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ESTIMATION OF THE GENERALIZED BESSEL – STRUVE TRANSFORM IN A CERTAIN SPACE OF GENERALIZED FUNCTIONS

ОЦІНКА УЗАГАЛЬНЕНОГО ПЕРЕТВОРЕННЯ БЕССЕЛЯ – СТРУВЕ В ДЕЯКОМУ ПРОСТОРІ УЗАГАЛЬНЕНИХ ФУНКЦІЙ

We investigate the so-called Bessel – Struve transform on certain class of generalized functions called Boehmians. By using different convolution products, we generate the Boehmian spaces, where the extended transform is well defined. We also show that the Bessel – Struve transform of a Boehmian is an isomorphism which is continuous with respect to a certain type of convergence.

Вивчається так зване перетворення Бесселя – Струве на деякому класі узагальнених функцій, що називаються бьоміанами. З використанням різних добутків типу згорток згенеровано простори Бьоміана, в яких розширене перетворення добре визначене. Також показано, що перетворення Бесселя – Струве для бьоміана є ізоморфізмом, який є неперервним відносно деякого виду збіжності.

1. Introduction. While special types of what would later be known as Bessel functions were studied by Euler, Lagrange, and the Bernoullis, the Bessel functions were first used by F. W. Bessel to describe three body motion, with the Bessel functions appearing in the series expansion on planetary perturbation and series solution to a second order differential equation that arise in many diverse situations. On the other hand, Struve functions occur in many places in physics and applied mathematics, e. g., in optics as the normalized line spread function, in fluid dynamics, and quite prominently in acoustics for impedance calculations as well.

The normalized Bessel and Struve functions of index α are, respectively, given by Watson [3] as

$$\jmath_\alpha(z)2^\alpha\Gamma(\alpha+1)\,z^{-\alpha}\jmath_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\dfrac{z}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}$$

and

$$k_\alpha(z)=2^\alpha\Gamma(\alpha+1)\,z^{-\alpha}H_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\dfrac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\dfrac{3}{2}\right)\Gamma\left(n+\alpha+\dfrac{3}{2}\right)}.$$

A kind of Fourier transforms named as Bessel – Struve transform was considered by S. Hamem et al. as [2]

$$\boldsymbol{f}^{\alpha}_{\beta,s}\left(f(x)\right)(\lambda)=\int\limits_{-\infty}^{\infty}f(x)\sigma_\alpha\left(-i\lambda x\right)d\mu_\alpha(x),$$

where $\alpha>-\dfrac{1}{2}$ and σ_α is the Bessel – Struve kernel given by the equation

$$\sigma_\alpha(x) = \jmath_\alpha(ix) - ik_\alpha(ix).$$

The Bessel – Struve kernel is the solution of the initial value problem $\ell_\alpha u(x) = \lambda^2 u(x)$, where $u(0) = 1$ and $u'(0) = \dfrac{\lambda\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{3}{2}\right)}$. It further satisfies the integral representation

$$\sigma_\alpha(\lambda x) = \frac{\lambda\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int\limits_0^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} e^{\lambda x t} dt,$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

Moreover, the Bessel – Struve transform is related to the Weyl integral transform [2]

$$\boldsymbol{w}_\alpha(f)(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int\limits_{|y|}^1 \left(x^2-y^2\right)^{\alpha-\frac{1}{2}} x f(\operatorname{sgn}(y)x)\, dx$$

and it satisfies

$$\boldsymbol{f}^\alpha_{\beta,s}(f) = \mathcal{F}f \circ \boldsymbol{w}_\alpha(f), \tag{1}$$

where $f \in \boldsymbol{l}^1_\alpha(\mathbb{R})$ and $\mathcal{F}f$ is the Fourier transform of f,

$$\mathcal{F}(f(x))(y) = \int\limits_{-\infty}^{\infty} f(x) e^{-ixy}\, dx.$$

The Mellin-type convolution product of first kind was given in terms of the integral equation [10]

$$f \times g(y) = \int\limits_0^\infty f\left(yx^{-1}\right) x^{-1} g(x)\, dx. \tag{2}$$

The space $\boldsymbol{l}^p_\alpha(\mathbb{R})$ consists of those real valued measurable functions f defined on $\mathbb{R}$ such that

$$\|f\|^p_\alpha := \begin{cases} \left(\displaystyle\int\limits_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty, \end{cases}$$

where

$$d\mu_\alpha(x) = A(x)dx \quad \text{and} \quad A(x) = |x|^{2\alpha+1}.$$

By $\boldsymbol{\kappa}(\mathbb{R})$ we denote the space of test functions of bounded supports over $\mathbb{R}$. Then, $\boldsymbol{\kappa}(\mathbb{R})$ is, indeed, a dense subspace of $\boldsymbol{l}^p(\mathbb{R})$ for every choice of p. Here $\boldsymbol{l}^1(0,\infty)$ denotes the Lebesgue space of complex valued integrable functions defined on $(0,\infty)$ and $\boldsymbol{l}^p_\alpha(0,\infty)$ denotes the restriction of $\boldsymbol{l}^p_\alpha(\mathbb{R})$ to the open interval $(0,\infty)$.

The following definition is very beneficial to our next investigation.

Definition 1. *Let* $\alpha > -\dfrac{1}{2}$ *and* $\mathtt{A}(t) = |t|^{2\alpha+1}$ *and* f, g *in* $\boldsymbol{l}^1(0,\infty)$. *Then we define the product* $\otimes$ *between* f *and* g *by the integral*

$$f \otimes g(y) = \int_0^\infty f(yt)\, g(t) d\mu(t), \tag{3}$$

where $d\mu(t) = \mathtt{A}(t)dt$.

An assistance of (2) and (3) leads to the following proposition.

Proposition 1. *Let* f, g *and* h *be integrable functions in* $\boldsymbol{l}^1(0,\infty)$ *and* $y > 0$. *Then we have*

$$f \otimes (g \times h)(y) = (f \otimes g) \otimes h(y).$$

Proof. Let the hypothesis of the theorem satisfy for f, g and h in $\boldsymbol{l}^1(0,\infty)$. Then, appealing to (2) and (3), we get

$$f \otimes (g \times h)(y) = \int_0^\infty f(yt) \int_0^\infty x^{-1} g\left(tx^{-1}\right) h(x) dx d\mu(t).$$

By Fubini's theorem, we obtain

$$f \otimes (g \times h)(y) = \int_0^\infty h(x) x^{-1} \int_0^\infty g\left(tx^{-1}\right) f(yt)\, d\mu(t) dx.$$

Setting variables reveals

$$f \otimes (g \times h)(y) = \int_0^\infty h(x) \int_0^\infty f(yxz) g(z) d\mu(z) d\mu(x).$$

Proposition 1 is proved.

By the benefit of Proposition 2.1 of [2], it follows that $\boldsymbol{w}_\alpha$ is a bounded operator from $\boldsymbol{l}^1_\alpha(\mathbb{R})$ into $\boldsymbol{l}^1(\mathbb{R})$. Hence, we have the following remark.

***Remark* 1.** Let $f \in \boldsymbol{l}^1_\alpha(\mathbb{R})$. Then we have $\boldsymbol{f}^\alpha_{\beta,s}(f) \in \boldsymbol{l}^1_\alpha(\mathbb{R})$.

Proof of this remark follows from equation (1) and the injectivity of $\mathcal{F}$. We therefore omit the details.

2. Generated spaces of Boehmians. Boehmians were used for all objects defined by an algebraic construction similar to that of field of quotients and in some cases, it just gives the field of quotients. The advent of Boehmians has recently brought drastic changes in the concept of applied functional analysis. The idea of construction of Boehmians was initiated by the concept of Mikusinski regular operators.

The minimal structure necessary for the construction of Boehmians consists of the following axioms:

$\mathbb{A}$(i) A nonempty set $\mathfrak{a}$.

$\mathbb{A}$(ii) A commutative semigroup $(\mathfrak{b}, \bullet)$.

$\mathbb{A}$(iii) An operation $\star\colon \mathfrak{a} \times \mathfrak{b} \to \mathfrak{a}$ such that for each $x \in \mathfrak{a}$ and $s_1, s_2, \in \mathfrak{b}$,

$$x \star (s_1 \bullet s_2) = (x \star s_1) \star s_2.$$

$\mathbb{A}$(iv) A collection $\Delta \subset \mathfrak{b}^{\mathbb{N}}$ such that:

(a) If $x, y \in \mathfrak{a}$, $(s_n) \in \Delta$, $x \bullet s_n = y \bullet s_n$ for all n, then $x = y$.

(b) If $(s_n), (t_n) \in \Delta$, then $(s_n \bullet t_n) \in \Delta$.

Elements of Δ are called delta sequences. Consider

$$Q = \left\{ (x_n, s_n) : x_n \in \mathfrak{a}, \ \ (s_n) \in \Delta, \ \ x_n \star s_m = x_m \star s_n \ \ \forall m, n \in \mathbb{N} \right\}.$$

If $(x_n, s_n), (y_n, t_n) \in Q$, $x_n \star t_m = y_m \star s_n$ $\forall m, n \in \mathbb{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence clases in Q is denoted by $\boldsymbol{b}$. Elements of $\boldsymbol{b}$ are called Boehmians.

Between $\mathfrak{a}$ and $\boldsymbol{b}$ there is a canonical embedding expressed as $x \to \dfrac{x \star s_n}{s_n}$. The operation $\star$ can be extended to $\boldsymbol{b} \times \mathfrak{a}$ by $\dfrac{x_n}{s_n} \star t = \dfrac{x_n \star t}{s_n}$. The relationship between the notion of convergence and the product $\star$ is given as:

(i) If $f_n \to f$ as $n \to \infty$ in $\mathfrak{a}$ and, $\phi \in \mathfrak{b}$ is any fixed element, then $f_n \star \phi \to f \star \phi$ in $\mathfrak{a}\,(\text{as } n \to \infty)$.

(ii) If $f_n \to f$ as $n \to \infty$ in $\mathfrak{a}$ and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \to f$ in $\mathfrak{a}$ (as $n \to \infty$).

The operation $\star$ is extended to $\boldsymbol{b} \times \mathfrak{b}$ as follows: If $\left[\dfrac{(f_n)}{(s_n)}\right] \in \boldsymbol{b}$ and $\phi \in \mathfrak{b}$, then $\left[\dfrac{(f_n)}{(s_n)}\right] \star \phi = \left[\dfrac{(f_n) \star \phi}{s_n}\right]$.

Convergence in $\boldsymbol{b}$ is defined as follows :

A sequence (h_n) in $\boldsymbol{b}$ is said to be δ **convergent** to h in $\boldsymbol{b}$, $h_n \xrightarrow{\delta} h$, if there is a sequence $(s_n) \in \Delta$ such that $(h_n \star s_n)$, $(h \star s_n) \in \mathfrak{a}$ $\forall k, n \in \mathbb{N}$, and $(h_n \star s_k) \to (h \star s_k)$ as $n \to \infty$, in $\mathfrak{a}$, for every $k \in \mathbb{N}$.

A sequence (h_n) in $\boldsymbol{b}$ is said to be Δ **convergent** to h in $\boldsymbol{b}$, $h_n \xrightarrow{\Delta} h$, if there is a sequence $(s_n) \in \Delta$ such that $(h_n - h) \star s_n \in \mathfrak{a}$ $\forall n \in \mathbb{N}$, and $(h_n - h) \star s_n \to 0$ as $n \to \infty$ in $\mathfrak{a}$.

Several integral transforms were extended to various spaces of Boehmians by many authors such as: Al-Omari and Kilicman [9, 15, 20], Al-Omari [13], Mikusinski and Zayed [16], Karunakaran and Roopkumar [17], Karunakaran and Vembu [18], Roopkumar [19], Nemzer [21], Al-Omari, Loonker, Banerji and Kalla [11] and many others to mention but a few. However, readers are assumed to be acquainted with the abstract construction of Boehmian spaces. If it were otherwise we refer to [4–9, 11, 13] and [15–21]. We need the following lemma to be established.

Lemma 1. *Let* $f \in \boldsymbol{l}^1_\alpha(0,\infty)$ *and* $\psi \in \boldsymbol{\kappa}(0,\infty)$. *Then we have*

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x); \lambda\big) = \big(\boldsymbol{f}^{\alpha}_{\beta,s} f \otimes \psi(x)\big)(\lambda).$$

Proof. Under the hypothesis of the theorem we write

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x); \lambda\big) = \int_0^\infty \int_0^\infty f\left(xt^{-1}\right) t^{-1} \psi(t) dt \sigma_\alpha\left(-i\lambda x\right) d\mu_\alpha(x).$$

By Fubini's theorem, this can be written as

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x); \lambda\big) = \int_0^\infty \psi(t) \int_0^\infty f\left(xt^{-1}\right) \sigma_\alpha\left(-i\lambda x\right) d\mu_\alpha(x) dt.$$

On setting variables we get

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \int\limits_0^\infty \big(\boldsymbol{f}^{\alpha}_{\beta,s} f(z);\lambda(t)\big)\,\psi(t)d\mu(t).$$

Hence, equation (3) reveals

$$\boldsymbol{f}^{\alpha}_{\beta,s}\big(f \times \psi(x);\lambda\big) = \big(\boldsymbol{f}^{\alpha}_{\beta,s} f \otimes \psi\big)\,(\lambda).$$

Lemma 1 is proved.

Spaces we are generating here are the space $\beta_1\big(\boldsymbol{l}^1_\alpha, \boldsymbol{\kappa}, \times), \times, \Delta\big)$ and the space $\beta_2(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta)$. Δ wherever it appears is the set of delta sequences (δ_n) from $\boldsymbol{\kappa}(0,\infty)$, where

$$\int\limits_0^\infty \delta_n(x)dx = 1, \tag{4}$$

$$\int\limits_0^\infty \big|\delta_n(x)\big|dx < m, \quad m \text{ is a positive real number,} \tag{5}$$

$$\operatorname{supp}\delta_n \subseteq [-\varepsilon,\varepsilon], \quad \varepsilon \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

It is of importance that we recall here some properties of the product $\times$ which we list as [12, 10]:

$$f \times g = g \times f, \tag{7}$$

$$f \times (g + h) = f \times g + f \times h, \tag{8}$$

$$f \times (g \times h) = (f \times g) \times h, \tag{9}$$

$$(\alpha f) \times g = \alpha\,(f \times g) = f \times (\alpha g)\,, \quad \alpha \in \mathbb{C}. \tag{10}$$

We merely generate the space $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ as the space $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$ can be generated similarly.

Theorem 1. *Let* $f \in \boldsymbol{l}^1_\alpha(0,\infty)$ *and* $\psi \in \boldsymbol{\kappa}(0,\infty)$, $\alpha > -\dfrac{1}{2}$. *Then we have* $f \times \psi \in \boldsymbol{l}^1_\alpha(0,\infty)$.

Proof. Let $f \in \boldsymbol{l}^1_\alpha(0,\infty)$ and $\psi \in \boldsymbol{\kappa}(0,\infty)$ be given. Let $K = [a,b]\,,\ 0 < a < b$, be a compact subset of $(0,\infty)$ such that $\operatorname{supp}\psi \subseteq K$. Then, for $\alpha > -\dfrac{1}{2}$, we have

$$\int\limits_0^\infty |f \times \psi(y)|\,d\mu(y) = \int\limits_0^\infty \left|\int\limits_0^\infty f\left(yt^{-1}\right)t^{-1}\psi(t)dt\right| d\mu(y) \leq$$

$$\leq \int\limits_a^b |\psi(t)|\,t^{-1}\int\limits_0^\infty \left|f\left(yt^{-1}\right)\right| d\mu(y)dt.$$

By setting the variables $z = yt^{-1}$ we get

$$\int\limits_{0}^{\infty} |f \times \psi(y)|\, d\mu(y) \le \int\limits_{a}^{b} |\psi(t)|\, t^{2\alpha} dt \int\limits_{0}^{\infty} |f(z)|\, d\mu(z).$$

That can be interpreted as

$$\left|f \times \psi(y)\right|_{\alpha}^{1} \le m^{*} \left\|f\right\|_{\alpha}^{1}, \tag{11}$$

where $m^{*} = \int_{a}^{b} |\psi(t)|\, t^{2\alpha} dt.$

Theorem 1 is proved.

Theorem 2. *Let $f \in \boldsymbol{l}_{\alpha}^{1}(0,\infty)$ and $\psi_1, \psi_2 \in \boldsymbol{\kappa}(0,\infty),\ \alpha > -\dfrac{1}{2}$. Then we have*

(i) $f \times (\psi_1 + \psi_2) = f \times \psi_1 + f \times \psi_2,$

(ii) $f \times (\psi_1 \times \psi_2) = (f \times \psi_1) \times (\psi_2)\,,$

(iii) $(\alpha f) \times \psi_1 = \alpha\,(f \times \psi_1) = f \times (\alpha \psi_1)\,,\ \alpha \in \mathbb{C}.$

Proof of identities (i) and (iii) follows from simple integral calculus. Identity (ii) directly follows from (9). This establishes the theorem.

Theorem 3. *Let $f_n \to f \in \boldsymbol{l}_{\alpha}^{1}(0,\infty)$ as $n \to \infty$ and $\psi \in \boldsymbol{\kappa}(0,\infty),\ \alpha > -\dfrac{1}{2}$. Then we have*

$$f_n \times \psi \to f \times \psi$$

as $n \to \infty$ in $\boldsymbol{l}_{\alpha}^{1}(0,\infty)$.

Proof of this theorem follows from simple integration. We, therefore, omit the details.

Theorem 4. *Let $f \in \boldsymbol{l}_{\alpha}^{1}(0,\infty)$ and $(\delta_n) \in \Delta, \alpha > -\dfrac{1}{2}$. Then we get*

$$f \times \delta_n \to f$$

as $n \to \infty$ in $\boldsymbol{l}_{\alpha}^{1}(0,\infty)$.

Proof. Let $f \in \boldsymbol{l}_{\alpha}^{1}(0,\infty)$ and $(\delta_n) \in \Delta$ be given. Since the space $\boldsymbol{\kappa}(0,\infty)$ is dense in $\boldsymbol{l}_{\alpha}^{1}(0,\infty)$ we find $\psi \in \boldsymbol{\kappa}(0,\infty)$ such that

$$\left\|f - \psi\right\|_{\alpha}^{1} < \varepsilon \tag{12}$$

for $\varepsilon > 0$.

Also, by (11) and the fact that $(\delta_n) \in \boldsymbol{\kappa}(0,\infty)$, we obtain

$$\left\|(f - \psi) \times \delta_n\right\|_{\alpha}^{1} \le m^{*} \left\|f - \psi\right\|_{\alpha}^{1}$$

for some real number m^{*}.

Hence, inserting (12) into above equation we get

$$\left\|(f - \psi) \times \delta_n\right\|_{\alpha}^{1} \le \varepsilon m^{*}. \tag{13}$$

Thus, we have obtained

$$\left\|\psi \times \delta_n - \psi\right\|_{\alpha}^{1} = \int\limits_{0}^{\infty} \left|(\psi \times \delta_n - \psi)\,(y)\right| d\mu(y) =$$

$$= \int_0^\infty \left| \int_0^\infty \psi\left(yt^{-1}\right) t^{-1}\delta_n(t)dt - \psi(y)\int_0^\infty \delta_n(t)dt \right| d\mu(y) \leq$$

$$\leq \int_0^\infty \int_0^\infty \left|\psi\left(yt^{-1}\right)t^{-1} - \psi(y)\right| |\delta_n(t)|\, dt d\mu(y). \tag{14}$$

Now, let $g_y(t) = \psi\left(yt^{-1}\right)t^{-1}$, then $g_y(t)$ is uniformly continuous function in $\boldsymbol{\kappa}(0,\infty)$. Therefore, we find $\delta > 0$ such that

$$\left|g_y(t) - g(1)\right| < \varepsilon \qquad \text{whenever} \quad |y-1| < \delta.$$

Thus, inventing (4) in (14) gives

$$\left\|\psi \times \delta_n - \psi\right\|_\alpha^1 \leq \int_0^\infty \int_0^\infty \left|g_y(t) - g_y(1)\right| |\delta_n(t)|\, dt\, d\mu(y) \leq \varepsilon \int_c^d d\mu(y), \tag{15}$$

where $[a,b]$ is an interval containing the support of g_y.

Therefore, (15) implies

$$\left\|\psi \times \delta_n - \psi\right\|_\alpha^1 \leq A\varepsilon, \tag{16}$$

where $A = \int_c^d d\mu(y)$.

On account of (13), (16) and (12), we reach to

$$\left\|f \times \delta_n - f\right\|_\alpha^1 \leq \left\|(f-\psi)\times\delta_n\right\|_\alpha^1 + \left\|\psi \times \delta_n - \psi\right\|_\alpha^1 t \left\|f - \psi\right\|_\alpha^1 \leq \varepsilon m^* + A\varepsilon + \varepsilon.$$

Hence, above equation gives

$$\left\|f \times \delta_n - f\right\|_\alpha^1 \leq B\varepsilon,$$

where $B = m^* + A + 1$.

Theorem 4 is proved.

The space $\beta_1\left(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ has therefore been generated.

The sum of two Boehmians in $\beta_1\left(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ and multiplication by a scalar can be defined as

$$\left[\frac{(f_n)}{(\delta_n)}\right] + \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(f_n)\times\psi_n + (g_n)\times(\delta_n)}{(\delta_n)\times(\psi_n)}\right] \qquad \text{and} \qquad \alpha\left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\alpha(f_n)}{(\delta_n)}\right],$$

where $\alpha \in \mathbb{C}$, $\mathbb{C}$ being the space of complex numbers.

The operation $\times$ and the differentiation are defined by

$$\left[\frac{(f_n)}{(\delta_n)}\right] \times \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(f_n)\times(g_n)}{(\delta_n)\times(\psi_n)}\right] \qquad \text{and} \qquad \mathcal{D}^\alpha\left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\mathcal{D}^\alpha(f_n)}{(\delta_n)}\right].$$

A sequence of Boehmians $(\beta_n) in \beta_1\left(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$ is said to be δ convergent to a Boehmian β in $\beta_1\left(\boldsymbol{l}_\alpha^1, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (δ_n) such that

$$(\beta_n \times \delta_k), (\beta \times \delta_k) \in \boldsymbol{l}^1_\alpha \quad \forall k, n \in \mathbb{N},$$

and

$$(\beta_n \times \delta_k) \to (\beta \times \delta_k) \quad \text{as} \quad n \to \infty, \quad \text{in} \quad \boldsymbol{l}^1_\alpha, \quad \text{for every} \quad k \in \mathbb{N}.$$

The equivalent statement for δ convergence:

$\beta_n \stackrel{\delta}{\to} \beta$ $(n \to \infty)$ in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ if and only if there is $(\varphi_{n,k}), (\varphi_k) \in \boldsymbol{l}^1_\alpha$ and $(\delta_k) \in \Delta$ such that $\beta_n = \left[\dfrac{(\varphi_{n,k})}{(\delta_k)}\right]$, $\beta = \left[\dfrac{(\varphi_k)}{(\delta_k)}\right]$ and for each $k \in \mathbb{N}, \varphi_{n,k} \to \varphi_k$ as $n \to \infty$ in $\boldsymbol{l}^1_\alpha$.

A sequence of Boehmians (β_n) in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ is said to be Δ convergent to a Boehmian β in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$, denoted by $\beta_n \stackrel{\Delta}{\to} \beta$, if there exists a $(\delta_n) \in \Delta$ such that $(\beta_n - \beta) \times \delta_n \in \boldsymbol{l}^1_\alpha$ $\forall n \in \mathbb{N}$, and $(\beta_n - \beta) \times \delta_n \to 0$ as $n \to \infty$ in $\boldsymbol{l}^1_\alpha$.

Similarly, the following theorems generate the Boehmian space $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$.

Theorem 5. *Let* $f \in \boldsymbol{l}^1(0,\infty)$ *and* $\psi \in \boldsymbol{\kappa}(0,\infty)$. *Then we have* $f \otimes \psi \in \boldsymbol{l}^1(0,\infty)$.

Theorem 6. *Let* $f \in \boldsymbol{l}^1(0,\infty)$ *and* $\psi_1, \psi_2 \in \boldsymbol{\kappa}(0,\infty)$. *Then we obtain*

(i) $f \otimes (\psi_1 + \psi_2) = f \otimes \psi_1 + f \otimes \psi_2$,

(ii) $(\alpha f) \otimes \psi_1 = \alpha\,(f \otimes \psi_1) = f \otimes (\alpha \psi_1)\,,\ \alpha \in \mathbb{C}$.

Theorem 7. *For* $f \in \boldsymbol{l}^1(0,\infty)$ *and* $\psi_1, \psi_2 \in \boldsymbol{\kappa}(0,\infty)$, *we get* $f \otimes (\psi_1 \times \psi_2) = (f \otimes \psi_1) \otimes \psi_2$.

Proof of Theorems 5 and 6 is, respectively, similar to that of Theorems 1 and 2. Proof of Theorem 7 follows from Proposition 1.

Theorem 8. (i) *Let* $f_n \to f$ *in* $\boldsymbol{l}^1(0,\infty)$ *as* $n \to \infty$ *and* $\psi \in \boldsymbol{\kappa}(0,\infty)$. *Then we have* $f_n \otimes \psi \to f \otimes \psi$ *as* $n \to \infty$.

(ii) *Let* $f_n \in \boldsymbol{l}^1(0,\infty)$ *and* $(\delta_n) \in \Delta$. *Then we have* $f_n \otimes \delta_n \to f$ *as* $n \to \infty$.

The proof of the Part (i) of the theorem follows from simple integration whereas proof of the second part is analogous to that of Theorem 3. Hence, we prefer we delete the details.

The sum of two Boehmians in $\beta_2\left(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\right)$ and multiplication by a scalar can also be defined as

$$\left[\frac{(f_n)}{(\delta_n)}\right] + \left[\frac{(g_n)}{(\varepsilon_n)}\right] = \left[\frac{(f_n) \otimes \varepsilon_n + (g_n) \otimes (\delta_n)}{(\delta_n) \times (\varepsilon_n)}\right] \qquad \text{and} \qquad \alpha\left[\frac{(f_n)}{(\delta_n)}\right] = \left[\alpha\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\alpha(f_n)}{(\delta_n)}\right],$$

$\alpha \in \mathbb{C}$, space of complex numbers.

The operation $\otimes$ and the differentiation are respectively defined by

$$\left[\frac{(f_n)}{(\delta_n)}\right] \otimes \left[\frac{(g_n)}{(\varepsilon_n)}\right] = \left[\frac{(f_n) \otimes (g_n)}{(\delta_n) \times (\varepsilon_n)}\right] \qquad \text{and} \qquad \mathcal{D}^k\left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{\mathcal{D}^k(f_n)}{(\delta_n)}\right].$$

The notion of δ and Δ convergence in $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ and $\beta_2\left(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\right)$ can be defined in a natural way as above.

3. The Bessel – Struve transform of a Boehmian. Let $\beta \in \beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$, $\beta = \big[(f_n)(\delta_n)\big]$, then, for every $\alpha > -\dfrac{1}{2}$, we define the Bessel – Struve transform of β as

$$\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\frac{(f_n)}{(\delta_n)}\right]\right) = \left[\frac{\left(\boldsymbol{f}^\alpha_{\beta,s} f_n\right)}{(\delta_n)}\right]. \tag{17}$$

The right-hand side of (17) belongs to $\beta_2\left(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\right)$ by the benefit of Remark 1. The above definition is, indeed, well-defined. Let $\left[\dfrac{(f_n)}{(\omega_n)}\right] = \left[\dfrac{(g_n)}{(\varepsilon_n)}\right] \in \beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$. Then, by the notion of equivalence classes of $\beta_1\left(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\right)$, we have

$$f_n \times \varepsilon_m = g_m \times \omega_n.$$

Employing (17) and the notion of equivalence classes of $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$ yield

$$\boldsymbol{f}^\alpha_{\beta,s} f_n \otimes \varepsilon_m = \boldsymbol{f}^\alpha_{\beta,s} g_m \otimes \omega_n.$$

Hence, it follows that $\dfrac{\left(\boldsymbol{f}^\alpha_{\beta,s} f_n\right)}{(\omega_n)} \sim \dfrac{\left(\boldsymbol{f}^\alpha_{\beta,s} g_n\right)}{(\varepsilon_n)}$ in $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$. Therefore, we get

$$\left[\frac{(\boldsymbol{f}^\alpha_{\beta,s} f_n)}{(\omega_n)}\right] = \left[\frac{\left(\boldsymbol{f}^\alpha_{\beta,s} g_n\right)}{(\varepsilon_n)}\right].$$

This proves the claim.

Theorem 9. *$\breve{\boldsymbol{f}}^\alpha_{\beta,s}$ is an isomorphism from $\beta_1\big(\boldsymbol{l}^1_\alpha, \boldsymbol{\kappa}, \times), \times, \Delta\big)$ into $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$.*

Proof. Let us first establish that $\breve{\boldsymbol{f}}^\alpha_{\beta,s}$ is injective. Given $\breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\dfrac{(f_n)}{(\omega_n)}\right]\right) = \breve{\boldsymbol{f}}^\alpha_{\beta,s}\left(\left[\dfrac{(g_n)}{(\varepsilon_n)}\right]\right)$. Then, by Lemma 1 and notion of equivalent classes of $\beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$, it follows that

$$\boldsymbol{f}^\alpha_{\beta,s} f_n \otimes \varepsilon_m = \boldsymbol{f}^\alpha_{\beta,s} g_m \otimes \omega_n.$$

Therefore, Lemma 1 implies $\boldsymbol{f}^\alpha_{\beta,s}(f_n \times \varepsilon_m) = \boldsymbol{f}^\alpha_{\beta,s}(g_m \times \omega_n)$. Employing $\boldsymbol{f}^\alpha_{\beta,s}$ gives

$$f_n \times \varepsilon_m = g_m \times \omega_n.$$

On the other hand, the notion of equivalent classes of $\beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ reveals that

$$\left[\frac{(f_n)}{(\omega_n)}\right] = \left[\frac{(g_n)}{(\varepsilon_n)}\right].$$

Now, we establish that $\breve{\boldsymbol{f}}^\alpha_{\beta,s}$ is a surjective mapping. Let $\left[\dfrac{(\boldsymbol{f}^\alpha_{\beta,s} f_n)}{(\omega_n)}\right] \in \beta_2\big(\boldsymbol{l}^1, (\boldsymbol{\kappa}, \times), \otimes, \Delta\big)$ be arbitrary. Then we have

$$\boldsymbol{f}^\alpha_{\beta,s} f_n \otimes \omega_m = \boldsymbol{f}^\alpha_{\beta,s} f_m \otimes \omega_n$$

for every choice of $m, n \in \mathbb{N}$. Hence $f_n, f_m \in \boldsymbol{l}^1_\alpha(0,\infty)$, for every $m, n \in \mathbb{N}$, are satisfy

$$\boldsymbol{f}^\alpha_{\beta,s}\big(f_n \times \omega_m\big) = \boldsymbol{f}^\alpha_{\beta,s}\big(f_m \times \omega_n\big).$$

That is, $\left[\dfrac{(f_n)}{(\omega_n)}\right] \in \beta_1\big(\boldsymbol{l}^1_\alpha, (\boldsymbol{\kappa}, \times), \times, \Delta\big)$ is such that

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)=\left[\frac{\left(\boldsymbol{f}^{\alpha}_{\beta,s}f_n\right)}{(\omega_n)}\right].$$

Theorem 9 is proved.

In addition to above, we derive the extension formula of $\times$ to $\beta_1\left(\boldsymbol{l}^1_\alpha,(\boldsymbol{\kappa},\times),\times,\Delta\right)$ as follows:

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Justification is as follows: by aid of (17) we write

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left[\frac{\left(\boldsymbol{f}^{\alpha}_{\beta,s}(f_n\times\phi)\right)}{(\omega_n)}\right].$$

Lemma 1 therefore gives

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left[\frac{\left(\boldsymbol{f}^{\alpha}_{\beta,s}f_n\otimes\phi\right)}{(\omega_n)}\right].$$

The definition of the product $\times$ implies

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\left[\frac{\left(\boldsymbol{f}^{\alpha}_{\beta,s}f_n\right)}{(\omega_n)}\right]\times\phi.$$

Once again, (17) yields

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Hence, we have reached to the conclusion that

$$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\times\phi\right)=\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}\left(\left[\frac{(f_n)}{(\omega_n)}\right]\right)\otimes\phi.$$

Theorem 10. *$\breve{\boldsymbol{f}}^{\alpha}_{\beta,s}:\beta_1\left(\boldsymbol{l}^1_\alpha,(\boldsymbol{\kappa},\times),\times,\Delta\right)\to\beta_2\left(\boldsymbol{l}^1,(\boldsymbol{\kappa},\times),\otimes,\Delta\right)$ is continuous with respect to δ and $\Delta-$ convergence.*

Proof of this theorem follows from similar technique to that followed below in the citations.

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