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DECAY ESTIMATES FOR A KIND OF LINEAR WAVE EQUATIONS*

ОЦІНКИ ЗАТУХАННЯ ДЛЯ ОДНОГУ ТИПУ ХВИЛЬОВИХ РІВНЯНЬ

We consider one kind of dissipative wave equations with exponential speed of propagation. An arbitrary power decay rate for the L^2-norm and energy is obtained by using the multiplier method.

Розглянуто один тип дисипативних хвильових рівнянь з експоненціальною швидкістю поширення. Довільний степеневий закон затухання отримано для L^2-норми та енергії за допомогою методу множників.

1. Introduction. Consider the following damped wave equation:

$$\begin{gathered} u_{tt} - \operatorname{div}(b(x)\nabla u) + a(x)u_t = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where $a(x) \in C^0(\mathbb{R}^n)$, $b(x) \in C^1(\mathbb{R}^n)$ are positive functions, and $(u_0(x), u_1(x)) \in H^1(\mathbb{R}^n) \times \times L^2(\mathbb{R}^n)$ have compact support

$$u_0(x) = 0 \text{ and } u_1(x) = 0 \text{ for } |x| > R.$$

Such a system appears in models for traveling waves in a nonhomogeneous gas with damping that changes with the position (see [1] and the references therein). It is well known that (1.1) admits a unique weak solution u satisfying $u \in C((0,\infty), H^1(\mathbb{R}^n))$ and $u_t \in C((0,\infty), L^2(\mathbb{R}^n))$ (see [2]).

The main quantities of interest are the L^2-norm and energy associated with u. In fact, the energy arises after multiplying equation (1.1) by u_t and applying the divergence theorem on $\mathbb{R}^n$:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^n} \left(u_t^2 + b(x)|\nabla u|^2\right) dx + \int\limits_{\mathbb{R}^n} a(x)u_t^2 dx = 0.$$

Hence the energy is a nonincreasing function of t. The important question is whether the energy decays as $t \to \infty$ and if so, how fast it decays. This problem has been studied intensively when b is constant (see [3 – 15] and the references therein). When b is space dependent, the problem (1.1) does not have constant speed of propagation anymore (see Radu et al. [1]). In [1], they considered the case

$$b_0(1+|x|)^\beta \le b(x) \le b_1(1+|x|)^\beta, \qquad a_0(1+|x|)^{-\alpha} \le a(x) \le a_1(1+|x|)^{-\alpha} \tag{1.2}$$

with $a_0,\ a_1,\ b_0,\ b_1 > 0$ and $\alpha < 1,\ 0 \le \beta < 2,\ 2\alpha + \beta \le 2$. Using the multiplier method, they obtained the following decay estimates of L^2-norm and energy:

$$\int\limits_{\mathbb{R}^n} u^2 dx \le C_\delta \left(\|\nabla u_0(x)\|_{L^2}^2 + \|u_1(x)\|_{L^2}^2\right) \begin{cases} t^{\delta - \frac{n-2\alpha}{2-\alpha-\beta}}, & \alpha > 0, \\ t^{\delta - \frac{n-\alpha}{2-\alpha-\beta}}, & \alpha \le 0, \end{cases}$$

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$$\int_{\mathbb{R}^n} \left(u_t^2 + b(x)|\nabla u|^2\right) dx \le C_\delta \left(\|\nabla u_0(x)\|_{L^2}^2 + \|u_1(x)\|_{L^2}^2\right) t^{\delta - \frac{n-\alpha}{2-\alpha-\beta} - 1}.$$

Note that the above decay rates go to infinity when $\beta \to 2^-$ and $\alpha \to 0^+$. But one can not propose limit to go to the case $\beta = 2$ and $\alpha = 0$. This is because for the case $\beta = 2$ and $\alpha = 0$, the problem has exponential speed of propagation, while power speed of propagation for the case (1.2), see Lemma 2.1. In this paper, we consider the case $\beta = 2$ and $\alpha = 0$. By modifying the method in [1], we prove arbitrary power decay rate of the L^2-norm and energy. For simplicity, we only consider the radial case

$$b(x) = b_0(1 + |x|)^2, \qquad a(x) = 1. \tag{1.3}$$

And the results in this paper can be generalized to the general case

$$b_0(1 + |x|)^2 \le b(x) \le b_1(1 + |x|)^2, \qquad a_0 \le a(x) \le a_1.$$

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove our main results.

2. Preliminaries. First, we state a result about the support of solutions for a wave equation with variable coefficients (see [1]).

Lemma 2.1. *Assume that* $b(x) = b_0(1 + |x|)^\beta$ *and that* $u(x,t)$ *satisfies* (1.1) *with* $u_0(x) = u_1(x) = 0$ *for* $|x| > R$. *Then* $u(x,t) = 0$ *whenever* $|x| > R_t$, *where*

$$R_t = (1 + R)e^{t\sqrt{b_0}} - 1 \quad \text{for} \quad \beta = 2, \tag{2.1}$$

$$R_t = \left((1 + R)^{(2-\beta)/2} + t\sqrt{b_0}\right)^{2/(2-\beta)} \quad \text{for} \quad \beta < 2. \tag{2.2}$$

Proof. (2.2) is a direct result of Proposition 2.1 in [1] for the radial case $b(x)$. (2.1) can be obtained in a similar way. We only point out the differences of the proof. Following the proof of Proposition 2.1 in [1], for $\beta = 2$, we obtain

$$q(r) = -\int_{r_0}^{r} \frac{1}{\sqrt{b_0(1 + s)^2}} ds = \frac{1}{\sqrt{b_0}} \left(\ln(1 + r_0) - \ln(1 + r)\right).$$

Hence $q^{-1}(y) = (1 + r_0)e^{-y\sqrt{b_0}} - 1$ and

$$R_t = q^{-1}(q(R) - t) = (1 + R)e^{t\sqrt{b_0}} - 1.$$

Lemma 2.2. *Under assumptions* (1.3) *there exists a solution* $A(x)$ *which satisfies*

$$\operatorname{div}(b(x)\nabla A(x)) = a(x) = 1, \quad x \in \mathbb{R}^n, \tag{2.3}$$

with the following properties:

(a$_1$) $A(x) \ge 0, \ x \in \mathbb{R}^n$;

(a$_2$) $A(x) = O(\ln(1 + |x|))$ *for large* $|x|$;

(a$_3$) $\lim_{|x|\to\infty} \dfrac{A(x)}{b(x)|\nabla A(x)|^2} = +\infty$.

Proof. As in [1], we obtain a radial solution for (2.3). In this case (2.3) becomes

$$b(r)\left(A_{rr}+\frac{n-1}{r}A_r\right)+b_rA_r=1,\quad r=|x|.$$

Multiplying the above equation by r^{n-1}, we have

$$\left(r^{n-1}b(r)A_r\right)_r=r^{n-1}.$$

After integration, we obtain

$$A_r=\frac{r^{1-n}}{b(r)}\left(C_0+\int\limits_0^r\rho^{n-1}d\rho\right).$$

Integrate again, and then we find a solution in the form

$$A(r)=C_1+\int\limits_0^r\frac{s^{1-n}}{b(s)}\left(C_0+\int\limits_0^s\rho^{n-1}d\rho\right)ds.$$

By letting $C_0=C_1=0$, we obtain the solution which satisfies $A(0)=A_r(0)=0$. By condition (1.3), we get

$$A(r)=\frac{1}{nb_0}\left(\ln(1+r)+\frac{1}{1+r}-1\right),\qquad A'(r)=\frac{r}{nb_0(1+r)^2}. \tag{2.4}$$

It is easy to see that $(\mathrm{a}_1)-(\mathrm{a}_3)$ are satisfied by such choice of $A(x)=A(|x|)$.

Lemma 2.3. *Under assumptions of Lemmas* 2.1 *and* 2.2. *Define*

$$G(t)=\sup\{A(x)|x\in\operatorname{supp}u(\cdot,t)\}.$$

Then $G(t)\leq G_0t$, *where* $G_0>0$ *is a constant.*

Proof. By (2.1), the support of u is contained in the set

$$|x|\leq(1+R)e^{t\sqrt{b_0}}-1.$$

Therefore $A(x)$ defined in (2.4) satisfies $A(x)\leq G_0t$.

3. Main results. In this section, we use the multiplier method to obtain weighted L^2 and energy estimates for the solution to (1.1) under conditions (1.3).

As in [1], let $u=\varphi\hat{u}$, then we obtain the equation for $\hat{u}$:

$$\hat{u}_{tt}-\hat{b}_1\Delta\hat{u}-\hat{b}_2\cdot\nabla\hat{u}+\hat{a}_1\hat{u}_t+\hat{a}_2\hat{u}=0, \tag{3.1}$$

where $\hat{b}_1=b,\ \hat{b}_2=\nabla b+2b\varphi^{-1}\nabla\varphi,\ \hat{a}_1=1+2\varphi^{-1}\varphi_t$ and $\hat{a}_2=\varphi^{-1}(\varphi_{tt}-\operatorname{div}(b\nabla\varphi)+\varphi_t)$.

Multiply equation (3.1) by $\varphi\hat{u}+\theta\hat{u}_t$ and integrate on $\mathbb{R}^n$ using the divergence theorem. The boundary terms vanish since $u(x,t)$ has compact support with respect to x. Then one has the following lemma [1].

Lemma 3.1. *Let u be a solution of* (1.1). *Assume that φ and $\theta > 0$ are C^2-functions. Then $\hat{u}$ satisfies*

$$\frac{d}{dt}E(\hat{u}_t, \nabla\hat{u}, \hat{u}) + F(\hat{u}_t, \nabla\hat{u}) + G(\hat{u}) = 0,$$

where

$$E(\hat{u}_t, \nabla\hat{u}, \hat{u}) = \frac{1}{2}\int \left(\theta\left(\hat{u}_t^2 + b|\nabla\hat{u}|^2\right) + 2\varphi\hat{u}_t\hat{u} + \left(\hat{a}_2\theta + \varphi_t + \varphi\right)\hat{u}^2\right)dx,$$

$$F(\hat{u}_t, \nabla\hat{u}) = \frac{1}{2}\int\left(-\theta_t + 2\left(1 + 2\varphi^{-1}\varphi_t\right)\theta - 2\varphi\right)\hat{u}_t^2 dx +$$

$$+\int b\left(\nabla\theta - 2\theta\varphi^{-1}\nabla\varphi\right)\cdot\hat{u}_t\nabla\hat{u}dx +$$

$$+\frac{1}{2}\int b(-\theta_t + 2\varphi)|\nabla\hat{u}|^2 dx,$$

$$G(\hat{u}) = \frac{1}{2}\int\left(\hat{a}_2\varphi - (\hat{a}_2\theta)_t\right)\hat{u}^2 dx.$$

In the following, we choose appropriate functions φ and θ to estimate $E,\ F$ and G. Given any large $m > 0$, define

$$\sigma(x) = 2mA(x) + \sigma_0, \tag{3.2}$$

where $A(x)$ is defined in (2.4), and $\sigma_0 = \sigma_0(m) > 0$ is chosen such that $\sigma(x) - b(x)|\nabla\sigma(x)|^2 \geq$ $\geq 0,\ x \in \mathbb{R}^n$. Then we have $\mathrm{div}(b(x)\nabla\sigma(x)) \geq 2m$. Set

$$\varphi(x,t) = t^{-m}e^{-\frac{\sigma(x)}{t}}, \qquad \theta(x,t) = \frac{1}{2}\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1}\varphi(x,t). \tag{3.3}$$

By direct calculation, we have

$$\varphi_t = \left(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\right)\varphi,\ \varphi_{tt} = \left(\frac{m}{t^2} - \frac{2\sigma(x)}{t^3}\right)\varphi + \left(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\right)^2\varphi,$$

$$\nabla\varphi = -\frac{\nabla\sigma(x)}{t}\varphi,\ \mathrm{div}(b(x)\nabla\varphi) = -\frac{1}{t}\mathrm{div}(b(x)\nabla\sigma)\varphi + \frac{b(x)|\nabla\sigma|^2}{t^2}\varphi,$$

$$\ln\theta = \ln\frac{1}{2} - \ln\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right) + \ln\varphi,$$

$$\frac{\theta_t}{\theta} = \left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1}\left(\frac{2}{t^2} + \frac{2\sigma(x)}{t^3}\right) + \left(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\right),$$

$$\frac{\nabla\theta}{\theta} = -\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1}\frac{\nabla\sigma(x)}{t^2} - \frac{\nabla\sigma(x)}{t}.$$

Lemma 3.2. *Let φ and θ be defined in* (3.3), *then there exists $T = T(m) > 0$ such that $F \geq 0$ and $G \geq 0$ for $t \geq T$.*

Proof. First, we use (3.3) and Lemma 2.3 to calculate the coefficient $\hat{a}_2$,

$$\hat{a}_2 = \frac{\operatorname{div}(b(x)\nabla\sigma) - m}{t} + \frac{\sigma(x) - b(x)|\nabla\sigma|^2}{t^2} + \frac{m}{t^2} - \frac{2\sigma(x)}{t^3} + \left(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\right)^2 \geq$$

$$\geq \frac{m}{t} - \frac{2\sigma(x)}{t^3} \geq \frac{m}{t} - \frac{C(m)t}{t^3} \geq 0$$

for sufficiently large t, where $C(m) > 0$ is a constant depending on m. In a similar way, we can show that $(\hat{a}_2)_t < 0$ for sufficiently large t.

Next, we calculate $-\theta_t + \varphi$. By (3.3), we obtain

$$-\theta_t + \varphi \geq \theta\left(\frac{m}{t} - \frac{\sigma(x)}{t^2} - \frac{2}{t} + 2\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)\right) =$$

$$= \theta\left(\frac{m+2}{t} + \frac{\sigma(x)}{t^2}\right).$$

By the above arguments, and considering the definition of G, we have $G \geq 0$ for sufficiently large t. In order to show $F \geq 0$, we argue as follows. By (3.3) and Lemma 2.3, we get

$$-\theta_t + 2\left(1 + 2\varphi^{-1}\varphi_t\right)\theta - 2\varphi \geq \theta\left(\frac{-3m-10}{t} - \frac{\sigma(x)}{t^2} + 2\right) \geq$$

$$\geq \theta\left(\frac{-3m-10}{t} - \frac{C(m)t}{t^2} + 2\right) \geq \theta$$

for sufficiently large t. On the other hand,

$$b\left|\nabla\theta - 2\theta\varphi^{-1}\nabla\varphi\right|^2 = \theta^2\frac{b|\nabla\sigma|^2}{t^2}\left|\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1}\frac{1}{t} - 1\right| \leq$$

$$\leq \theta^2\frac{b|\nabla\sigma|^2}{t^2}.$$

Since $\sigma(x) \geq b(x)|\nabla\sigma|^2$, we obtain

$$\left(-\theta_t + 2\left(1 + 2\varphi^{-1}\varphi_t\right)\theta - 2\varphi\right)\left(-\theta_t + 2\varphi\right) \geq b\left|\nabla\theta - 2\theta\varphi^{-1}\nabla\varphi\right|^2,$$

which implies that $F \geq 0$ for sufficiently large t.

By Lemmas 3.1 and 3.2, we have

$$E(\hat{u}_t, \nabla\hat{u}, \hat{u}) \leq E(\hat{u}_t, \nabla\hat{u}, \hat{u})|_{t=T} \overset{\text{df}}{=} E_T, \qquad t \geq T. \tag{3.4}$$

Considering the definition of E, we get

$$\frac{1}{2}\frac{d}{dt}\int \varphi\hat{u}^2 dx + \frac{1}{2}\int \varphi\hat{u}^2 dx \leq E_T.$$

The above inequality implies

$$\int \varphi \hat{u}^2 dx \le H_T + 2E_T, \qquad t \ge T, \tag{3.5}$$

where $H_T = \int \varphi \hat{u}^2 dx|_{t=T}$. Using $|2\varphi \hat{u}_t \hat{u}| \le \frac{1}{2}\theta \hat{u}_t^2 + 2\theta^{-1}\varphi^2 \hat{u}^2$, we obtain from (3.4)

$$\int \theta \left(\hat{u}_t^2 + b|\nabla \hat{u}|^2\right) dx \le 4E_T + 4\int \left(-\hat{a}_2\theta - \varphi_t - \varphi + 2\theta^{-1}\varphi^2\right) \hat{u}^2 dx.$$

By the choices of $\theta,\ \varphi,\ \sigma(x)$ and Lemma 2.3, we have

$$-\hat{a}_2\theta - \varphi_t - \varphi + 2\theta^{-1}\varphi^2 \le -\varphi_t - \varphi + 2\theta^{-1}\varphi^2 =$$

$$= \varphi\left(\frac{8+m}{t} + \frac{3\sigma(x)}{t^2} - 1\right) \le$$

$$\le \varphi\left(\frac{8+m}{t} + \frac{C(m)t}{t^2} - 1\right) \le 0$$

for sufficiently large t. Therefore, we obtain the following lemma.

Lemma 3.3. *Let φ and θ be defined in* (3.3), *then we have*

$$\int \theta \left(\hat{u}_t^2 + b|\nabla \hat{u}|^2\right) dx \le 4E_{T_1}, \qquad t \ge T_1,$$

where $T_1 = T_1(m) > T(m)$ is a constant.

Note that

$$\hat{u}_t^2 = \left(-\varphi^{-2}\varphi_t u + \varphi^{-1}u_t\right)^2 \ge \frac{1}{2}\varphi^{-2}u_t^2 - \varphi^{-4}\varphi_t^2 u^2,$$

$$|\nabla \hat{u}|^2 = \left|-\varphi^{-2}\nabla\varphi u + \varphi^{-1}\nabla u\right|^2 \ge \frac{1}{2}\varphi^{-2}|\nabla u|^2 - \varphi^{-4}|\nabla\varphi|^2 u^2.$$

Combined with the estimate in Lemma 3.3, these inequalities imply

$$\frac{1}{2}\int \theta\varphi^{-2}\left(u_t^2 + b|\nabla u|^2\right) dx \le 4E_{T_1} + \int \theta\varphi^{-4}\left(\varphi_t^2 + b|\nabla\varphi|^2\right) u^2 dx.$$

By the choices of $\varphi,\ \theta$ and Lemma 2.3, we get

$$\theta\varphi^{-4}\left(\varphi_t^2 + b|\nabla\varphi|^2\right) =$$

$$= \varphi^{-1}\frac{1}{2}\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1}\left(\left(\frac{-m}{t} + \frac{\sigma(x)}{t^2}\right)^2 + \frac{b(x)|\nabla\sigma(x)|^2}{t^2}\right) \le$$

$$\le \varphi^{-1}$$

for sufficiently large t. And in terms of u, (3.5) can be stated as

$$\int \varphi^{-1}u^2 dx \le H_{T_1} + 2E_{T_1}, \qquad t \ge T_1.$$

Therefore, we obtain the estimates of u as follows.

Lemma 3.4. *Let φ and θ be defined in* (3.3), *then for $t \geq T_0 \geq T_1$ we have*

$$\int \varphi^{-1} u^2 dx \leq C(H_{T_0} + E_{T_0}),$$

$$\int \theta \varphi^{-2} \left(u_t^2 + b|\nabla u|^2\right) dx \leq C(H_{T_0} + E_{T_0}),$$

where C and $T_0 = T_0(m)$ are positive constants, $H_{T_0} = \int \varphi \hat{u}^2 dx|_{t=T_0}$ and $E_{T_0} = E(\hat{u}_t, \nabla \hat{u}, \hat{u})|_{t=T_0}$.

From Lemma 3.4, we have for any $t \geq T_0(m)$,

$$\int t^m e^{\frac{\sigma(x)}{t}} u^2 dx \leq C(H_{T_0} + E_{T_0}), \tag{3.6}$$

$$\int \left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1} t^m e^{\frac{\sigma(x)}{t}} \left(u_t^2 + b|\nabla u|^2\right) dx \leq C(H_{T_0} + E_{T_0}). \tag{3.7}$$

Note that

$$\left(\frac{2}{t} + \frac{\sigma(x)}{t^2}\right)^{-1} = \frac{t^2}{2t + \sigma(x)} \geq C_1(m)t, \qquad t \geq T_0,$$

and

$$H_{T_0} + E_{T_0} \leq C_2(m) \left(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2\right),$$

where $C_1(m)$ and $C_2(m) > 0$ depends also on R, $b(x)$ and n. Note also that the estimates (3.6) and (3.7) are trivial for small $t \geq 1$, thus we have our main results.

Theorem 3.1. *Assume that $a(x)$ and $b(x)$ satisfy condition* (1.3). *Then for any large $m > 0$ the solution of* (1.1) *satisfies*

$$\int e^{\frac{\sigma(x)}{t}} u^2 dx \leq C_0(m) \left(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2\right) t^{-m},$$

$$\int e^{\frac{\sigma(x)}{t}} \left(u_t^2 + b|\nabla u|^2\right) dx \leq C_0(m) \left(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2\right) t^{-m-1}$$

for all $t \geq 1$. Here $\sigma(x) > 0$ is defined in (3.2). *The constant $C_0(m)$ depends also on R, $b(x)$ and n.*

References

1. *Radu P., Todorova G., Yordanov B.* Decay estimates for wave equations with variable coefficients // Trans. Amer. Math. Soc. – 2010. – **362**. – P. 2279 – 2299.
2. *Barbu V.* Nonlinear semigroups and differential equations in Banach spaces. – Editura Acad., Bucuresti, Romania and Noordhoff Intern. Publ., Leyden, the Netherlands, 1976.
3. *Matsumura A.* On the asymptotic behavior of solutions of semi-linear wave equations // Publ. Res. Inst. Math. Sci. – 1976. – **12**. – P. 169 – 189.
4. *Nishihara K.* Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping // J. Different. Equat. – 1997. – **137**. – P. 384 – 395.
5. *Han Y., Milani A.* On the diffusion phenomenon of quasilinear hyperbolic waves // Bull. Sci. Math. – 2000. – **124**, № 5. – P. 415 – 433.

6. *Volkmer H.* Asymptotic expansion of L^2-norms of solutions to the heat and dissipative wave equations // Asymptot. Anal. – 2010. – **67**. – P. 85 – 100.
7. *Rauch J., Taylor M.* Decaying states of perturbed wave equations // J. Math. Anal. and Appl. – 1976. – **54**. – P. 279 – 285.
8. *Mochizuki K.* Scattering theory for wave equations with dissipative terms // Publ. Res. Inst. Math. Sci. – 1976. – **12**. – P. 383 – 390.
9. *Matsumura A.* Energy decay of solutions of dissipative wave equations // Proc. Jap. Acad. Ser. A. Math. Sci. – 1977. – **53**. – P. 232 – 236.
10. *Mochizuki K., Nakazawa H.* Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation // Publ. Res. Inst. Math. Sci. – 1996. – **32**. – P. 401 – 414.
11. *Uesaka H.* The total energy decay of solutions for the wave equation with a dissipative term // J. Math. Kyoto Univ. – 1979. – **20**. – P. 57 – 65.
12. *Ikehata R., Todorova G., Yordanov B.* Optimal decay rate of the energy for wave equations with critical potentia // J. Math. Soc. Jap. – 2013. – **65**. – P. 183 – 236.
13. *Todorova G., Yordanov B.* Weighted L^2-estimates for dissipative wave equations with variable coefficients // J. Different. Equat. – 2009. – **246**. – P. 4497 – 4518.
14. *Wirth J.* Solution representations for a wave equation with weak dissipation // Math. Meth. Appl. Sci. – 2004. – **27**. – P. 101 – 124.
15. *Kenigson J. S., Kenigson J. J.* Energy decay estimates for the dissipative wave equation with space-time dependent potential // Math. Meth. Appl. Sci. – 2011. – **34**, № 1. – P. 48 – 62.

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