at every $\lambda \notin \sigma_{uw}(T)$, and dually, a -Browder's (equivalently, generalized a -Browder's) theorem holds for T^* if and only if T^* has SVEP at every $\lambda \notin \sigma_{lw}(T)$, see [3, 8].

From the identity theorem for analytic function it easily follows that $T \in L(X)$, as well as its dual T^* , has SVEP at every point of the boundary of the spectrum $\sigma(T) = \sigma(T^*)$, so both T and T^* have SVEP at every isolated point of the spectrum.

According to [3] (Theorem 1.2), if $T \in L(X)$ and suppose that $\lambda_0 I - T \in \Phi_{\pm}(X)$. Then the following statements are equivalent:

- (i) T has SVEP at λ_0 ;
- (ii) $p(T - \lambda_0 I) < \infty$;
- (iii) $\sigma_a(T)$ doesn't cluster at λ_0 .

Dually, if $\lambda_0 I - T \in \Phi_{\pm}(X)$, then the following statements are equivalent:

- (iv) T^* has SVEP at λ_0 ;
- (v) $q(T - \lambda_0 I) < \infty$;
- (vi) $\sigma_s(T)$ doesn't cluster at λ_0 .

A bounded operator T is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T . T is said to be hereditarily polaroid if every part of T is polaroid. T is said to be a -polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T . And T is said to be a -isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T .

Let $T \in L(X)$ and $d \in \mathbb{N}$. Then T has a uniform descent for $n \geq d$ if $R(T) + N(T^n) = R(T) + N(T^d)$ for $n \geq d$. If in addition, $R(T) + N(T^d)$ is closed, then T is said to have a topological uniform descent for $n \geq d$, see [14].

If $\lambda \in \Pi_a(T)$ or $T - \lambda I$ is a semi- B -Fredholm operator, then $T - \lambda$ is an operator of topological uniform descent.

In Section 2, we introduce and study the new property (gaR) in connection with Weyl type theorems. We prove that an operator T possessing property (gaR) possesses property (aR) , but the converse is not true in general as shown by Example 2.2. We prove also that if T^* has SVEP at every $\lambda \notin \sigma_{uw}(T)$, then property (gaR) , property (gaw) , generalized Weyl's theorem and generalized a -Weyl's theorem are equivalent. In Section 3, in Theorem 3.3 we prove that if $T \in L(X)$ and M is a nilpotent operator commuting with T , then T possesses property (gaR) if and only if $T+M$ possesses property (gaR) . And we provide a condition under which the new property (gaR) is preserved under commuting finite-dimensional operator, we prove in Theorem 3.2 that if $\text{iso } \sigma_a(T) = \phi$ and K is a finite-dimensional operator commuting with T , then $T + K$ satisfies property (gaR) . In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems, extending a similar diagram given [9].

2. Property (gaR) .

Definition 2.1. An operator T is said to satisfy property (gaR) if $\Pi(T) = E_a(T)$.

Lemma 2.1 [4, 9]. Suppose that $T \in L(X)$. Then we have:

- (i) T satisfies generalized Weyl's theorem if and only if generalized Browder's theorem holds for T and $\Pi(T) = E(T)$.
- (ii) T satisfies generalized a -Weyl's theorem if and only if generalized a -Browder's theorem holds for T and $\Pi_a(T) = E_a(T)$.

Lemma 2.2 [14]. Suppose that T is a bounded linear operator and that λ belongs to the boundary of the spectrum of T . If $T - \lambda$ has topological uniform descent, then λ is a pole of T .

Theorem 2.1. Let T satisfy property (gaR) . Then $E_a(T) = \Pi_a(T) = E(T) = \Pi(T)$.

Proof. Observe that $\Pi(T) \subseteq E(T) \subseteq E_a(T)$ holds for every operator T . As T satisfies property (gaR) , $E_a(T) = \Pi(T)$, and hence $\Pi(T) = E(T) = E_a(T)$. As $\Pi(T) \subseteq \Pi_a(T) \subseteq E_a(T)$ holds for every operator T and $E_a(T) = \Pi(T)$, then $\Pi(T) = \Pi_a(T) = E_a(T)$, i.e., $E_a(T) = \Pi_a(T) = E(T) = \Pi(T)$.

The following example shows neither of the two equalities $E_a(T) = \Pi_a(T)$, $E(T) = \Pi(T)$ can imply $\Pi(T) = E_a(T)$.

Example 2.1. Let $R: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \dots) = \left(\frac{1}{3}x_1, x_2, x_3, \dots \right)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$. Define $T = R \oplus Q$. Then $\sigma(T) = \sigma(T^*) = D$, $\sigma_a(T) = \partial D \cup \left\{ \frac{1}{3} \right\}$ and $\sigma_{uw}(T) = \partial D$, where D denotes the closed unit disc and ∂D denotes the unit circle, and hence $\Pi(T) = E(T) = \phi$. We show that T does not satisfy property (gaR) . Since T has SVEP at the points of ∂D and T has SVEP at $\frac{1}{3}$. Hence T has SVEP and a -Browder's theorem holds for T , i.e., $\sigma_{uw}(T) = \sigma_{ub}(T) = \partial D$. It follows that $p_{00}^a(T) = \sigma_a(T) \setminus \sigma_{ub}(T) = \left\{ \frac{1}{3} \right\}$.

Observe that the operator T satisfies the equality $\Pi_a(T) = E_a(T)$. Indeed, $\frac{1}{3}$ is an isolated point of $\sigma_a(T)$, and hence $E_a(T) = \left\{ \frac{1}{3} \right\} = \Pi_a(T)$. While T does not satisfy property (gaR) since $E_a(T) = \left\{ \frac{1}{3} \right\} \neq \Pi(T)$.

As noted in Example 2.1 the condition $\Pi_a(T) = E_a(T)$ is strictly weaker than property (gaR) . However, we have the following theorem.

Theorem 2.2. *T satisfies property (gaR) if and only if the following two conditions hold:*

- (i) $E_a(T) \subseteq \text{iso } \sigma(T)$,
- (ii) $\Pi_a(T) = E_a(T)$.

Proof. If T satisfies property (gaR) , then $E_a(T) = \Pi(T) \subseteq \text{iso } \sigma(T)$ and by Theorem 2.1 we have $\Pi_a(T) = E_a(T)$. Conversely, suppose that both (i) and (ii) hold. As $\Pi(T) \subseteq E_a(T)$ holds for every operator T . To show the opposite inclusion, let $\lambda \in E_a(T)$. Then $\lambda \in E_a(T) = \Pi_a(T)$, and hence $T - \lambda$ has topological uniform descent, since $E_a(T) \subseteq \text{iso } \sigma(T)$, then λ is a pole of T , thus $\lambda \in \Pi(T)$. Therefore $\Pi(T) = E_a(T)$.

Theorem 2.3. *Let T satisfy property (gaR) . Then property (aR) holds for T .*

Proof. Since $p_{00}(T) \subseteq \pi_{00}^a(T)$ holds for every operator T . To show the opposite inclusion, let $\lambda \in \pi_{00}^a(T)$. Then $\lambda \in E_a(T)$ and $\alpha(T - \lambda) < \infty$. Since T satisfies property (gaR) , then $E_a(T) = \Pi(T)$. And hence $\lambda \in \Pi(T)$ and $\alpha(T - \lambda) < \infty$, i.e., $\lambda \in p_{00}(T)$.

The following example shows that property (aR) is weaker than property (gaR) .

Example 2.2. Let $R: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \dots) = \left(\frac{1}{2}x_1, \frac{1}{2}x_2, \dots \right)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$. Define $T := R \oplus Q$. Then $\sigma(T) = D$, $\sigma_a(T) = \partial D \cup \left\{ \frac{1}{2} \right\}$. It follows that $\Pi(T) = \phi$, $E_a(T) = \left\{ \frac{1}{2} \right\}$, then T does not satisfy property (gaR) . But T satisfies property (aR) since $p_{00}(T) = \pi_{00}^a(T) = \phi$.

In the following theorem we give a condition for the equivalence of property (gaR) and property (gaw) .

Theorem 2.4. *T satisfies property (gaw) if and only if generalized Browder's theorem holds for T and T has property (gaR) .*

Proof. If generalized Browder's theorem holds for T and T has property (gaR) , then $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$ and $\Pi(T) = E_a(T)$, and hence $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$.

Conversely, it is easy to prove that property (gaw) implies generalized Browder's theorem by [12] (Corollary 2.7, Theorem 3.5), then $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. Since T satisfies property (gaw) , then $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$, hence $\Pi(T) = E_a(T)$, i.e., T has property (gaR) .

The following example shows that property (gaR) is weaker than property (gaw) .

Example 2.3. Let $R: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$ and $L: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be the unilateral left shift operator defined by $L(x_1, x_2, \dots) = (x_2, x_3, \dots)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$. Define $T := R \oplus L$. Then $\sigma(T) = \sigma_a(T) = D$. It follows that $\Pi(T) = E_a(T) = \phi$, then T satisfies property (gaR) . While T doesn't satisfy property (gaw) , since $0 \in \sigma(T) \setminus \sigma_{BW}(T) \neq \phi = E_a(T)$.

The following example shows property (gaR) for an operator is not transmitted to the dual T^* .

Example 2.4. Let $L : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be the unilateral left shift operator defined by

$$L(x_1, x_2, \dots) = (x_2, x_3, \dots) \quad \text{for all } x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$$

and

$$Q(x_1, x_2, \dots) = (0, x_2, x_3, \dots) \quad \text{for all } x = (x_1, x_2, \dots) \in l^2(\mathbb{N}).$$

Define $T := L \oplus Q$. Then $\sigma(T) = \sigma(T^*) = \sigma_a(T) = D$ and $\sigma_a(T^*) = \partial D \cup \{0\}$. It follows that $\Pi(T) = E_a(T) = \phi$, then T satisfies property (gaR) . While T^* doesn't satisfy property (gaR) , since $0 \in E_a(T^*) \neq \phi = \Pi(T^*)$.

The following example shows that generalized a -Weyl's theorem does not entail property (gaR) .

Example 2.5. Let T be defined as in Example 2.1. As already observed, T does not satisfy property (gaR) , while T has SVEP and hence generalized a -Browder's theorem holds for T . Since $\Pi_a(T) = E_a(T)$, by part (ii) of Lemma 2.1, then generalized a -Weyl's theorem holds for T .

The following example shows that property (gaR) does not entail generalized a -Weyl's theorem.

Example 2.6. Let T be defined as in Example 2.3. We have $\alpha(T) = \beta(T) = 1$ and $p(T) = \infty$. Therefore, $0 \notin \sigma_w(T)$, while $0 \in \sigma_b(T)$, so Browder's theorem (and hence generalized a -Weyl's theorem) does not hold for T . On the other hand, since $\sigma(T) = \sigma_a(T) = D$, we have $\Pi(T) = E_a(T) = \phi$, and hence property (gaR) holds for T .

Theorem 2.5. Let T satisfy both generalized a -Browder's theorem and property (gaR) . Then T satisfies generalized a -Weyl's theorem and $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$.

Proof. If T satisfies generalized a -Browder's theorem and property (gaR) , then $E_a(T) = \Pi_a(T)$ by Theorem 2.1. Therefore generalized a -Weyl's theorem holds for T by (ii) of Lemma 2.1, i.e., $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$. Property (gaR) then implies $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$.

In [11] an operator T is said to have property (gb) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$.

The following example shows that property (gaR) does not entail property (gb) .

Example 2.7. Let T be defined as in Example 2.3. Then T satisfies property (gaR) , while property (gb) does not hold for T , since $0 \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ and $\Pi(T) = \phi$. This example also shows that without the assumption that T satisfies generalized a -Browder's theorem, the result of Theorem 2.5 does not hold.

The following example shows that property (gb) does not entail property (gaR) .

Example 2.8. Let $Q(x_1, x_2, \dots) = \left(\frac{x_2}{2^2}, \frac{x_3}{2^3}, \dots\right)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$. Clearly, Q is quasinilpotent and hence $\sigma(Q) = \sigma_a(Q) = \sigma_{SBF_+^-}(Q) = \{0\}$. We have $\alpha(Q) = 1$, then $0 \in E_a(Q)$, $\Pi(Q) = \phi$, it then follows that Q does not satisfy property (gaR) . On the other hand, since $\sigma_a(Q) \setminus \sigma_{SBF_+^-}(Q) = \Pi(Q) = \phi$, Q has property (gb) .

The next result shows that the equivalence of property (gaR) , property (gaw) , generalized Weyl's theorem and generalized a -Weyl's theorem is true whenever we assume that T^* has SVEP at the points $\lambda \notin \sigma_{uw}(T)$.

Theorem 2.6. Let T^* have SVEP at every $\lambda \notin \sigma_{uw}(T)$. Then the following statements are equivalent:

- (i) $E(T) = \Pi(T)$;
- (ii) $E_a(T) = \Pi_a(T)$;
- (iii) $E_a(T) = \Pi(T)$.

Consequently, property (gaR) , property (gaw) , generalized Weyl's theorem and generalized a -Weyl's theorem are equivalent for T .

Proof. It is easy to see that $\sigma(T) = \sigma_a(T)$, then we have $E(T) = E_a(T)$. The following we would show $\Pi_a(T) = \Pi(T)$, observe first that $\Pi(T) \subseteq \Pi_a(T)$ holds for every operator T . To show the opposite inclusion, let $\lambda \in \Pi_a(T)$. Then $T - \lambda$ has topological uniform descent and $\lambda \in \text{iso } \sigma_a(T) = \text{iso } \sigma(T)$, it follows from Lemma 2.2 that λ is a pole of T , i.e., $\lambda \in \Pi(T)$. From which the equivalence of (i), (ii) and (iii) easily be obtained. To show the last statement observe that the SVEP of T^* at the points $\lambda \notin \sigma_{uw}(T)$ entails that generalized a -Browder's theorem (and hence generalized Browder's theorem) holds for T , see [3] (Theorem 2.3). By Lemma 2.1 and Theorem 2.4, then property (gaR) , property (gaw) , generalized Weyl's theorem and generalized a -Weyl's theorem are equivalent for T .

Dually, we have the following theorem.

Theorem 2.7. *Let T have SVEP at every $\lambda \notin \sigma_{lw}(T)$. Then the following statements are equivalent:*

- (i) $E(T^*) = \Pi(T^*)$;
- (ii) $E_a(T^*) = \Pi_a(T^*)$;
- (iii) $E_a(T^*) = \Pi(T^*)$.

Consequently, property (gaR) , property (gaw) , generalized Weyl's theorem and generalized a -Weyl's theorem are equivalent for T^ .*

Proof. It is clear from Theorem 2.6.

Theorem 2.8. *Let T be a -polaroid. Then T satisfies property (gaR) .*

Proof. Since $\Pi(T) \subseteq E_a(T)$ holds for every operator T . To show the opposite inclusion, let $\lambda \in E_a(T)$. Then λ is an isolated point of $\sigma_a(T)$. Since T is a -polaroid, λ is a pole of the resolvent of T , $\lambda \in \Pi(T)$, i.e., T satisfies property (gaR) .

Corollary 2.1 [18]. *Let T be a -polaroid. Then T satisfies property (aR) .*

The next example shows that under a weaker condition of being polaroid the result of Theorem 2.8 does not hold.

Example 2.9. Let $R: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be the unilateral right shift operator defined by $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$ and $Q(x_1, x_2, \dots) = \left(\frac{x_2}{2^2}, \frac{x_3}{2^3}, \dots\right)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$. Define $T := R \oplus Q$. Then $\sigma(T) = D$, hence $\text{iso } \sigma(T) = \Pi(T) = \emptyset$. Therefore, T is polaroid. Moreover, $\sigma_a(T) = \partial D \cup \{0\}$, $\text{iso } \sigma_a(T) = \{0\}$, $0 < \alpha(T) = 1 < \infty$ implies $0 \in E_a(T)$, and hence $E_a(T) \neq \Pi(T)$, thus T doesn't satisfy property (gaR) .

From the proof of Theorem 2.6 we know that if T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Therefore if T^* has SVEP, then T is a -polaroid $\Leftrightarrow T$ is polaroid.

Corollary 2.2. *Let T be polaroid and T^* have SVEP. Then T satisfies property (gaR) .*

Note that the result of Corollary 2.2 does not hold if we replace the SVEP for T^* by the SVEP for T .

Example 2.10. Let T be defined as in Example 2.9. Then T has SVEP and is polaroid, while T does not satisfy property (gaR) .

3. Property (gaR) under perturbations.

Theorem 3.1 [13]. *If T is a -isoloid and satisfies a -Weyl's theorem, then $T+K$ satisfies a -Weyl's theorem for every finite-dimensional operator K commuting with T .*

The following example shows that an analogous result of Theorem 3.1 does not hold for property (gaR) , even with the class of a -isoloid operators.

Example 3.1. Let $T: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \dots) = (2x_1, 2x_2, 0, x_3, x_4, \dots) \quad \text{for all } x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$$

and

$$K(x_1, x_2, \dots) = (-2x_1, -2x_2, 0, 0, 0, \dots) \quad \text{for all } x = (x_1, x_2, \dots) \in l^2(\mathbb{N}).$$

Then $\sigma(T) = D \cup \{2\}$ and $\sigma_a(T) = \partial D \cup \{2\}$, it follows that $E_a(T) = \Pi(T) = \{2\}$. Therefore, T is a -isoloid operator, $KT = TK$ and satisfies property (gaR) . While $\sigma(T + K) = D$ and $\sigma_a(T + K) = \partial D \cup \{0\}$, it follows that $\Pi(T + K) = \text{iso } \sigma(T + K) = \phi \neq \{0\} = E_a(T + K)$. Therefore, $T + K$ does not satisfy property (gaR) .

Theorem 3.2. Let $T \in L(X)$ and $\text{iso } \sigma_a(T) = \phi$. If K is a finite-dimensional operator commuting with T , then $T + K$ satisfies property (gaR) .

Proof. Since $\text{iso } \sigma_a(T) = \phi$ and K is a finite-dimensional operator commuting with T , by the proof of [2] (Theorem 2.8), $\sigma_a(T) = \sigma_a(T + K)$, then $\text{iso } \sigma_a(T + K) = \phi$. Since $\text{iso } \sigma(T + K) \subseteq \subseteq \text{iso } \sigma_a(T + K)$, $\text{iso } \sigma(T + K) = \phi$. It follows that $\Pi(T + K) = E_a(T + K) = \phi$, i.e., $T + K$ satisfies property (gaR) .

Corollary 3.1 [18]. Let $T \in L(X)$ and $\text{iso } \sigma_a(T) = \phi$. If K is a finite-dimensional operator commuting with T , then $T + K$ satisfies property (aR) .

The next result shows that property (gaR) for T is transmitted to $T + M$, when M is a nilpotent operator which commutes with T . Recall first that the equality $\sigma_a(T) = \sigma_a(T + Q)$ holds for every quasinilpotent operator Q which commutes with T .

Theorem 3.3. Let $T \in L(X)$ and $M \in L(X)$ be a nilpotent operator which commutes with T . Then we have:

- (i) $E_a(T + M) = E_a(T)$.
- (ii) T satisfies property (gaR) if and only if $T + M$ satisfies property (gaR) .
- (iii) If T is a -polaroid, then $T + M$ satisfies property (gaR) .

Proof. (i) Let $\lambda \in E_a(T + M)$. We can assume $\lambda = 0$. Clearly, $0 \in \text{iso } \sigma_a(T + M) = \text{iso } \sigma_a(T)$. Let $p \in \mathbb{N}$ be such that $M^p = 0$. If $x \in N(T + M)$, then $T^p x = (-1)^p M^p x = 0$, thus $N(T + M) \subseteq N(T^p)$, since by assumption $\alpha(T + M) > 0$, it then follows that $\alpha(T^p) > 0$ and this obviously implies that $\alpha(T) > 0$. Therefore, $0 \in E_a(T)$, and consequently $E_a(T + M) \subseteq E_a(T)$. $E_a(T) \subseteq E_a(T + M)$ follows by symmetry.

(ii) Suppose that T has property (gaR) . Then $E_a(T + M) = E_a(T) = \sigma(T) \setminus \sigma_D(T) = \sigma(T + M) \setminus \sigma_D(T + M) = \Pi(T + M)$, therefore $T + M$ has property (gaR) . The converse follows by symmetry.

(iii) Obviously, by part (ii), since T satisfies property (gaR) by Theorem 2.8.

This example shows that the commutativity hypothesis in (ii) of Theorem 3.3 is essential.

Example 3.2. Let $Q: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be defined by

$$Q(x_1, x_2, \dots) = \left(0, 0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \dots\right) \quad \text{for all } x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$$

and

$$M(x_1, x_2, \dots) = \left(0, 0, -\frac{x_1}{2}, 0, 0, \dots\right) \quad \text{for all } x = (x_1, x_2, \dots) \in l^2(\mathbb{N}).$$

Clearly M is a nilpotent operator and $\Pi(Q) = E_a(Q) = \phi$, i.e., Q satisfies property (gaR) . While $\Pi(Q + M) = \phi$ and $E_a(Q + M) = \{0\}$, it follows that $\Pi(Q + M) \neq E_a(Q + M)$, i.e., $Q + M$ does not satisfy property (gaR) .

The previous theorem does not extend to commuting quasinilpotent operators as shown by the following example.

Example 3.3. Let $Q: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be defined by $Q(x_1, x_2, \dots) = \left(\frac{x_2}{2^2}, \frac{x_3}{2^3}, \frac{x_4}{2^4}, \dots\right)$ for all $x = (x_1, x_2, \dots) \in l^2(\mathbb{N})$ and $T = 0$. Clearly T satisfies property (gaR) . While Q is quasinilpotent and $TQ = QT$, so $\sigma(Q) = \sigma_D(Q) = \{0\}$ and hence $\{0\} = E_a(Q) \neq \sigma(Q) \setminus \sigma_D(Q) = \Pi(Q) = \phi$, i.e., $T + Q = Q$ does not satisfy property (gaR) .

Theorem 3.4. Let T be a -polaroid and finite-isoloid and Q be a quasinilpotent operator which commutes with T . Then $T + Q$ has property (gaR) .

Proof. Clearly by the proof of [2] (Theorem 2.13).

Recall that a bounded operator T is said to be algebraic if there exists a nonconstant polynomial h such that $h(T) = 0$.

Theorem 3.5. Let $T \in L(X)$ and $K \in L(X)$ be an algebraic operator which commutes with T :

- (i) If T is hereditarily polaroid and has SVEP, then $T^* + K^*$ satisfies property (gaR) .
- (ii) If T^* is hereditarily polaroid and has SVEP, then $T+K$ satisfies property (gaR) .

Proof. Since $T^* + K^*$ is a -polaroid by the proof of [2] (Theorem 2.15), property (gaR) for $T^* + K^*$ follows from Theorem 2.8.

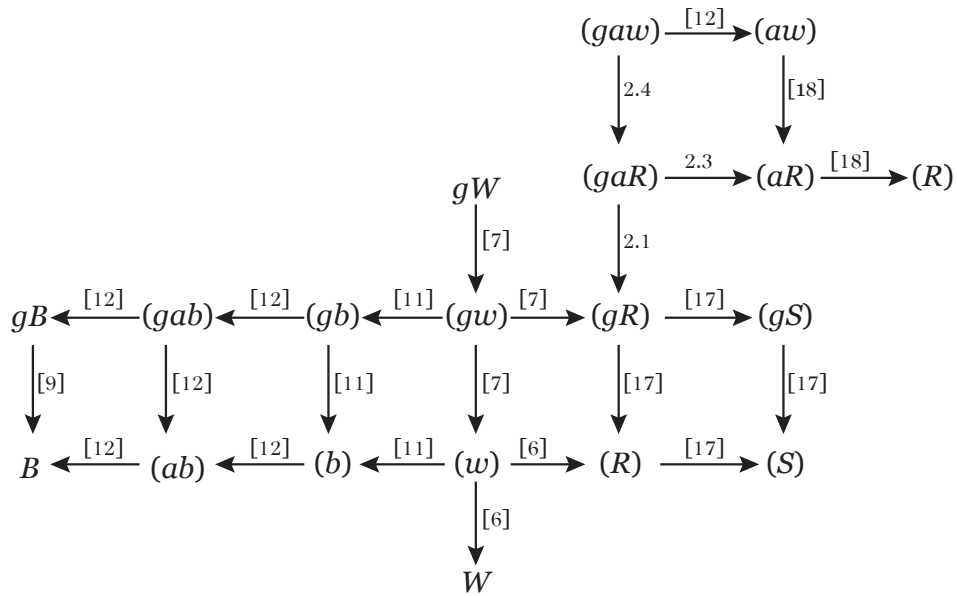
- (ii) The proof is similar to (i).

4. Conclusion. In the last part, we give a summary of the known Weyl type theorems as in [9], including the properties introduced in [5–7, 11, 12, 17, 18], and in this paper. We use the abbreviations gW ; W ; (gw) ; (w) ; (gaw) ; (aw) ; (gR) ; (R) ; (gaR) ; (aR) ; (gS) and (S) to signify that an operator $T \in L(X)$ obeys generalized Weyl’s theorem, Weyl’s theorem, property (gw) , property (w) , property (gaw) , property (aw) , property (gR) , property (R) , property (gaR) , property (aR) , property (gS) and property (S) . Similarly, the abbreviations gB ; B ; (gb) ; (b) ; (gab) and (ab) have analogous meaning with respect to Browder’s theorem.

The following table summarizes the meaning of various theorems and properties.

gW	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$	(aR)	$p_{00}(T) = \pi_{00}^a(T)$
W	$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$	gB	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	B	$\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$
(w)	$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(gaw)	$\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$	(b)	$\sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}(T)$
(aw)	$\sigma(T) \setminus \sigma_w(T) = \pi_{00}^a(T)$	(gab)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$
(gR)	$\Pi_a(T) = E(T)$	(ab)	$\sigma(T) \setminus \sigma_w(T) = p_{00}^a(T)$
(R)	$p_{00}^a(T) = \pi_{00}(T)$	(gS)	$\Pi(T) = E(T)$
(gaR)	$\Pi(T) = E_a(T)$	(S)	$\pi_{00}(T) = p_{00}(T)$

In the following diagram, which extends the similar diagram presented in [9], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw) , property (gaw) , property (gR) , property (gaR) , property (gS) , property (w) , property (aw) , property (R) , property (aR) and property (S) . The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).



References

1. Aiena P. Fredholm and local spectral theory, with application to multipliers. – New York: Kluwer Acad. Press, 2004.
2. Aiena P., Aponte E., Guillén J. R., Peña P. Property (R) under perturbations // *Mediterr. J. Math.* – 2013. – **10**, № 1. – P. 367–382.
3. Aiena P., Carpintero C., Rosas E. Some characterizations of operators satisfying a -Browder’s theorem // *J. Math. Anal. and Appl.* – 2005. – **311**, № 2. – P. 530–544.
4. Aiena P., García O. Generalized Browder’s theorem and SVEP // *Mediterr. J. Math.* – 2007. – **4**, № 2. – P. 215–228.
5. Aiena P., Guillén J. R., Peña P. Property (R) for bounded linear operators // *Mediterr. J. Math.* – 2011. – **8**, № 4. – P. 491–508.
6. Aiena P., Peña P. Variations on Weyl’s theorem // *J. Math. Anal. and Appl.* – 2006. – **324**. – P. 566–579.
7. Amouch M., Berkani M. On the property (gw) // *Mediterr. J. Math.* – 2008. – **5**, № 3. – P. 371–378.
8. Amouch M., Zguitti H. On the equivalence of Browder’s and generalized Browder’s theorem // *Glasgow Math. J.* – 2006. – **48**, № 1. – P. 179–185.
9. Berkani M., Koliha J. J. Weyl type theorems for bounded linear operators // *Acta Sci. Math. (Szeged)*. – 2003. – **69**, № 1-2. – P. 359–376.
10. Berkani M., Sarih M. On semi B -Fredholm operators // *Glasgow Math. J.* – 2001. – **43**, № 3. – P. 457–465.
11. Berkani M., Zariouh H. Extended Weyl type theorems // *Math. Bohemica*. – 2009. – **134**, № 4. – P. 369–378.
12. Berkani M., Zariouh H. New extended Weyl type theorems // *Mat. Vesnik*. – 2010. – **62**, № 2. – P. 145–154.
13. Djordjević D. S. Operators obeying a -Weyl’s theorem // *Publ. Math. Debrecen*. – 1999. – **55**, № 3-4. – P. 283–298.
14. Grabiner S. Uniform ascent and descent of bounded operators // *J. Math. Soc. Jap.* – 1982. – **34**, № 2. – P. 317–337.
15. Heuser H. Functional analysis. – New York: John Wiley & Sons Inc., 1982.
16. Laursen K. B., Neumann M. M. Introduction to local spectral theory. – Oxford: Clarendon Press, 2000.
17. Rashid M. H. M. Properties (S) and (gS) for bounded linear operators // *Filomat*. – 2014. – **28**, № 8. – P. 1641–1652.
18. Shen J. L., Chen A. A new variation of Weyl type theorem and perturbations // *Filomat*. – 2014. – **28**, № 9. – P. 1899–1906.

Received 23.09.13,
after revision — 25.02.17