UDC 517.9

Y. Guo (School Math. Sci., Qufu Normal Univ. and School Control Sci. and Eng., Shandong Univ., China)

GLOBALLY ROBUST STABILITY ANALYSIS FOR STOCHASTIC COHEN – GROSSBERG NEURAL NETWORKS WITH IMPULSE CONTROL AND TIME-VARYING DELAYS *

ГЛОБАЛЬНО РОБАСТНИЙ АНАЛІЗ СТАБІЛЬНОСТІ СТОХАСТИЧНИХ НЕЙРОННИХ СІТОК КОЕНА – ГРОССБЕРГА З ІМПУЛЬСНИМ УПРАВЛІННЯМ ТА ЗАТРИМКАМИ, ЩО ЗАЛЕЖАТЬ ВІД ЧАСУ

By constructing suitable Lyapunov functionals, in combination with the matrix-inequality technique, a new simple sufficient linear matrix inequality condition is established for the globally robustly asymptotic stability of the stochastic Cohen – Grossberg neural networks with impulsive control and time-varying delays. This condition contains and improves some previous results from the earlier references.

За допомогою побудови відповідних функціоналів Ляпунова, в комбінації з технікою матричної нерівності, встановлено нову просту достатню лінійну умову матричної нерівності для глобально робастно асимптотичної стабільності стохастичних нейронних сіток Коена – Гроссберга з імпульсним управлінням та затримками, що залежать від часу. Ця умова містить та покращує деякі відомі результати, що отримані раніше.

1. Introduction. Notations. Let R denotes the set of real numbers, R_+ the set of nonnegative real numbers, Z_+ the set of positive integers and R^n the n-dimensional Euclidean space, $|\cdot|$ the Euclidean norm. For any $\ell \subset R$, let $PC(\ell, R^n) = \{\varphi\colon\ \ell \to R^n$ is continuous everywhere except at a finite number of points t_k, at which $\varphi(t_k^+), \varphi(t_k^-)$ exist and $\varphi(t_k^+) = \varphi(t_k)\}$.

In this paper, we are concerned with the model of continuous-time neural networks described by the following systems of the form:

$$x_i'(t) =$$

$$= c_i(x_i(t))\left[-d_i(x_i(t)) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j\big(x_j(t-\tau_j(t))\big) + J_i\right], \quad t \geq 0, \quad t \neq t_k,$$

$$\triangle x_i|_{t=t_k} = x_i(t_k) - x_i(t_k^-) = I_{ik}(x_i(t_k^-)), \qquad k \in Z_+, \tag{1}$$

$$x_i(s) = \phi_i(s), \qquad s \in [t_0 - \tau, t_0], \quad i = 1, 2, \ldots, n,$$

or equivalently

$$x'(t) = C(x(t))\Big[-D(x(t)) + Af(x(t)) + Bf\big(x(t-\tau(t))\big) + J\Big], \qquad t \geq 0, \quad t \neq t_k,$$

$$\triangle x|_{t=t_k} = I_k(x(t_k^-)), \quad k \in Z_+, \tag{1*}$$

$$x(s) = \phi(s), \quad s \in [t_0 - \tau, t_0],$$

where n denotes the number of the neurons in the network, $x_i(t)$ is the state of the ith neuron at time $t,\ x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n,\ f(x(t)) = \big(f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))\big)^T \in R^n$

* This work was supported financially by the Open Research Project of the State Key Laboratory of Industrial Control Technology, Zhejiang University, China (No. ICT170289) and the Postdoctoral Science Foundation of China (2014M551738).

denote the activation functions of the jth neuron at time t, $D(x(t)) = (d_1(x_1(t)), d_2(x_2(t)), \ldots$ $\ldots, d_n(x_n(t)))^T$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ are the feedback matrix and the delayed feedback matrix, respectively, $C(x(t)) = \operatorname{diag}\big(c_1(x_1(t)), c_2(x_2(t)), \ldots, c_n(x_n(t))\big) > 0$, $J = (J_1, J_2, \ldots$ $\ldots, J_n)^T \in R^n$ be a constant external input vector, the time delay $\tau(t)$ is any nonnegative continuous function with $0 \leq \tau_j(t) \leq \tau$, and $0 < \tau_j'(t) \leq \delta < 1$, where τ, δ is a constant. $0 \leq t_0 < t_1 <$ $< t_2 < \ldots < t_k < t_{k+1} < \ldots$, $\lim_{k\to\infty} t_k = \infty$, with $\sup_{k\in Z_+}\{t_{k+1} - t_k\} < \infty$ and x' denotes the right-hand derivative of x. The functions $I_{ik}(t)$ represents the abrupt change of the state $x_i(t)$ at the impulsive moment t_k, $\phi_i(s)$, $i = 1, 2, \ldots, n$, are bounded and continuous for $s \in [-\tau, 0]$.

The past few decades have witnessed tremendous developments in the research field of neural networks [1 – 7]. Various neural networks, such as Hopfield neural networks [1], cellular neural networks [2, 3], bidirectional associative neural networks [4] and Cohen – Grossberg neural networks [5], have been widely investigated and successfully applied in many areas. Among them, system (1) is one of the most popular and generic neural network models. The Cohen – Grossberg neural network models were firstly proposed and studied by Cohen and Grossberg [5], which have been widely applied in various engineering and scientific fields such as neural biology, population biology, and computing technology. In such applications, it is important to know the convergence properties of the designed neural networks. Usually, this kind of neural networks can be described by the system (1).

On the other hand, a real system is usually affected by external perturbations which in many cases are of great uncertainty and hence may be treated as random, as pointed out by Haykin [8] that in real nervous systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Friedman [9] issues stochastic differential equations and their applications. It has also been known that a neural network could be stabilized or destabilized by certain stochastic inputs. Hence, the stability analysis problem for stochastic neural network becomes increasingly significant, and some results on stability have been derived (see, for example, [10 – 17]).

However, taking more factors into account leads to the development of the theory of impulsive differential equations [18], where the wide range of topics of the impulse systems theory, in particular stability theory, are considered. The authors [19, 20] investigate the impulsive stabilization of delay differential systems. Samoilenko and Stanzhytskyi [21] consider the stability of stochastic systems with impulse acting. According to Arbib [22] and Haykin [23], when a stimulus from the body or the external environment is received by receptors the electrical impulses will be conveyed to the neural net and impulsive effects arise naturally in the net. Therefore, neural network model with stochastic and impulsive effects should be more accurate to describe the evolutionary process of the systems. Since randomness and impulses can affect the dynamical behaviors of the system [24 – 26], it is necessary to investigate both randomness and impulsive effects on the stability of neural networks.

In this paper, we will consider the global asymptotic stability of the Cohen – Grossberg neural networks with distributed delays described by (1). The organization of this paper is as follows. In Section 2, problem formulation and preliminaries are given. In Section 3, some new results are given to the Cohen – Grossberg neural networks with distributed delays described by (1) based on Lyapunov method. Section 4 gives an example to illustrate the effectiveness of our results.

2. Preliminaries. In our analysis, we assume that the following conditions are satisfied:

(H_1) there exist constant scalers $l_i > 0$ such that

$$0 \le \frac{f_i(\eta_1) - f_i(\eta_2)}{\eta_1 - \eta_2} \le l_i \qquad \forall \eta_1, \eta_2 \in R, \quad \eta_1 \ne \eta_2;$$

(H$_2$) $0 < \underline{\alpha_i} \le c_i(x_i(t)) \le \overline{\alpha_i}$, $\underline{c_i}$ and $\overline{c_i}$ are constant scalers, $i = 1, 2, \ldots, n$;

(H$_3$) for all $\eta_1, \eta_2 \in R$, $\eta_1 \ne \eta_2$, there exist constant scalers $\mu_i > 0$ such that

$$\frac{d_i(\eta_1) - d_i(\eta_2)}{\eta_1 - \eta_2} \ge \mu_i > 0.$$

In the following, we will use the notation $A > 0$ (or $A < 0$) to denote the matrix A is a symmetric and positive definite (or negative definite) matrix. The notation A^T and A^{-1} means the transpose of and the inverse of a square matrix A. If A, B are symmetric matrices, $A > B$ $(A \le B)$ means that $A - B$ is positive definite (positive semidefinite). Next we give the results about existence and uniqueness of equilibriums of system (1).

Assume $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an equilibrium of equation (1), one can derive from (1) that the transformation $y_i(t) = x_i(t) - x_i^*$ transforms system (1) into the following system:

$$y_i'(t) = \alpha_i(y_i(t)) \left[-\beta_i(y_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} g_j\big(y_j(t - \tau_j(t))\big) \right], \qquad t \ge 0, \quad t \ne t_k,$$

$$\triangle y_i|_{t=t_k} = J_{ik}\big(y_i(t_k^-)\big), \quad k \in Z_+, \tag{2}$$

or

$$y'(t) = \alpha(y(t)) \big[-\beta(y(t)) + Ag(y(t)) + Bg\big(y(t - \tau(t))\big) \big], \qquad t \ge 0, \quad t \ne t_k,$$
$$\triangle y|_{t=t_k} = J_k(y(t_k^-)), \quad k \in Z_+,$$

where

$$\alpha_i(y_i(t)) = c_i\big(y_i(t) + x_i^*\big), \ \alpha\big(y(t)\big) = \mathrm{diag}\,\big(\alpha_1(y_1(t)), \alpha_2(y_2(t)), \ldots, \alpha_n(y_n(t))\big),$$
$$\beta_i(y_i(t)) = d_i\big(y_i(t) + x_i^*\big) - d_i(x_i^*), \ \beta\big(y(t)\big) = \mathrm{diag}\,\big(\beta_1(y_1(t)), \beta_2(y_2(t)), \ldots, \beta_n(y_n(t))\big),$$
$$g_j(y_j(t)) = f_j\big(y_j(t) + x_j^*\big) - f_j(x_j^*), \ g\big(y(t)\big) = \mathrm{diag}\,\big(g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t))\big),$$
$$J_{jk}(y_j(t_k^-)) = I_{jk}\big(y_j(t_k^-) + x_j^*\big), \ J_k\big(y(t_k^-)\big) = \mathrm{diag}\,\big(Q_{1k}(y_1(t_k^-)), \ldots, \alpha_{nk}(y_n(t_k^-))\big).$$

Note that since each function $f_j(\cdot)$ satisfies the hypothesis (H$_1$), hence, each $g_j(\cdot)$ satisfies

$$0 \le \frac{g_j(y_j)}{y_j} \le l_j \quad \forall y_j \in R, \quad y_j \ne 0, \qquad \text{and} \qquad g_j(0) = 0, \quad j = 1, 2, \ldots, n,$$

and since each function $d_j(\cdot)$ satisfies the hypothesis (H$_3$), hence, each $\beta_j(\cdot)$ satisfies

$$\frac{\beta_j(y_j)}{y_j} \ge \mu_j > 0 \quad \forall y_j \in R, \quad y_j \ne 0, \qquad \text{and} \qquad \beta_j(0) = 0, \quad j = 1, 2, \ldots, n.$$

To prove the stability of x^* of equation (1), it is sufficient to prove the stability of the trivial solution of equation (2).

As discussed in Section 1, in the real world, the neural network is often disturbed by environmental noises that affect the stability of the equilibrium. In this paper, impulsive Cohen – Grossberg neural network with stochastic perturbations is introduced as follows:

$$\begin{aligned} dy(t) &= \alpha(y(t))\big[-\beta(y(t)) + Ag(y(t)) + Bg\big(y(t-\tau(t))\big)\big]\,dt+ \\ &+\sigma\big(t, y(t), y(t-\tau(t))\big)\,dw(t), \qquad t \geq 0, \quad t \neq t_k, \\ \triangle y\,|_{t=t_k} &= J_k(y(t_k^-)), \qquad k \in Z_+, \\ y(t) &= \phi(t), \qquad -\tau \leq t \leq 0, \quad \phi \in L^2_{\mathcal{F}_0}\big([-\tau, 0], R^n\big), \end{aligned} \tag{3}$$

where $w(t) = \big(w_1(t), w_2(t), \ldots, w_n(t)\big)^T$ is an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\{w(s) : 0 \leq s \leq t\}$. $L^2_{\mathcal{F}_t}(\ell; R^n)$ is the family of all bounded $\mathcal{F}_t$-measurable, $PC(\ell; R^n)$-valued stochastic variables satisfying $\|\varphi\| := \sup_{s\in\ell} \mathbb{E}|\varphi(s)|^2 < \infty$. $\mathbb{E}$ denotes the mathematical expectation operator. $\sigma(t, x, y) : R^+ \times R^n \times R^n \to R^{n\times m}$ is locally Lipschitz continuous and satisfies the linear growth condition as well, $\sigma(t, 0, 0) = 0$. Furthermore, σ satisfies:

(H_4) $\operatorname{trace}\big[\sigma^T\big(t, y(t), y(t-\tau(t))\big)\sigma\big(t, y(t), y(t-\tau(t))\big)\big] \leq |\Theta_1 y(t)|^2 + |\Theta_2 y(t-\tau(t))|^2$,
where Θ_1 and Θ_2 are known constant matrices with appropriate dimensions. In addition, we always assume that $J_k(y(t)) = 0$ if and only if $y = 0, t \geq t_0, k \in Z_+$. Let $y(t; \phi)$ denote the solution of the neural network (3) from the initial data $y(s) = \phi(s)$ on $-\tau \leq s \leq 0$ in $L^2_{\mathcal{F}_0}\big([-\tau, 0]; R^n\big)$, then system (3) admits a trivial solution $y(t; 0) = 0$ corresponding to the initial data $\phi = 0$.

***Remark* 1.** If $n = 2$, $\sigma_1 = 0.5y_1(t) + 0.5y_1(t-\tau_1(t))$, $\sigma_2 = 0.4y_2(t) + 0.4y_2(t-\tau_2(t))$, where $\tau_1(t) = 0.3 + 0.5\sin t$, $\tau_2(t) = 0.3 + 0.5\cos t$. Then $\sigma = [\sigma_1, \sigma_2]^T$ satisfies the condition (H_4) for

$$\Theta_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.25 \end{bmatrix}, \qquad \Theta_2 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.16 \end{bmatrix}.$$

Definition 1. *The function* $V : [0, \infty) \times PC\big([0, \infty), R^n\big) \to R_+$ *belongs to class* $\mathcal{V}$ *if*

(1) V *is continuous on each of the sets* $[t_{k-1}, t_k) \times PC([0, \infty), R^n)$ *and*

$$\lim_{(t,\varphi_1)\to(t_k^-,\varphi_2)} V(t, \varphi_1) = V(t_k^-, \varphi_2)$$

exists;

(2) $V(t, y)$ *is locally Lipschitzian in* y *and* $V(t, 0) \equiv 0$.

Definition 2. *Suppose* $V \in \mathcal{V}$; *for any* $(t, y) \in [t_{k-1}, t_k) \times PC([0, \infty), R^n)$, *the upper right-hand Dini derivative of* $V(t, y(t))$ *along the solution of* (3) *is defined by*

$$D^+V(t, y(t)) = \limsup_{s\to 0^+} \frac{1}{s}\big\{V\big(t+s, y(t) + sh(t, y(t), y(t-\tau(t)))\big) - V(t, y(t))\big\},$$

where

$$h\big(t, y(t), y(t-\tau(t))\big) = \alpha(y(t))\big[-\beta(y(t)) + Ag(y(t)) + Bg\big(y(t-\tau(t))\big)\big].$$

Definition 3. *For the system* (3) *and every* $\xi \in L^2_{\mathcal{F}_0}\big([-\tau, 0]; R^n\big)$, *the trivial solution* (*equilibrium point*) *is robustly*, *globally*, *asymptotically stable in the mean square if*, *the following holds*:

$$\lim_{t\to\infty} \mathbb{E}\big|y(t; \xi)\big|^2 = 0.$$

Lemma 1. *For any vectors* $a, b \in R^n$, *the inequality*

$$2a^T b \le \varepsilon a^T a + \varepsilon^{-1} b^T b$$

holds for any scalar $\varepsilon > 0$.

Lemma 2 [28]. *Given constant matrices* $\Sigma_1,\ \Sigma_2,\ \Sigma_3$, *where* $\Sigma_1 = \Sigma_1^T$ *and* $0 < \Sigma_2 = \Sigma_2^T$, *then*

$$\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$$

if and only if

$$\begin{pmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{pmatrix} < 0 \qquad \text{or} \qquad \begin{pmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{pmatrix} < 0.$$

3. Existence and uniqueness theorem. By [31, 32], we have the following definition.

Definition 4. *A function* $y(\cdot) \in L^2_{\mathcal{F}_t}\big([-\tau, \infty), R^n\big)$ *is said to be a solution of* (3) *if* $y(s) = \phi(s)$, *for* $s \in [-\tau, 0]$, *and the following integral equation is satisfied*:

$$y(t) = \phi(0) + \int_0^t h(s, y(s))\, ds + \int_0^t \sigma\big(t, y(t), y(t - \tau(t))\big)\, dw(s) +$$
$$+ \sum_{0 < t_k < t} J_k(y(t_k)), \quad t \ge 0.$$

Theorem 1. *Suppose that both* h *and* σ *satisfy the local Lipschitz condition and the linear growth condition. That is, for each* $k = 1, 2, \ldots$, *there is a constant* $p_k > 0$ *such that for* $t \in [t_{k-1}, t_k)$,

(H_5) $\big|h(t, u, v) - h(t, \overline{u}, \overline{v})\big| \vee \big|\sigma(t, u, v) - \sigma(t, \overline{u}, \overline{v})\big| \le p_k\big(|u - \overline{u}| + |v - \overline{v}|\big)$

and there is a constant $q_k > 0$ *such that*

(H_6) $\big|h(t, u, v)\big| \vee \big|\sigma(t, u, v)\big| \le q_k\big(1 + |u| + |v|\big)$,

then system (3) *has a unique global solution.*

Proof. Without loss of generality, we assume that $t_0 > 0$. For $t \in [0, t_0^-]$, it is well-known [33] that for any initial condition $\phi(0)$, system (3) has a unique global continuous solution $y(t) = y(t; \phi(0))$ that is defined on segment $t \in [0, t_0^-]$. At $t = t_0$, there exists an impulse which transfers solution $y(t) = y(t; \phi(0))$ into $J_0(y(t_0^-))$. By induction one get a unique global continuous solution $y(t) = y(t; t_0^-)$ of system (3) that is defined on segment $t \in [t_0, t_1^-]$ and so on. Thus we infer that system (3) has a unique global solution.

Theorem 1 is proved.

4. Impulsive stability analysis. In the section, we present and prove our main results.

Theorem 2. *Assume that* (H_1) – (H_6) *are satisfied and there exist real scalars* $\rho > 0$, $\varepsilon_1 > 0, \varepsilon_2 > 0$, *matric* X, *positive define matrices* $P = P^T > 0$, $Q = Q^T > 0$ *such that the three LMIs*

$$P < \rho I, \tag{4}$$

$$\begin{bmatrix} -P\underline{\alpha}\mu-\underline{\alpha}\mu P+\dfrac{1}{\tau}Q & 0 & \overline{\alpha}PA & \overline{\alpha}PB & \rho\Theta_1^T & 0 & 0 & \varepsilon_1 l \\ * & -\dfrac{1-\delta}{\tau}Q & 0 & 0 & 0 & \rho\Theta_2^T & \varepsilon_2 l & 0 \\ * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\rho I & 0 & 0 & 0 \\ * & * & * & * & * & -\rho I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_1 I \end{bmatrix} < 0, \quad (5)$$

$$\begin{bmatrix} X+X^T & X^T \\ * & -P \end{bmatrix} \leq 0 \tag{6}$$

hold, where $\underline{\alpha} = \mathrm{diag}(\underline{\alpha_i})_{n\times n}$, $\overline{\alpha} = \mathrm{diag}(\overline{\alpha_i})_{n\times n}$, $l = \mathrm{diag}(l_i)_{n\times n}$, $\mu = \mathrm{diag}(\mu_i)_{n\times n}$. *Then system* (3) *can be robustly, globally, asymptotically stable in the mean square via an impulsive controller*:

$$I_k u = P^{-1}Xu. \tag{7}$$

Proof. Let $V(t) = y^T(t)Py(t) + \dfrac{1}{\tau}\displaystyle\int_{t-\tau(t)}^{t} y^T(s)Qy(s)\,ds$. By it's differential formula (see [29]), the stochastic derivative of $V(t)$ along (3) can be obtained as follows:

(1) for $t \in [t_k, t_{k+1})$,

$$D^+V(t) = \mathcal{L}V(t)dt + 2y^T(t)P\sigma\big(t, y(t), y(t-\tau(t))\big)\,dw(t),$$

where

$$\mathcal{L}V(t) = -2y^T(t)P\alpha(y(t))\big[\beta(y(t)) - Ag(y(t)) - Bg\big(y(t-\tau(t))\big)\big]+$$
$$+\mathrm{trace}\big[\sigma^T(t, y(t), y\big(t-\tau(t)\big))\, P\sigma\big(t, y(t), y(t-\tau(t))\big)\big]+$$
$$+\frac{1}{\tau}y^T(t)Qy(t) - \frac{1-\tau'(t)}{\tau}y^T(t-\tau(t))Qy(t-\tau(t)).$$

Since

$$\mathcal{L}V(t) \leq -2y^T(t)P\underline{\alpha}\mu y(t)+$$
$$+2y^T(t)P\alpha(y(t))\,Ag(y(t)) + 2y^T(t)P\alpha(y(t))\,Bg\big(y(t-\tau(t))\big)\big]+$$
$$+\mathrm{trace}\big[\sigma^T\big(t, y(t), y(t-\tau(t))\big)\, P\sigma\big(t, y(t), y(t-\tau(t))\big)\big]+$$
$$+\frac{1}{\tau}y^T(t)Qy(t) - \frac{1-\delta}{\tau}y^T(t-\tau(t))Qy(t-\tau(t)).$$

Recall that the inequality $2ab \leq \dfrac{1}{\varepsilon}a^2 + \varepsilon b^2$ holds for any $a, b \in R$ and for any $\varepsilon > 0$. Then

$$2y^T(t)P\alpha(y(t))Ag(y(t)) \leq$$

$$\leq \frac{1}{\varepsilon_1}y^T(t)P\alpha(y(t))AA^T\alpha(y(t))Py(t) + \varepsilon_1 g^T(y(t))g(y(t)) \leq$$

$$\leq \frac{1}{\varepsilon_1}y^T(t)\big(PAA^TP\overline{\alpha}^2\big)y(t) + \varepsilon_1 y^T(t)l^2y(t),$$

$$2y^T(t)P\alpha(y(t))Bg\big(y(t-\tau(t))\big) \leq \frac{1}{\varepsilon_2}y^T(t)P\alpha(y(t))BB^T\alpha(y(t))y(t)+$$

$$+\varepsilon_2 g^T\big(y(t-\tau(t))\big)\,g\big(y(t-\tau(t))\big) \leq$$

$$\leq \frac{1}{\varepsilon_2}y^T(t)\big(PBB^TP\overline{\alpha}^2\big)y(t) + \varepsilon_2 y^T(t-\tau(t))l^2y(t-\tau(t)),$$

and

$$\mathrm{trace}\big[\sigma^T\big(t, y(t), y(t-\tau(t))\big)\,P\sigma\big(t, y(t), y(t-\tau(t))\big)\big] \leq$$

$$\leq \lambda_{\max}(P)\mathrm{trace}\big[\sigma^T\big(t, y(t), y(t-\tau(t))\big)\sigma\big(t, y(t), y(t-\tau(t))\big)\big] \leq$$

$$\leq \rho\big[y^T(t)(\Theta_1^T\Theta_1)y(t) + y^T(t-\tau(t))\Theta_2^T\Theta_2 y(t-\tau(t))\big],$$

thus

$$\mathcal{L}V(t) \leq \begin{bmatrix} y(t) \\ y(t-\tau(t)) \end{bmatrix}^T \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \begin{bmatrix} y(t) \\ y(t-\tau(t)) \end{bmatrix},$$

$$\Omega_1 = -P\underline{\alpha}\mu - \underline{\alpha}\mu P + \frac{1}{\tau}Q + \frac{1}{\varepsilon_1}PAA^TP\overline{\alpha}^2 + \varepsilon_1 l^2 + \frac{1}{\varepsilon_2}PBB^TP\overline{\alpha}^2 + \rho\Theta_1^T\Theta_1,$$

$$\Omega_2 = -\frac{1-\delta}{\tau}Q + \varepsilon_2 l^2 + \rho\Theta_2^T\Theta_2.$$

By Lemma 2, it is obvious from (5) that $\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} < 0$. There must exist a scalar $\eta > 0$ such that

$$\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} + \begin{bmatrix} \eta I & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

Define

$$\xi(t) := \begin{bmatrix} y(t) \\ y(t-\tau(t)) \end{bmatrix}, \qquad \Xi := \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix}.$$

So

$$D^+V(t) \le \xi^T(t)\Xi\xi(t)\,dt + 2y^T(t)P\sigma\big(t, y(t), y(t-\tau(t))\big)\,dw(t).$$

Then, we have

$$\frac{d\mathbb{E}V(t)}{dt} \le \mathbb{E}\xi^T(t)\Xi\xi(t) \le -\eta\mathbb{E}|y(t)|^2. \tag{8}$$

(2) For $t = t_k$, using the condition (6) we get

$$V(t_k) - V(t_k^-) = y^T(t_k)Py(t_k) - y^T(t_k^-)Py(t_k^-) +$$

$$+ \int\limits_{t_k-\tau(t_k)}^{t_k} y^T(s)Qy(s)\,ds - \int\limits_{t_k^- - \tau(t_k^-)}^{t_k^-} y^T(s)Qy(s)\,ds =$$

$$= y^T(t_k^-)\big[(I + P^{-1}X)^T P(I + P^{-1}X) - P\big]y(t_k^-) =$$

$$= y^T(t_k^-)\big(X^T + X + X^T P^{-1}X\big)\,y(t_k^-) \le 0,$$

which gives

$$V(t_k) \le V(t_k^-).$$

This and (1) imply that (8) holds for $t = t_k$. Then, by (1) and (2), the system (3) can be robustly, globally, asymptotically stable in the mean square.

***Remark* 2.** In this paper, we do not need any restriction on the time interval of impulsive, however, for the impulsive delay differential systems containing model (3) discussed in [24–26, 30], the time interval of impulsive is necessary. Such as $\sup_{k\in Z_+}\{t_k - t_{k-1}\} < \dfrac{\ln q}{c}$ (see, for example, [30]), where $q,\ c$ are constants. Therefore, the results of this paper are new and they complement previously known results.

***Remark* 3.** In Theorem 1, we do not need the assumptions of boundedness, monotonicity, and differentiability for the activation functions, moreover, the model discussed is with time-varying delays. Clearly, the proposed results are different from those in [1–5, 13–15, 27, 28] and the references cited therein. Therefore, the results of this paper are new and they complement previously known results.

***Remark* 4.** In this paper, we need the assumptions of differentiability for the time-varying delay functions $\tau(t)$, but the assumptions of differentiability for $\tau(t)$ is not necessary. This can be seen from that when we choose $V(t) = y^T(t)Py(t) + \dfrac{1}{\tau}\displaystyle\int_{t-\tau}^{t} y^T(s)Qy(s)ds.$

When the time-varying delay functions $\tau(t)$ is not differentiable, we can choose $V(t) = y^T(t)Py(t) + \dfrac{1}{\tau}\displaystyle\int_{t-\tau}^{t} y^T(s)Qy(s)ds$. Then we have the following theorem.

Theorem 3. *Assume that* (H_1)–(H_6) *are satisfied and there exist real scalars* $\rho > 0,\ \varepsilon_1 > 0,\ \varepsilon_2 > 0$, *matric* X, *positive define matrices* $P = P^T > 0,\ Q = Q^T > 0$ *such that the three LMIs* (4), (6) *and*

$$\begin{bmatrix} -P\underline{\alpha}\mu-\underline{\alpha}\mu P+\dfrac{1}{\tau}Q & 0 & \overline{\alpha}PA & \overline{\alpha}PB & \rho\Theta_1^T & 0 & 0 & \varepsilon_1 l \\ * & -\dfrac{1}{\tau}Q & 0 & 0 & 0 & \rho\Theta_2^T & \varepsilon_2 l & 0 \\ * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\rho I & 0 & 0 & 0 \\ * & * & * & * & * & -\rho I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_1 I \end{bmatrix} < 0$$

hold. Then system (3) *can be robustly, globally, asymptotically stable in the mean square via an impulsive controller* (7).

In the following, we first consider the Cohen – Grossberg neural network without stochastic perturbations (2). By the Theorem 1, we obtain the following results.

***Corollary* 1.** *Assume that* (H_1) – (H_3), (H_5) *and* (H_6) *are satisfied and there exist real scalars* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, *matric* X, *positive define matrices* $P = P^T > 0$, $Q = Q^T > 0$ *such that the two LMIs* (6) *and*

$$\begin{bmatrix} -P\underline{\alpha}\mu-\underline{\alpha}\mu P+\dfrac{1}{\tau}Q & 0 & \overline{\alpha}PA & \overline{\alpha}PB & 0 & \varepsilon_1 l \\ * & -\dfrac{1-\delta}{\tau}Q & 0 & 0 & \varepsilon_2 l & 0 \\ * & * & -\varepsilon_1 I & 0 & 0 & \\ * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & -\varepsilon_1 I \end{bmatrix} < 0$$

hold. Then system (2) *can be robustly, globally, asymptotically stable in the mean square via an impulsive controller* (7).

Second, if there appears only stochastic cellular neural networks, i.e., $\alpha(y(t)) = I$ in system (3), the model (3) can now be simplified to

$$\begin{gathered} dy(t) = \big[-\beta(y(t)) + Ag(y(t)) + Bg\big(y(t-\tau(t))\big)\big]\,dt+ \\ +\sigma\big(t, y(t), y(t-\tau(t))\big)\,dw(t), \qquad t \geq 0, \quad t \neq t_k, \\ \triangle y|_{t=t_k} = J_k(y(t_k^-)), \quad k \in Z_+. \end{gathered} \tag{9}$$

In this case, $\underline{\alpha} = \overline{\alpha} = I$ in Theorem 1, then we have the following corollary.

***Corollary* 2.** *Assume that* (H_1) – (H_6) *are satisfied and there exist real scalars* $\rho > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, *matric* X, *positive define matrices* $P = P^T > 0$, $Q = Q^T > 0$ *such that the three LMIs* (4), (6) *and*

$$\begin{bmatrix} -P\mu-\mu P+\frac{1}{\tau}Q & 0 & PA & PB & \rho\Theta_1^T & 0 & 0 & \varepsilon_1 l \\ * & -\frac{1-\delta}{\tau}Q & 0 & 0 & 0 & \rho\Theta_2^T & \varepsilon_2 l & 0 \\ * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\rho I & 0 & 0 & 0 \\ * & * & * & * & * & -\rho I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_1 I \end{bmatrix} < 0,$$

hold. Then system (9) *can be robustly*, *globally*, *asymptotically stable in the mean square via an impulsive controller* (7).

Third, we consider the following system:

$$dy(t) = \big[-\beta(y(t)) + Ag(y(t)) + Bg\big(y(t-\tau(t))\big)\big]\,dt, \qquad t \geq 0, \quad t \neq t_k,$$
$$\triangle y|_{t=t_k} = J_k(y(t_k^-)), \quad k \in Z_+. \tag{10}$$

***Corollary* 3.** *Assume that* (H_1)–(H_3) *are satisfied and there exist real scalars* $\rho > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, *matric* X, *positive define matrices* $P = P^T > 0$, $Q = Q^T > 0$ *such that the two LMIs* (6) *and*

$$\begin{bmatrix} -P\mu-\mu P+\frac{1}{\tau}Q & 0 & PA & PB & 0 & \varepsilon_1 l \\ * & -\frac{1-\delta}{\tau}Q & 0 & 0 & \varepsilon_2 l & 0 \\ * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & -\varepsilon_1 I \end{bmatrix} < 0,$$

hold. Then system (10) *can be robustly*, *globally*, *asymptotically stable in the mean square via an impulsive controller* (7).

5. Numerical example. In this section, an example is used to demonstrate that the method presented in this paper is effective.

***Example*.** Consider the following three state neural networks (3) with

$$A = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.2 & -0.2 \\ 0.1 & 0.2 & -0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.3 & 0.2 & -0.6 \\ 0.5 & 0.2 & -0.3 \end{bmatrix},$$

$$\underline{\alpha} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, \qquad \overline{\alpha} = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.9 \end{bmatrix},$$

$$l = 0.3I, \qquad \mu = 0.9I, \qquad \delta = 0.5, \qquad \tau = 2, \qquad \Theta_1 = 0.08I, \quad \Theta_2 = 0.09I.$$

By solving the LMIs (4), (5) and (6) for $\rho > 0, \varepsilon_i > 0,\ i = 1, 2,\ P > 0,\ Q > 0$ we obtain

$$P = \begin{bmatrix} 1.4201 & -0.1001 & -0.1629 \\ -0.1001 & 1.4362 & -0.2535 \\ -0.1629 & -0.2535 & 1.4319 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.2734 & -0.0198 & -0.0705 \\ -0.0198 & 0.4202 & -0.1834 \\ -0.0705 & -0.1834 & 0.4591 \end{bmatrix},$$

$$X = \begin{bmatrix} -0.5364 & 0.0125 & 0.0189 \\ 0.0125 & -0.5375 & 0.0289 \\ 0.0189 & 0.0289 & -0.5367 \end{bmatrix}, \qquad K = \begin{bmatrix} -0.3834 & -0.0242 & -0.0348 \\ -0.0242 & -0.3849 & -0.0507 \\ -0.0347 & -0.0507 & -0.3878 \end{bmatrix},$$

$$\rho = 2.2035, \qquad \varepsilon_1 = 1.5345, \qquad \varepsilon_2 = 1.7937,$$

which implies from Theorem 1 that the above delayed stochastic Cohen – Grossberg neural network is robustly, globally, asymptotically stable in the mean square via an impulsive controller $I_k(u) = Ku$.

6. Conclusion. In this paper, we have dealt with the problem of global asymptotic stability analysis for stochastic Cohen – Grossberg neural networks with impulsive. We have removed the monotonicity and smoothness assumptions on the activation function. A LMI approach has been developed to solve the problem addressed. The stability criteria have been derived in terms of the positive definite solution to three LMIs involving several scalar parameters, which can be easily solved by using the Matlab toolbox. A simple example has been used to demonstrate the effectiveness of the main results.

References

1. *Guo S., Huang L.* Stability analysis of a delayed Hopfield neural network // Phys. Rev. E. – 2003. – **67**. – P. 1 – 7.
2. *Civalleri P. P., Gilli M., Pandolfi L.* On stability of cellular neural networks with delay // IEEE Trans. Circuits Syst. I. – 1993. – **40**. – P. 157 – 165.
3. *Roska T., Chua L. O.* Cellular neural networks with delay-type template elements nonuniform grid // Int. J. Circuit Theory and Appl. – 1992. – **20**. – P. 469 – 481.
4. *Li Y.* Global exponential stability of BAM neural networks with delays and impulses // Chaos, Solitons and Fractals. – 2005. – **24**. – P. 279 – 285.
5. *Cohen M. A., Grossberg S.* Absolute stability of global parallel memory storage by competitive neural networks // IEEE Trans. Syst. Man Cybermet. – 1983. – **13**. – P. 815 – 826.
6. *Guo Y.* Global asymptotic stability analysis for integro-differential systems modeling neural networks with delays // Z. angew. Math. und Phys. – 2010. – **61**. – S. 971 – 978.
7. *Guo Y., Liu S. T.* Global exponential stability analysis for a class of neural networks with time delays // Int. J. Robust and Nonlinear Control. – 2012. – **22**. – P. 1484 – 1494.
8. *Haykin S.* Neural networks. – Prentice-Hall, NJ, 1994.
9. *Friedman A.* Stochastic differential equations and their applications. – New York: Academic, 1976.
10. *Guo Y.* Mean square global asymptotic stability of stochastic recurrent neural networks with distributed delays // Appl. Math. and Comput. – 2009. – **215**. – P. 791 – 795.
11. *Yang J., Zhong S., Luo W.* Mean square stability analysis of impulsive stochastic differential equations with delays // J. Comput. and Appl. Math. – 2008. – **216**. – P. 474 – 483.
12. *Guo Y.* Global stability analysis for a class of Cohen – Grossberg neural network models // Bull. Korean Math. Soc. – 2012. – **49**, № 6. – P. 1193 – 1198.
13. *Wang Z., Liu Y., Li M., Liu X.* Stability analysis for stochastic Cohen – Grossberg neural networks with mixed time delays // IEEE Trans. Neural Networks. – 2006. – **17**, № 3. – P. 814 – 820.

14. *Blythe S., Mao X., Liao X.* Stability of stochastic delay neural networks // J. Franklin Inst. – 2001. – **338**. – P. 481 – 495.
15. *Guo Y.* Mean square exponential stability of stochastic delay cellular neural networks // Electron. J. Qual. Theory Different. Equat. – 2013. – **34**. – P. 1 – 10.
16. *Yang Z., Xu D., Xiang L.* Exponential p-stability of impulsive stochastic differential equations with delays // Phys. Lett. A. – 2006. – **359**. – P. 129 – 137.
17. *Wan L., Sun J.* Mean square exponential stability of stochastic delayed Hopfield neural networks // Phys. Lett. A. – 2005. – **343**, № 4. – P. 306 – 318.
18. *Samoilenko A. M., Perestyuk M. O.* Impulsive differential equations. – Singapore etc.: World Sci., 1995.
19. *Wang Q., Liu X.* Impulsive stabilization of delay differential systems via the Lyapunov – Razumikhin method // Appl. Math. Lett. – 2007. – **20**. – P. 839 – 845.
20. *Luo Z., Shen J.* Impulsive stabilization of functional differential equations with infinite delays // Appl. Math. Lett. – 2003. – **61**. – P. 695 – 701.
21. *Samoilenko A., Stanzhytskyi O.* Qualitative and asymptotic analysis of differential equations with random perturbations. – Singapore: World Sci., 2011.
22. *Arbib M. A.* Branins, machines, and mathematics. – New York: Springer-Verlag, 1987.
23. *Haykin S.* Neural networks: a comprehensive foundation. – Englewood Cliffs, NJ: Prentice-Hall, 1998.
24. *Wang P., Li B., Li Y.* Square-mean almost periodic solutions for impulsive stochastic shunting inhibitory cellular neural networks with delays // Neurocomputing. – 2015. – **167**, № 1. – P. 76 – 82.
25. *Cheng P., Deng F.* Global exponential stability of impulsive stochastic functional differential systems // Statist. and Probab. Lett. – 2010. – **80**. – P. 1854 – 1862.
26. *Yao F., Cao J., Cheng P. et al.* Generalized average dwell time approach to stability and input-to-state stability of hybrid impulsive stochastic differential systems // Nonlinear Anal: Hybrid Syst. – 2016. – **22**. – P. 147 – 160.
27. *LaSalle J. P.* The stability of dynamical system. – Philadelphia: SIAM, 1976.
28. *Blythe S., EI Ghaoui L., Feron E., Balakrishnan V.* Linear matrix inequalities in system and control theory. – Philadelphia, PA: SIAM, 1994.
29. *Karatzas I., Shreve S. E.* Brownian motion and stochastic calculus. – New York: Springer, 1991.
30. *Xu L., Xu D.* Mean square exponential stability of impulsive control stochastic systems with time-varying delay // Phys. Lett. A. – 2009. – **373**, № 3. – P. 328 – 333.
31. *Lakshmikantham V., Bainov D. D., Simeonov P. S.* Theory of impulsive differential equations. – Teaneck, NJ: World Sci. Publ., 1989. – Vol. 6.
32. *Oksendal B.* Stochastic differential equations. – 5th ed. – Berlin: Springer, 2002.
33. *Mao X.* Ruzumikhin-type theorems on exponential stability of stochastic functional differential equations // Stochast. Proces. and Appl. – 1996. – **65**. – P. 233 – 250.

Received 23.09.13,
after revision — 16.03.17