UDC 512.5
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EXPRESSING INFINITE MATRICES AS SUMS OF IDEMPOTENTS

ЗОБРАЖЕННЯ НЕСКІНЧЕННИХ МАТРИЦЬ У ВИГЛЯДІ СУМ ІДЕМПОТЕНТІВ

Let $\mathcal{M}_{Cf}(F)$ be the set of all column-finite $\mathbb{N} \times \mathbb{N}$ matrices over a field F. The following problem is studied: what elements of $\mathcal{M}_{Cf}(F)$ can be expressed as a sum of idempotents? The result states that every element of $\mathcal{M}_{Cf}(F)$ can be represented as the sum of 14 idempotents.

Нехай $\mathcal{M}_{Cf}(F)$ — множина всіх $\mathbb{N} \times \mathbb{N}$ матриць зі скінченними стовпчиками над полем F. Вивчається наступна проблема: які елементи $\mathcal{M}_{Cf}(F)$ можна зобразити у вигляді суми ідемпотентів? Показано, що кожний елемент $\mathcal{M}_{Cf}(F)$ можна зобразити у вигляді суми 14 ідемпотентів.

1. Introduction. It is a classical question whether the elements of a ring or a group can be expressed as sums or products of elements of some particular set. One of the most known problems is expressing the elements as a sum of a unit and an idempotent [1, 3, 5]. Rings in which every element can be written in such way are called clean and have a special place in ring theory [4]. However, even more often we are interested in situation when the elements can be written as sums of the elements sharing some property. One example of such property is being square-zero [6, 22]. Another example, that will be of our interest in this paper, is idempotency. The first result in this field is due to Stampfli [20] who proved that any bounded linear operator on a Hilbert space is a sum of at most 8 projections. Further research in this direction showed that in some cases this number can be even less (see [12, 14]).

In [7] Hartwig and Putcha considered the matrices over a field $\mathbb{F}$ and posed the following questions:

(Qf1) When is A an $\mathbb{F}$-linear combination of idempotents?

(Qf2) When is A a ± 1-combination of idempotents?

(Qf3) When is A a sum of idempotents?

(Qf4) When is A a positive linear combination of idempotents ($\mathbb{F} = \mathbb{R}$)?

In particular they answered (Qf3) and proved the following theorem.

Theorem 1.1 ([7], Theorem 1)**.** *Let $M \in \mathbb{F}_{n\times n}$. Then M is a sum of idempotents if and only if* $\mathrm{tr}(M) = ke$, *where $k \in \mathbb{Z}$ and $k \geq \rho(M)$.*

(The symbol ρ stands here for the rank and tr stands for the trace.)

Let us note that the same result was also proved by Wu in [23] who considered also the minimal number of required idempotents. The problem of finding minimal number was studied further in [21]. In particular, it is of interest how we can describe matrices that are sums of some fixed, usually quite small, number of idempotents [10]. We should also mention that from [8] we know the form of any commutative ring in which every element is a sum of two idempotents and from [9] we know the form of any algebra generated by two idempotents.

As we can easily observe not every matrix is a sum of idempotents. Because of that the answer to question (Qf1) seems to be very interesting as well. In [13] it is proved that if characteristic of the field is equal to 0, then every matrix is a linear combination of at most 3 idempotents. It is also known that for the operators acting on a Hilbert space we need 5 projections [11]. Sometimes this number can be even smaller [15].

All the result we have mentioned so far hold for the matrices over fields with characteristic 0. This problem for the fields of positive characteristic was solved only in 2010 by de Seguins Pazzis [18]. Let us present the solution.

Theorem 1.2 ([18], Theorem 4). *A matrix $A \in M_n(\mathbb{K})$ is a sum of idempotents iff $\operatorname{tr} A \in \mathbb{F}_p$. In particular, every matrix of $M_n(\mathbb{F}_p)$ is a sum of idempotents.*

This author found also the form of the matrices that are linear combinations of 2 and of 3 idempotents over an arbitrary field [16, 17].

In the present paper we would like to study the problem whether a $\mathbb{N} \times \mathbb{N}$ matrix can be written as a sum of idempotents. We will consider the matrices with the property that each column contains only a finite number of nonzero entries and we will call them column-finite. The set of all column-finite matrices over a field F will be denoted by $\mathcal{M}_{Cf}(F)$.

Our main result is the following theorem.

Theorem 1.3. *Assume that F is a field. Any matrix $a \in \mathcal{M}_{Cf}(F)$ can be expressed as a sum of at most 14 idempotents $\mathcal{M}_{Cf}(F)$.*

2. Proofs. We start with introducing the notation.

We write e_∞ and e_k for $\mathbb{N} \times \mathbb{N}$ and $k \times k$ identity matrices, respectively, and e_{nm} for the $\mathbb{N} \times \mathbb{N}$ matrix with 1 in the position (n, m) and 0 in every other position.

By $\mathcal{T}_\infty(F)$ we denote the subring of $\mathcal{M}_{Cf}(F)$ consisting of all upper triangular matrices, and by $\mathcal{LT}_{Cf}(F)$ — the subring of all column-finite lower triangular matrices. The symbols $\mathcal{T}_n(F)$ and $\mathcal{LT}_n(F)$ will be used for the rings of all $n \times n$ upper or lower triangular matrices respectively, whereas $\mathcal{D}_n(F)$ will denote the ring of all $n \times n$ diagonal matrices. The full $n \times n$ matrix ring will be denoted by $\mathcal{M}_{n\times n}(F)$.

If a is any matrix and b is invertible, then by a^b we mean $b^{-1}ab$.

If we write that an integer $k \in F$ then by k we mean the element

$$\underbrace{1 + 1 + \ldots + 1}_{k},$$

where 1 is the identity in F. By $\langle 1 \rangle$ we will denote a subring of F that is generated by 1. The symbol F^* stands for $F \setminus \{0\}$.

If an infinite matrix a has a nonzero entries only in the positions (i, j) from some set I, then we will write

$$a = \sum_{(i,j)\in I} a_{ij} e_{ij}.$$

Note that in this context Σ is a notation, not a traditional sum.

In the whole paper we will use without a reference the following.

***Remark* 2.1.** The matrix $a \in \mathcal{M}_{Cf}(F)$ is a sum/linear combination of idempotents if and only if any b similar to a (in $\mathcal{M}_{Cf}(F)$) is a sum/linear combination of idempotents.

2.1. Triangular matrices. Obviously we would like to deal with matrices of quite simple structure, for instance containing as many zeros as possible. In order to do that we start with a lemma that will help us with replacing the matrices by their conjugacies.

Lemma 2.1. *Let F be a field and $t \in \mathcal{T}_\infty(F)$. If $t = e_\infty + \sum_{m-n\geq 1} t_{nm} e_{nm}$ and for all $n \in \mathbb{N}$ we have $t_{n,n+1} \neq 0$, then t is similar to $e_\infty + \sum_{n=1}^{\infty} t_{n,n+1} e_{n,n+1}$.*

Proof. We will show that it is possible to find an invertible $x \in \mathcal{T}_\infty(F)$ such that

$$tx = x\left(e_\infty + \sum_{n=1}^{\infty} t_{n,n+1}e_{n,n+1}\right).$$

From the above equation we conclude that the entries of x satisfy the system

$$t_{n,n+1}x_{n+1,m} + \sum_{r=n+2}^{m} t_{nr}x_{rm} = x_{n,m-1}t_{m-1,m} \quad \text{for all} \quad m-n \geq 1. \tag{1}$$

It can be solved, for instance, as below.

In the first step we substitute $m = n+1$ to (1) and get

$$t_{n,n+1}x_{n+1,n+1} = x_{nn}t_{n,n+1}.$$

This means that we can choose x_{11} arbitrarily from F^* and for $n \geq 2$ we have $x_{nn} = x_{11}$.

In the second step we substitute $m = n+2$ to (1) and get

$$t_{n,n+1}x_{n+1,n+2} + t_{n,n+2}x_{n+2,n+2} = x_{n,n+1}t_{n+1,n+2}.$$

Since $t_{n,n+1} \neq 0$ we have

$$x_{n+1,n+2} = t_{n,n+1}^{-1}\big(x_{n,n+1}t_{n+1,n+2} - t_{n,n+2}x_{n+2,n+2}\big). \tag{2}$$

Hence, we can choose x_{12} arbitrarily from F and find the next entries from the first diagonal using (2).

In the sth step we substitute $m = n+1+s$ to (1) and have

$$t_{n,n+1}x_{n+1,n+1+s} + t_{n,n+2}x_{n+2,n+1+s} + t_{n,n+3}x_{n+3,n+1+s} + \ldots + t_{n,n+1+s}x_{n+1+s,n+1+s} =$$
$$= x_{n,n+s}t_{n+s,n+1+s}.$$

Again we set $x_{1,1+s}$ arbitrarily and find the next elements of the sth diagonal using

$$x_{n+1,n+1+s} = t_{n,n+1}^{-1}\big(x_{n,n+s}t_{n+s,n+1+s} -$$
$$- t_{n,n+2}x_{n+2,n+1+s} - t_{n,n+3}x_{n+3,n+1+s} - \ldots - t_{n,n+1+s}x_{n+1+s,n+1+s}\big).$$

Performing this way we find all diagonals of x and consequently x itself.

We will use Lemma 2.1 to prove the following proposition.

Proposition 2.1. *Let F be a field. If $t \in \mathcal{T}_\infty(F)$ satsifies the condition $t_{nn} = 2$ for all $n \in \mathbb{N}$, then t is a sum of at most 4 idempotents.*

Proof. First we define two matrices t_1 and t_2 as follows:

$$(t_1)_{nm} = \begin{cases} 1 & \text{if} \quad m = n, \\ t_{nm} & \text{if} \quad m-n > 1, \\ t_{nm} & \text{if} \quad m-n = 1 \quad \text{and} \quad t_{nm} \neq 0, \\ 1 & \text{if} \quad m-n = 1 \quad \text{and} \quad t_{nm} = 0, \end{cases} \qquad t_2 = t - t_1.$$

One can see that t_1 fulfills the assumptions of Lemma 2.1. Hence, for some x we have

$$t_1^x = e_\infty + \sum_{n=1}^{\infty} (t_1)_{n,n+1} e_{n,n+1}.$$

We write t_1^x as $u + u'$, where

$$u = \sum_{n=1}^{\infty} \big(e_{2n-1,2n-1} + (t_1)_{2n-1,2n} e_{2n-1,2n}\big), \qquad u' = \sum_{n=1}^{\infty} \big(e_{2n,2n} + (t_1)_{2n,2n+1} e_{2n,2n+1}\big).$$

It is easy to check that u and u' are idempotents.

We deal similarly with t_2. We write it as $v' + v''$ where v' is defined by the following inductive rule:

(1) $v'_{11} = (t_2)_{11}$, $v'_{12} = (t_2)_{12}$;

(2) if $v'_{n,n+1} = 0$, then put we $v'_{n+1,n+1} = 1$, $v'_{n+1,n+2} = (t_2)_{n+1,n+2}$;

(3) if $v'_{n,n+1} \neq 0$, then we put $v'_{n+1,n+1} = 0$, $v'_{n+1,n+2} = 0$.

The idea of this decomposition is depicted in Fig. 1.

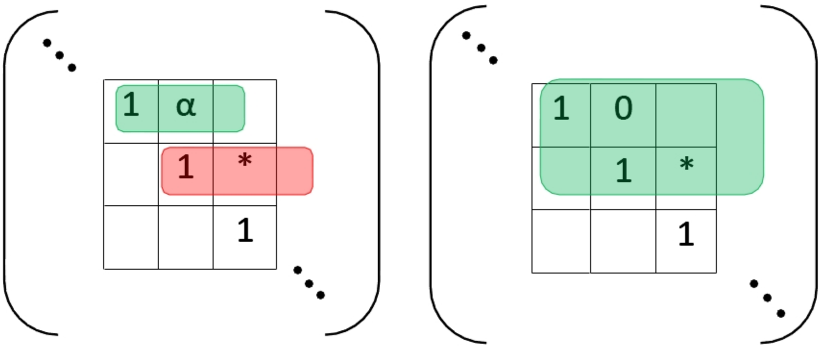

Fig. 1. Picture to the proof of Proposition 2.1. If $v'_{n,n+1} = \alpha \neq 0$, then the entries from the next row 'go' to v''. If $v'_{n,n+1} = 0$, then the next row can 'stay' in v'.

From the construction of v' and v'' it follows that they are idempotents. Thus, the result follows.

Now we will focus on lower triangular matrices. Note that we consider only column-finite lower triangular matrices. This restriction may cause some difficulties. In particular we can not make use of Lemma 2.1, because since t is lower triangular, the matrix x may turn out not to be column-finite. Therefore, we will first decompose t to a sum of two block matrices and then we will focus on those two. In particular we will need some more information about finite matrices. It can be noticed that from the method of proof of Proposition 2.1 we can derive the following corollary.

***Corollary* 2.1.** *Let F be a field and $k \in \mathbb{N}$. If a is either from $\mathcal{T}_k(F)$ or $\mathcal{LT}_k(F)$ and $a_{nn} = 2$ for all n, $1 \leq n \leq k$, then a is a sum of at most 4 idempotents.*

The above corollary will be useful in the proof of the below result.

Proposition 2.2. *If F is a field and $t \in \mathcal{LT}_{Cf}(F)$ is such that $t_{nn} = 3$ for all $n \in \mathbb{N}$, then t can be written as a sum of at most 6 idempotents.*

Proof. We apply the method used in the proof of Proposition 2.2 from [19].
First we construct the sequence $(l_m)_{m\in\mathbb{N}}$ as follows:

$$l_m = \max\{i : t_{im} \neq 0\}.$$

Now we define another sequence $(l'_m)_{m\in\mathbb{N}}$ as follows:

$$l'_m = \begin{cases} l_m & \text{if for all} \quad j < m \quad \text{we have} \quad l_j \leq l_m, \\ \max\{l_j : j < m \wedge l_j > l_m\} & \text{otherwise.} \end{cases}$$

The sequence $(l'_m)_{m\in\mathbb{N}}$ is nondecreasing. It can be observed that for all $m \in \mathbb{N}$ and all $i > l'_m$ we have $t_{im} = 0$. Thus, we can say that t has staircase structure and that the 'stairs' are determined by the sequence (l'_m). (See the 'stairs' in Fig. 2.)

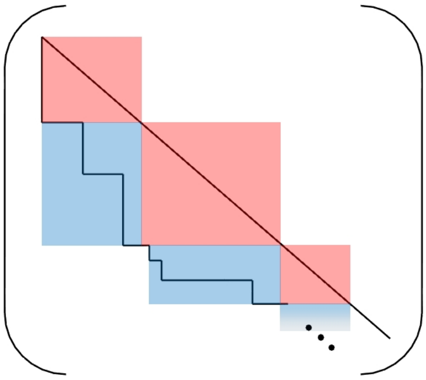

Fig. 2. Picture to the proof of Proposition 2.2. The polygon arc depicts the staircase structure of t. It the red area there are the nonzero coefficients of u, whereas in the blue area we have the nonzero coefficients of v.

Now we will write t as a sum of two matrices: u and v. The matrix u is constructed as follows. For all $i \in \mathbb{N}$ we put $u_{ii} = 2$. Now let m be equal to 1 and let n be equal to l'_1. For all i, j such that $m \leq j < i \leq n$ we put $u_{ij} = t_{ij}$.

Let now m be equal to the preceding n increased by 1 and let the new n be equal to l'_m. Then for all i, j, $m \leq i, j \leq n$ we put $u_{ij} = t_{ij}$.

Proceeding the same way we obtain a block diagonal matrix u (in Fig. 2 it is marked by red color). Clearly $v = t - u$.

By Corollary 2.1 each block of u is a sum of at most 4 idempotents. Thus, so is u. From the construction of v it follows that

$$v = \begin{pmatrix} e_{k_1} & & & & & \\ v_1 & e_{k_2} & & & & \\ 0 & v_2 & e_{k_3} & & & \\ 0 & 0 & v_3 & e_{k_4} & & \\ 0 & 0 & 0 & v_4 & e_{k_5} & \\ \vdots & \vdots & & & \ddots & \ddots \end{pmatrix}.$$

We put

$$v' = \begin{pmatrix} e_{k_1} & & & & & \\ v_1 & 0 & & & & \\ 0 & 0 & e_{k_3} & & & \\ 0 & 0 & v_3 & 0 & & \\ 0 & 0 & 0 & 0 & e_{k_5} & \\ \vdots & \vdots & & & \ddots & \ddots \end{pmatrix},$$

$$v'' = v - v' = \begin{pmatrix} 0 & & & & & \\ 0 & e_{k_2} & & & & \\ 0 & v_2 & 0 & & & \\ 0 & 0 & 0 & e_{k_4} & & \\ 0 & 0 & 0 & v_4 & 0 & \\ \vdots & \vdots & & & \ddots & \ddots \end{pmatrix}.$$

It is easy to check that v', v'' are idempotents.

Proposition 2.2 is proved.

2.2. Diagonal matrices. We start with an observation about 2×2 matrices that follows from the proof of Theorem 1.1.

Lemma 2.2. *If $x \in F$, where F is a field, then the matrix*

$$\begin{pmatrix} x & 0 \\ 0 & 2 - x \end{pmatrix}$$

is a sum of two idempotents.

Now we can give proof of the result that holds for $\mathcal{D}_\infty(F)$.

Proposition 2.3. *Let F be any field. Any $d \in \mathcal{D}_\infty(F)$ is a sum of 4 idempotents from $\mathcal{M}_{Cf}(F)$.*

Proof. We write d as a sum of $x, y \in \mathcal{D}_\infty(F)$, where the entries of x, y are defined inductively according to the scheme as below:

$$\begin{aligned} x_{11} &= d_{11}, & y_{11} &= 0, \\ x_{22} &= 2 - x_{11}, & y_{22} &= d_{22} - x_{22}, \\ y_{33} &= 2 - y_{22}, & x_{33} &= d_{33} - y_{33}, \\ x_{44} &= 2 - x_{33}, & y_{44} &= d_{44} - x_{44}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \end{aligned}$$

Then x and y are of the forms:

$$x = \begin{pmatrix} x_{11} & 0 & & & \\ 0 & 2 - x_{11} & & & \\ & & x_{33} & 0 & \\ & & 0 & 2 - x_{33} & \\ & & & & \ddots \end{pmatrix},$$

$$y = \begin{pmatrix} \boxed{0} & & & & & \\ & y_{22} & 0 & & & \\ & 0 & 2 - y_{22} & & & \\ & & & y_{44} & 0 & \\ & & & 0 & 2 - y_{44} & \\ & & & & & \ddots \end{pmatrix},$$

and by Lemma 2.2 both x and y are sums of two idempotents.

Proposition 2.3 is proved.

2.3. Consequences. In this section we join the results from the two preceding ones and prove our main theorem.

***Proof of Theorem* 1.3.** Define $t_1,\ t_2$ and d as follows:

$$(t_1)_{ij} = \begin{cases} 2 & \text{if} \quad j = i, \\ a_{ij} & \text{if} \quad i < j, \\ 0 & \text{if} \quad i > j, \end{cases} \qquad (t_2)_{ij} = \begin{cases} 3 & \text{if} \quad j = i, \\ a_{ij} & \text{if} \quad i > j, \\ 0 & \text{if} \quad i < j, \end{cases} \qquad d_{ij} = \begin{cases} a_{ij} - 5 & \text{if} \quad j = i, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Clearly $a = t_1 + t_2 + d$. From Propositions 2.1 and 2.2 we know that t_1 and t_2 are sums of at most 4 and 6 idempotents, respectively. By Proposition 2.3 the matrix d is a sum of 4 idempotents. This proves the claim.

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Received 22.10.15,
after revision — 08.04.17