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J.-L. Wu (Macau Univ. Sci. and Technology and Mudanjiang Normal Univ., China)

BOUNDEDNESS FOR RIESZ-TYPE POTENTIAL OPERATORS ON VARIABLE EXPONENT HERZ – MORREY SPACES*

ОБМЕЖЕНІСТЬ ПОТЕНЦІАЛЬНИХ ОПЕРАТОРІВ ТИПУ РІСА НА ПРОСТОРАХ ХЕРЦА – МОРРЕЯ ЗІ ЗМІННИМ ПОКАЗНИКОМ

We show the boundedness of the Riesz-type potential operator of variable order $\beta(x)$ from the variable exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$ into the weighted space $M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n,\omega)$, where $\alpha(x)\in L^\infty(\mathbb{R}^n)$ is log-Hölder continuous both at the origin and at infinity, $\omega=(1+|x|)^{-\gamma(x)}$ with some $\gamma(x)>0$, and $1/q_1(x)-1/q_2(x)=\beta(x)/n$ when $q_1(x)$ is not necessarily constant at infinity. It is assumed that the exponent $q_1(x)$ satisfies the logarithmic continuity condition both locally and at infinity and $1<(q_1)_\infty\le q_1(x)\le(q_1)_+<\infty,\ x\in\mathbb{R}^n$.

Встановлено обмеженість потенціального оператора типу Ріса змінного порядку $\beta(x)$, що діє з просторів Херца–Моррея зі змінним показником $M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$ у зважений простір $M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n,\omega)$, де $\alpha(x)\in L^\infty(\mathbb{R}^n)$ є log-Гельдер неперервним як на початку координат, так і на нескінченності, $\omega=(1+|x|)^{-\gamma(x)}$ з деяким $\gamma(x)>0$ та $1/q_1(x)-1/q_2(x)=\beta(x)/n$, коли $q_1(x)$ не обов'язково є сталою на нескінченності. Вважаємо, що показник $q_1(x)$ задовольняє умову логарифмічної неперервності як локально, так і на нескінченності та $1<(q_1)_\infty\le q_1(x)\le$ $\le(q_1)_+<\infty,\ x\in\mathbb{R}^n$.

1. Introduction. Last decade, there is an evident increase of investigations related to both the theory of the variable exponent function spaces and the operator theory in these spaces. This is caused with keen interest not in real analysis but also in partial differential equations and in applied mathematics, because they are applicable to the modeling for electrorheological fluids, mechanics of the continuum medium and image restoration (see, for example, [1 – 7] and references therein) etc.

The theory of function spaces with variable exponent has rapidly made progress in the past twenty years since some elementary properties were established by Kováčik and Rákosník [8]. One of the main problems on the theory is the boundedness of the Hardy – Littlewood maximal operator on variable Lebesgue spaces.

In 2012, Almeida and Drihem [9] discuss the boundedness of a wide class of sublinear operators on Herz spaces $K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponent $\alpha(\cdot)$ and $q(\cdot)$. Meanwhile, they also established Hardy – Littlewood – Sobolev theorems for fractional integrals on variable Herz spaces. In 2013, Samko [10, 11] introduced a new Herz type function spaces with variable exponent, where all the three parameters are variable, and proved the boundedness of some sublinear operators. And in 2015, Rafeiro and Samko [12] considered the validity of Sobolev type theorem for the Riesz potential operator in continual variable exponents Herz spaces. In recently, Wu [13, 14] also considers the boundedness for fractional Hardy-type operator and Riesz-type potential operator on Herz – Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ but fixed $\alpha\in\mathbb{R}$ and $p\in(0,\infty)$.

Motivated by the above results, and based on some facts in [9, 15], the author will investigate mapping properties of the operator $I_{\beta(\cdot)}$ within the framework of the variable exponent Herz – Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, where the Riesz-type potential operator of variable order

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$$I_{\beta(\cdot)}(f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta(x)}} dy, \quad 0 < \beta(x) < n.$$

2. Preliminaries. In this section, we define some function spaces with variable exponent, and give basic properties and useful lemmas. Throughout this paper we will use the following notation:

denote by $|S|$ the Lebesgue measure and by χ_S the characteristic function for a measurable set $S \subset \mathbb{R}^n$;

f_S denotes the mean value of f on measurable set S, namely $f_S := \frac{1}{|S|} \int_S f(x)dx$;

$B(x,r)$ is the ball cenetered at x and of radius r; $B_0 = B(0,1)$;

C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line;

for any exponent $1 < q(x) < \infty$, we denote by $q'(x)$ its conjugate exponent, namely, $1/q(x) + + 1/q'(x) = 1$.

2.1. Function spaces with variable exponent. Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first define Lebesgue spaces with variable exponent.

Definition 2.1. *Let $q(\cdot): \Omega \to (1,\infty)$ be a measurable function.*

(i) *The variable Lebesgue spaces $L^{q(\cdot)}(\Omega)$ is defined by*

$$L^{q(\cdot)}(\Omega) = \{f \textit{ is measurable function: } F_q(f/\eta) < \infty \textit{ for some constant } \eta > 0\},$$

where $F_q(f) := \int_\Omega |f(x)|^{q(x)}dx$. The Lebesgue space $L^{q(\cdot)}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : F_q(f/\eta) = \int\limits_\Omega \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx \le 1 \right\}.$$

(ii) *The space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by*

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega) = \{f \textit{ is measurable function: } f \in L^{q(\cdot)}(\Omega_0) \textit{ for all compact subsets } \Omega_0 \subset \Omega\}.$$

(iii) *The weighted Lebesgue space $L^{q(\cdot)}_\omega(\Omega)$ is defined by as the set of all measurable functions for which*

$$\|f\|_{L^{q(\cdot)}_\omega(\Omega)} = \|\omega^{1/q(\cdot)} f\|_{L^{q(\cdot)}(\Omega)} < \infty.$$

Next we define some classes of variable exponent functions. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int\limits_{B(x,r)} |f(y)|dy \quad \forall\, x \in \mathbb{R}^n,$$

where and what follows $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Definition 2.2. *Given a measurable function $q(\cdot)$ defined on $\mathbb{R}^n$, we write*

$$q_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q(x), \qquad q_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x).$$

(i) $q'_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q'(x) = \dfrac{q_+}{q_+ - 1}$, $q'_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q'(x) = \dfrac{q_-}{q_- - 1}$.

(ii) *Denote by* $\mathscr{P}_0(\mathbb{R}^n)$ *the set of all measurable functions* $q(\cdot)\colon \mathbb{R}^n \to (0,\infty)$ *such that* $0 < q_- \le q(x) \le q_+ < \infty$.

(iii) *Denote by* $\mathscr{P}(\mathbb{R}^n)$ *the set of all measurable functions* $q(\cdot)\colon \mathbb{R}^n \to (1,\infty)$ *such that* $1 < q_- \le q(x) \le q_+ < \infty$.

(iv) $\mathscr{B}(\Omega) = \big\{q(\cdot) \in \mathscr{P}(\mathbb{R}^n)\colon$ *the maximal operator* M *is bounded on* $L^{q(\cdot)}(\Omega)\big\}$.

Definition 2.3. *Let* $q(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ *be a real-valued function.*

(1) *Denote by* $\mathscr{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ *the set of all local* log*-Hölder continuous functions* $q(\cdot)$ *which satisfies*

$$|q(x) - q(y)| \le \frac{-C}{\ln(|x-y|)}, \qquad |x-y| \le 1/2, \quad x, y \in \mathbb{R}^n.$$

(2) *Denote by* $\mathscr{C}^{\log}_0(\mathbb{R}^n)$ *the set of all* log*-Hölder continuous functions* $q(\cdot)$ *at origin satisfies*

$$|q(x) - q(0)| \le \frac{C}{\ln\left(e + \dfrac{1}{|x|}\right)}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

(3) *Denote by* $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ *the set of all* log*-Hölder continuous functions* $q(\cdot)$ *at infinity satisfies*

$$\big|q(x) - q_\infty\big| \le \frac{C_\infty}{\ln(e+|x|)}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $q_\infty = \lim_{|x|\to\infty} q(x)$.

(4) *Denote by* $\mathscr{C}^{\log}(\mathbb{R}^n) := \mathscr{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ *the set of all global* log*-Hölder continuous functions* $q(\cdot)$.

***Remark* 2.1.** The $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ condition is equivalent to the uniform continuity condition

$$|q(x) - q(y)| \le \frac{C}{\ln(e+|x|)}, \qquad |y| \ge |x|, \quad x, y \in \mathbb{R}^n.$$

The $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ condition was originally defined in this form in [16].

Now, we define variable exponent Herz–Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$. Let $B_k = \{x \in \mathbb{R}^n\colon |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

Definition 2.4. *Suppose that* $0 \le \lambda < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ *and* $\alpha(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. *The variable exponent Herz–Morrey space* $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ *is definded by*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \Big\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})\colon \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty\Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}.$$

Compare the variable Herz – Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with the variable Herz space [9] $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$, where

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \sum_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}.$$

Obviously, $M\dot{K}^{\alpha(\cdot),0}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$. When $\alpha(\cdot)$ is constant, we have $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ (see [13]). If both $\alpha(\cdot)$ and $q(\cdot)$ are constants, and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$ are classical Herz spaces.

2.2. Recent results for Riesz-type potential $I_{\beta(\cdot)}$. In this subsection we recall some recent results for Riesz-type potential operator $I_{\beta(\cdot)}$. The order $\beta(x)$ of the potential is not assumed to be continuous. We assume that it is a measurable function on Ω satisfying the following assumptions:

$$\begin{gathered} \beta_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \beta(x) > 0, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)\beta(x) < n, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p_\infty \beta(x) < n. \end{gathered} \tag{2.3}$$

The open problem, the boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ from the variable exponent space $L^{p(\cdot)}(\mathbb{R}^n)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$ with the limiting Sobolev exponent $\dfrac{1}{q(x)} = \dfrac{1}{p(x)} - \dfrac{\beta(x)}{n}$, was first solved in the case of bounded domains $\Omega \subset \mathbb{R}^n$(see [17]). After Diening [18] proved the boundedness of the maximal operator over bounded domains, the validity of the Sobolev theorem for bounded domains became an unconditional statement.

In 2008, in the case of bounded sets, Almeida, Hasanov and Samko [19] proved the boundedness of the maximal operator in variable exponent Morrey spaces, and in 2009, Hästö [20] used his new "local-to-global" approach to extend the result of [19] about the maximal operator to the whole space $\mathbb{R}^n$. In 2010, in the case of bounded sets, Guliyev, Hasanov and Samko [21] considered the boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ on the generalized variable exponent Morrey type spaces.

For the whole space $\mathbb{R}^n$, under the condition that the exponent $p(x)$ is constant outside some ball of large radius, the Sobolev theorem was proved by Diening [22].

Another version of the Sobolev theorem for the space $\mathbb{R}^n$ was proved in [23] for the exponents $p(x)$ not necessarily constant in a neigbourhood of infinity, but with some extra power weight fixed to infinity and under the assumption that $p(x)$ takes its minimal value at infinity.

Theorem A. *Let $\beta(x)$ meet conditions* (2.3) *which $q_1(\cdot)$ instead of $p(\cdot)$. Suppose that $q_1(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ and*

$$1 < (q_1)_\infty \le q_1(x) \le (q_1)_+ < \infty. \tag{2.4}$$

Then the following weighted Sobolev-type estimate is valid for the operator $I_{\beta(\cdot)}$:

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)},$$

where $q_2(x)$ *is defined by*

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\beta(x)}{n}, \tag{2.5}$$

and

$$\gamma(x) = C_\infty \beta(x)\left(1 - \frac{\beta(x)}{n}\right) \le \frac{n}{4} C_\infty, \tag{2.6}$$

C_∞ *being the Dini – Lipschitz constant from* (2.2) *which* $q(\cdot)$ *is replaced by* $q_1(\cdot)$.

In 2013, in the case of unbounded sets, Guliyev and Samko [24] considered the boundedness of the Riesz-type potential operator $I_{\beta(\cdot)}$ on the generalized variable exponent Morrey type spaces. And recently, the author [14] obtain the similar results of Theorem A on the variable exponent Herz – Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$.

***Remark* 2.2.** The fractional maximal operator is defined as

$$M_{\beta(\cdot)}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{n-\beta(x)}} \int\limits_{B(x,r)} |f(y)|\, dy. \tag{2.7}$$

The pointwise estimate for (2.7) is also valid which yields Theorem A.

2.3. Auxiliary propositions and lemmas. In this subsection we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

Proposition 2.1. *Let* $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

(i) *If* $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, *then we have* $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

(ii) $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ *if and only if* $q'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

The first part in Proposition 2.1 is independently due to Cruz – Uribe et al. [16] and to Nekvinda [25] respectively. The second of Proposition 2.1 belongs to Diening [26] (see Theorem 8.1 or Theorem 1.2 in [27]).

***Remark* 2.3.** Since

$$\left|q'(x) - q'(y)\right| \le \frac{|q(x) - q(y)|}{(q_- - 1)^2},$$

it follows at once that if $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then so does $q'(\cdot)$, i.e., if the condition hold, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$ and $L^{q'(\cdot)}(\mathbb{R}^n)$. Furthermore, Diening has proved general results on Musielak – Orlicz spaces.

The next proposition is the generalization of variable exponents Herz spaces in [9], and it was proved in [15].

Proposition 2.2. *Let* $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $p \in (0,\infty)$, *and* $\lambda \in [0,\infty)$. *If real-valued function* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathscr{C}^{\log}_0(\mathbb{R}^n) \cap \mathscr{C}^{\log}_\infty(\mathbb{R}^n)$, *then*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p} \approx$$

$$\approx \max\left\{ \sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{\tilde{k}_1} 2^{k\alpha(0)p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}, \right.$$

$$\sup_{\substack{k_0\ge 0\\ k_0\in\mathbb{Z}}}\left(2^{-k_0\lambda}\left(\sum_{k=-\infty}^{\tilde{k}_2}2^{k\alpha(0)p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}+\right.$$

$$\left.+\,2^{-k_0\lambda}\left(\sum_{k=0}^{\tilde{k}_3}2^{k\alpha_\infty p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}\right)\Bigg\},$$

where $\tilde{k}_1 = k_0,\ \tilde{k}_2 = -1,\ \tilde{k}_3 = k_0$.

The next lemma is known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [8].

Lemma 2.1 (generalized Hölder's inequality)**.** *Suppose that* $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, *then, for any* $f\in$ $\in L^{q(\cdot)}(\mathbb{R}^n)$ *and any* $g\in L^{q'(\cdot)}(\mathbb{R}^n)$, *we have*

$$\int_{\mathbb{R}^n}\big|f(x)g(x)\big|dx\le C_q\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where $C_q = 1 + 1/q_- - 1/q_+$.

The following lemma can be found in [28].

Lemma 2.2. (I) *Let* $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. *Then there exist positive constants* $\delta\in(0,1)$ *and* $C>0$ *such that*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\le C\left(\frac{|S|}{|B|}\right)^{\delta}$$

for all balls B *in* $\mathbb{R}^n$ *and all measurable subsets* $S\subset B$.

(II) *Let* $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. *Then there exists a positive constant* $C>0$ *such that*

$$C^{-1}\le\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\le C$$

for all balls B *in* $\mathbb{R}^n$.

***Remark* 2.4.** (i) If $q_1(\cdot),\ q_2(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$, then we see that $q_1'(\cdot),\ q_2(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. Hence we can take positive constants $0<\delta_1<1/(q_1')_+,\ 0<\delta_2<1/(q_2)_+$ such that

$$\frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\le C\left(\frac{|S|}{|B|}\right)^{\delta_1},\qquad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}\le C\left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{2.8}$$

hold for all balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$ (see [28, 29]).

(ii) On the other hand, Kopaliani [30] has proved the conclusion: If the exponent $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ equals to a constant outside some large ball, then $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$ if and only if $q(\cdot)$ satisfies the Muckenhoupt type condition

$$\sup_{Q:\ \text{cube}}\frac{1}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)}<\infty.$$

3. Main result and its proof. Our main result can be stated as follows.

Theorem 3.1. *Suppose that* $q_1(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ *and* $\beta(x)$ *meet conditions* (2.3) *which* $q_1(\cdot)$ *instead of* $p(\cdot)$. *Define the variable exponent* $q_2(\cdot)$ *by* (2.5). *Let* $q_1(\cdot)$, $q_2'(\cdot)$ *satisfies condition* (2.4), *and* $0 < p_1 \leq p_2 < \infty$, $\lambda \geq 0$, *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *be log-Hölder continuous both at the origin and at infinity, with* $\lambda - n\delta_2 < \alpha(0) \leq \alpha_\infty < \lambda + n\delta_1$, *where* $\delta_1 \in (0, 1/(q_1')_+)$ *and* $\delta_2 \in (0, 1/(q_2)_+)$ *are the constants appearing in* (2.8). *Then*

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x)$ *is defined as in* (2.6), *and the Dini–Lipschitz constant is* $\max\left\{C_\infty, \dfrac{2C_\infty}{\left((q_1)_- - 1\right)^2}\right\}$ *when the* $q(\cdot)$ *in* (2.2) *is replaced by* $q_1(\cdot)$.

***Remark* 3.1.** (i) Under the assumptions of Theorem 3.1, the similar result of Theorem 3.1 is also valid for the fractional maximal operator $M_{\beta(\cdot)}(f)$ defined by (2.7) (partly detail ref. [14]).

(ii) If $\alpha(\cdot)$ be constant exponent, then the above result can be founded in [14].

(iii) When $\lambda = 0$, the above result is also valid.

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$, if we denote $f_j := f\chi_j = f\chi_{A_j}$ for each $j \in \mathbb{Z}$, then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Because of $0 < p_1/p_2 \leq 1$, applying inequality

$$\left(\sum_{i=-\infty}^{\infty} |a_i|\right)^{p_1/p_2} \leq \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2}, \tag{3.1}$$

and Proposition 2.2, we obtain

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\right\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \approx$$

$$\approx \max\left\{\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_2}\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1/p_2},\right.$$

$$\sup_{\substack{k_0\geq 0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left[\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_2}\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1/p_2} +\right.$$

$$\left.\left.+\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_2}\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1/p_2}\right]\right\} \leq$$

$$\leq \max\left\{\sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right),\right.$$

$$\sup_{\substack{k_0\geq 0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left[\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)+\right.$$

$$\left.+\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1}\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)\right]\Bigg\} \equiv: \max\{E_1, E_2+E_3\},$$

where

$$E_1 = \sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right),$$

$$E_2 = \sup_{\substack{k_0\geq 0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right),$$

$$E_3 = \sup_{\substack{k_0\geq 0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty p_1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f)\chi_k\right\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right).$$

It is not difficult to found that the estimate of E_1 is analogous to that of E_2, therefore, the estimates for E_1 and E_3 will be considered here.

To E_1, we have

$$E_1 \leq C\sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=-\infty}^{k-2}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)+$$

$$+C\sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=k-1}^{k+1}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)+$$

$$+C\sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=k+2}^{\infty}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)\equiv$$

$$\equiv: C(E_{11}+E_{12}+E_{13}).$$

First we estimate E_{12}. Using Theorem A and Proposition 2.2, we get

$$E_{12} \leq C\sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\left(\sum_{j=k-1}^{k+1}\|f_j\chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right)\leq$$

$$\leq C\sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|f\chi_k\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)\leq C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

For E_{11}. Note that when $x\in A_k$, $j\leq k-2$, and $y\in A_j$, then $|x-y|\backsim|x|$, $2|y|\leq|x|$. Therefore, using the generalized Hölder's inequality, we obtain

$$\left|I_{\beta(\cdot)}(f_j)(x)\chi_k(x)\right| \leq \int\limits_{A_j} \frac{|f(y)|}{|x-y|^{n-\beta(x)}} dy \chi_k(x) \leq$$

$$\leq C \cdot 2^{-kn} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |x|^{\beta(x)} \chi_k(x). \tag{3.2}$$

Notice that the fact

$$I_{\beta(\cdot)}(\chi_{B_k})(x) \geq I_{\beta(\cdot)}(\chi_{B_k})(x)\chi_{B_k}(x) \geq C|x|^{\beta(x)}\chi_k(x). \tag{3.3}$$

Using Theorem A, Lemma 2.2, (2.8), (3.2) and (3.3), we have

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j)\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq$$

$$\leq C \cdot 2^{-kn} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(\chi_{B_k})\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq$$

$$\leq C \cdot 2^{-kn} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq C \cdot 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.4}$$

On the other hand, note the following fact:
***Case* I** ($\tilde{k}_i < 0, \ i = 1,2,3$)**:**

$$\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \leq 2^{-j\alpha(0)} \left(\sum_{i=-\infty}^{j} 2^{i\alpha(0)p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{1/p_1} \leq$$

$$\leq 2^{j(\lambda-\alpha(0))} \left(2^{-j\lambda} \left(\sum_{i=-\infty}^{j} \|2^{i\alpha(\cdot)} f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{1/p_1}\right) \leq C \cdot 2^{j(\lambda-\alpha(0))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.5}$$

***Case* II** ($\tilde{k}_i \geq 0, \ i = 1,2,3$)**:**

$$\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \leq 2^{-j\alpha_\infty} \left(\sum_{i=0}^{j} 2^{i\alpha_\infty p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{1/p_1} \leq$$

$$\leq C \cdot 2^{j(\lambda-\alpha_\infty)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.6}$$

Definition 2.4, Proposition 2.2 and the condition of $\alpha(\cdot)$ are used in above facts.
Thus, combining (3.4) and (3.5), and using $\alpha(0) \leq \alpha_\infty < \lambda + n\delta_1$, it follows that

$$E_{11} \leq C \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\right) \leq$$

$$\leq C \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \sup_{\substack{k_0<0\\k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1}\right) \leq C \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Now, let us turn to estimate for E_{13}. Note that when $x \in A_k$, $j \geq k+2$, and $y \in A_j$, then $|x-y| \backsim |y|, 2|x| \leq |y|$. Therefore, using the generalized Hölder's inequality, we have

$$\left|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j)(x)\chi_k(x)\right| \leq (1+|x|)^{-\gamma(x)} \int\limits_{A_j} \frac{|f(y)|}{|x-y|^{n-\beta(x)}}\, dy\chi_k(x) \leq$$

$$\leq C \int_{A_j} |f(y)|(1+|x|)^{-\gamma(x)}|y|^{\beta(x)-n}\, dy\chi_k(x) \leq$$

$$\leq C \cdot 2^{-jn} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)}\chi_j(\cdot)\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \chi_k(x). \tag{3.7}$$

Similar to (3.3), we get

$$I_{\beta(\cdot)}(\chi_{B_j})(x) \geq I_{\beta(\cdot)}(\chi_{B_j})(x)\chi_{B_j}(x) \geq C|x|^{\beta(x)}\chi_j(x). \tag{3.8}$$

Using Theorem A, Lemma 2.2, (2.8), (3.7) and (3.8), we obtain

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f_j)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq$$

$$\leq C \cdot 2^{-jn} \|\chi_{B_j}\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C \cdot 2^{(k-j)n\delta_2} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \tag{3.9}$$

Therefore, combining (3.5) and (3.9), and using $\lambda - n\delta_2 < \alpha(0) \leq \alpha_\infty$, it follows that

$$E_{13} \leq C \sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_2} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \leq$$

$$\leq C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \sup_{\substack{k_0<0\\ k_0\in\mathbb{Z}}} 2^{-k_0\lambda p_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \right) \leq C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_{11}, E_{12} and E_{13} yields

$$E_1 \leq C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

For E_3, similar to the estimate of E_1, using Theorem A, Proposition 2.2, (2.8), (3.1) – (3.4), (3.6) – (3.9), we have

$$E_3 \leq C\|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Joint the estimate for E_1, E_2 and E_3 yields

$$\left\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Theorem 3.1 is proved.

References

1. *Růžička M.* Electrorheological fluids: modeling and mathematical theory // Lect. Notes Math. – 2000. – **1748**.
2. *Diening L., Růžička M.* Calderón – Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics // J. reine und angew. Math. – 2003. – **563**. – S. 197 – 220.
3. *Levine S., Chen Y., Stanich J.* Image restoration via nonstandard diffusion // Techn. Rept. № 04-01. – Dept. Math. and Comput. Sci., Duquesne Univ., 2004.

4. *Chen Y., Levine S., Rao M.* Variable exponent, linear growth functionals in image restoration // SIAM J. Appl. Math. – 2006. – **66**, № 4. – P. 1383 – 1406.
5. *Li F., Li Z., Pi L.* Variable exponent functionals in image restoration // Appl. Math. and Comput. – 2010. – **216**, № 3. – P. 870 – 882.
6. *Harjulehto P., Hästö P., Latvala V. et al.* Critical variable exponent functionals in image restoration // Appl. Math. Lett. – 2013. – **26**, № 1. – P. 56 – 60.
7. *Sakharkar M., Sarde M.* Image denoising based on variable exponent using adaptive perona malik model // Int. J. Eng. Sci. Invent. – 2014. – **3**, № 6. – P. 34 – 37.
8. *Kováčik O., Rákosník J.* On spaces $L^{p(x)}$ and $W^{k,p(x)}$ // Czechoslovak Math. J. – 1991. – **41**, № 4. – P. 592 – 618.
9. *Almeida A., Drihem D.* Maximal, potential and singular type operators on Herz spaces with variable exponents // J. Math. Anal. and Appl. – 2012. – **39**, № 2. – P. 781 – 795.
10. *Samko S.* Variable exponent Herz spaces // Mediterr. J. Math. – 2013. – **10**, № 4. – P. 2007 – 2025.
11. *Samko S.* Erratum to “Variable exponent Herz spaces" // Mediterr. J. Math. – 2013. – **10**, № 4. – P. 2027 – 2030.
12. *Rafeiro H., Samko S.* Riesz potential operator in continual variable exponents Herz spaces // Math. Nachr. – 2015. – **288**, № 4. – S. 465 – 475.
13. *Wu J. L.* Boundedness for fractional Hardy-type operator on Herz – Morrey spaces with variable exponent // Bull. Korean Math. Soc. – 2014. – **51**, № 2. – P. 423 – 435.
14. *Wu J. L.* Boundedness for Riesz-type potential operators on Herz – Morrey spaces with variable exponent // Math. Inequal. and Appl. – 2015. – **18**, № 2. – P. 471 – 484.
15. *Lu Y., Zhu Y. P.* Boundedness of multilinear Calderón – Zygmund singular operators on Morrey – Herz spaces with variable exponents // Math. Sinica (Engl. Ser.). – 2014. – **30**, № 7. – P. 1180 – 1194.
16. *Cruz-Uribe D., Fiorenza A., Neugebauer C.* The maximal function on variable L^p spaces // Ann. Acad. Sci. Fenn. Math. – 2003. – **28**, № 1. – P. 223 – 238.
17. *Samko S.* Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$ // Integral Transforms and Spec. Funct. – 1998. – **7**, № 3-4. – P. 261 – 284.
18. *Diening L.* Maximal functions on generalized Lebesgue spaces $L^{p(\cdot)}$ // Math. Inequal. and Appl. – 2004. – **7**, № 2. – P. 245 – 253.
19. *Almeida A., Hasanov J., Samko S.* Maximal and potential operators in variable exponent Morrey spaces // Georg. Math. J. – 2008. – **15**, № 2. – P. 195 – 208.
20. *Hästö P.* Local-to-global results in variable exponent spaces // Math. Res. Lett. – 2009. – **16**, № 2. – P. 263 – 278.
21. *Guliyev V., Hasanov J., Samko S.* Boundedness of maximal, potential type, and singular integral operators in the generalized variable exponent Morrey type spaces // J. Math. Sci. – 2010. – **170**, № 4. – P. 423 – 443.
22. *Diening L.* Riesz potential and Sobolev embeddings on generalized Lebesgue spaces and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$ // Math. Nachr. – 2004. – **268**. – S. 31 – 43.
23. *Kokilashvili V., Samko S.* On Sobolev theorem for Riesz type potentials in the Lebesgue spaces with variable exponent // Z. Anal. und Anwend. – 2003. – **22**, № 4. – S. 899 – 910.
24. *Guliyev V., Samko S.* Maximal, potential and singular operators in the generalized variable exponent Morrey spaces on unbounded sets // J. Math. Sci. – 2013. – **193**, № 2. – P. 228 – 248.
25. *Nekvinda A.* Hardy – Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$ // Math. Inequal. and Appl. – 2004. – **7**. – P. 255 – 265.
26. *Diening L.* Maximal functions on Musielak – Orlicz spaces and generalized Lebesgue spaces // Bull. Sci. Math. – 2005. – **129**. – P. 657 – 700.
27. *Cruz-Uribe D., Fiorenza A., Martell J., Pérez C.* The boundedness of classical operators on variable L^p spaces // Ann. Acad. Sci. Fenn. Math. – 2006. – **31**. – P. 239 – 264.
28. *Izuki M.* Fractional integrals on Herz – Morrey spaces with variavle exponent // Hiroshima Math. J. – 2010. – **40**. – P. 343 – 355.
29. *Zhang P., Wu J. L.* Boundedness of commutators of the fractional Hardy operators on Herz – Morrey spaces with variable exponent // Adv. Math. (China). – 2014. – **43**, № 4. – P. 581 – 589.
30. *Kopaliani T.* Infimal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable L^p spaces // Arch. Math. – 2007. – **89**, № 2. – S. 185 – 192.

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