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POINTS OF UPPER AND LOWER SEMICONTINUITY OF MULTIVALUED FUNCTIONS*

ТОЧКИ ВЕРХНЬОЇ ТА НИЖНЬОЇ НАПІВНЕПЕРЕРВНОСТІ ДЛЯ БАГАТОЗНАЧНИХ ФУНКЦІЙ

We investigate joint upper and lower semicontinuity of two-variable set-valued functions. More precisely. among other results, we show that, under certain conditions, a two-variable lower horizontally quasicontinuous mapping $F: X \times Y \to \to \mathcal{K}(Z)$ is jointly upper semicontinuous on sets of the from $D \times \{y_0\}$, where D is a dense G_δ subset of X and $y_0 \in Y$. A similar result is obtained for the joint lower semicontinuity of upper horizontally quasicontinuous mappings. These results improve some known results on the joint continuity of single-valued functions.

Вивчається спільна верхня та нижня напівнеперервність для багатозначних функцій двох змінних. Більш точно, серед інших результатів показано, що за деяких умов нижньо горизонтально квазінеперервне відображення від двох змінних $F: X \times Y \to \mathcal{K}(Z)$ є спільно верхньо напівнеперервним на множинах з $D \times \{y_0\}$, де D — щільна G_δ підмножина X та $y_0 \in Y$. Подібний результат отримано також для спільної нижньої напівнеперервності верхньо горизонтальних квазінеперервних відображень. Ці результати покращують деякі відомі результати про спільну неперервність однозначних функцій.

1. Introduction and preliminaries. Throughout the paper, we will assume that all topological spaces are T_1. For a topological space Z, we denote by $\mathcal{P}(Z)$, $\mathcal{C}(Z)$ and $\mathcal{K}(Z)$ the set of all nonempty subsets, the set of all nonempty closed subsets and the set of all nonempty compact subsets of Z respectively. If $F: X \to \mathcal{P}(Z)$ is a set-valued function. For a subset G of Z, we define $F^+(G)$ and $F^-(G)$ as follows:

$$F^+(G) = \{x \in X : F(x) \subseteq G\}, \qquad F^-(G) = \{x \in X : F(x) \cap G \neq \varnothing\}.$$

The function F is called:

(a) *upper* (resp. *lower*) *semicontinuous* if for every open subset G of Z, $F^+(G)$ (resp. $F^-(G)$) is an open subset of X.

(b) *upper* (resp. *lower*) *quasicontinuous* at $x_0 \in X$ if for any open set G, with $x_0 \in F^+(G)$ (resp. $x_0 \in F^-(G)$) and any neighborhood U of x_0, there exists a nonempty open set $V \subseteq U$ such that $V \subseteq F^+(G)$ (resp. $V \subseteq F^-(G)$).

(c) categorically upper (resp. lower) quasicontinuous at $x_0 \in X$, if for each neighborhood U of x_0 and neighborhood G containing $F(x_0)$, there exists a set $A \subseteq U$ of the second category such that $F(a) \subseteq G$ (resp. $F(a) \cap G \neq \varnothing$) for all $a \in A$.

Let Z be a topological space and $\{\mathcal{G}_n\}$ be a sequence of open covers of Z. For every $z \in Z$ and $n \in \mathbb{N}$, let $St(z, \mathcal{G}_n) = \cup\{G \in \mathcal{G}_n : z \in \mathcal{G}_n\}$. The sequence $\{\mathcal{G}_n\}$ is called:

(a) *A development*, if for every $z \in Z$, the sequence $\{St(z, \mathcal{G}_n)\}$ is a base at z. A space with a developable space is called *a developable space*. A regular developable space is called a *Moore space*.

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(b) *A weak k-development* if for every compact subset $K \subset Z$ and $\{\mathcal{H}_n\}$ such that for every n, $\mathcal{H}_n \subseteq \mathcal{G}_n$, $\mathcal{H}_n$ is finite, $K \cap H \neq \varnothing$ for every $H \in \mathcal{H}_n$ and $K \subseteq \bigcup \mathcal{H}_n$, we have that the sequence $\left\{ \bigcap_{i \le n} \left(\bigcup \mathcal{H}_i \right) \right\}$ is a base at K. A space with a weak k-development is called a *weakly k-developable space*.

We refer the reader to [1] for more information about k-developable topological spaces.

The aim of this paper is to obtain points of joint upper and lower semicontinuity of two variable set-valued functions. One of the first results in this direction is the following.

Theorem 1.1 [4]. *Let X be a Baire space, Y a second countable space, Z be a metrizable space and $F\colon X \times Y \to \mathcal{K}(Z)$ be a compact-valued mapping which is lower semicontinuous with respect to the first variable and upper semicontinuous with respect to the second variable. Then there exists a dense in X G_δ-set $A \subseteq X$ such that F is jointly upper semicontinuous at every point of the set $A \times Y$.*

Let X and Y be topological spaces and let $f\colon X \to Y$ be a function. The function f is called quasicontinuous if for each $x \in X$ and neighborhoods U of x and V of $f(y)$, there is a nonempty open subset U' of U such that $f(U') \subseteq V$.

The notion of quasicontinuity turned out to be a useful tool in some mathematical problems (see, e.g., [2, 3, 10, 12, 13, 17, 18, 20, 21]).

In 1975, V. Popa [22] generalized the notion of quasicontinuity for set-valued functions. Since then, some authors investigated various types of continuity of set-valued functions [5 – 9].

A function $f\colon X \times Y \to Z$ is said to be *horizontally quasicontinuous* at $(x, y) \in X \times Y$ if for any neighborhoods U, V and W of x, y and $f(x, y)$ respectively, there is a nonempty open subset $U' \subset$ of X and $y' \in V$ such that $f(U' \times \{y'\}) \subseteq W$. In [14 – 16] some properties of horizontally quasicontinuous functions are investigated. In Section 2, we will define upper and lower horizontal quasicontinuity for two-variable set-valued functions. We will show that if F is a lower horizontally quasicontinuous function, $F_x\colon Y \to \mathcal{K}(Z)$ is upper semicontinuous and $F^y\colon X \to \mathcal{K}(Z)$ is categorically upper quasicontinuous for each $(x, y) \in X \times Y$. Then there is a dense G_δ subset D of X such that F is jointly upper semicontinuous on $D \times \{y_0\}$ provided that X is Baire, $y_0 \in Y$ has a countable base and Z is a normal weakly k-developable space.

Finally, in Section 3, we will investigate lower semicontinuity of a two variable set-valued function $F\colon X \times Y \to \mathcal{C}(Z)$. In fact, by using a different method from what we used in Section 2, we will prove a similar result for upper horizontally quasicontinuous functions. In fact, we will show that if $F\colon X \times Y \to \mathcal{C}(Z)$ is an upper horizontally quasicontinuous function such that $F_x\colon Y \to \mathcal{C}(Z)$ is lower semicontinuous and $F^y\colon X \to \mathcal{C}(Z)$ is categorically lower quasicontinuous for each $(x, y) \in X \times Y$. Then F is symmetrically lower quasicontinuous at each point of $X \times \{y_0\}$ provided that X is Baire, $y_0 \in Y$ has a countable base and Z is regular. Our results can be considered as generalizations of corresponding results in [5, 11, 15, 16].

2. Upper semicontinuity of two variable set-valued functions. In this section, we will show that under some circumstances, a two-variable function set-valued function $F\colon X \times Y \to \mathcal{K}(Z)$, for each $y_0 \in Y$, is jointly upper-semicontinuous on a set of the form $D \times \{y_0\}$, where D is a dense G_δ subset of X.

We begin with the following definitions.

Definition 2.1. *Let* $F\colon X\times Y\to\mathcal{P}(Z)$ *be a function and* $(x_0,y_0)\in X\times Y$. *The function* F *is called*:

(a) *symmetrically upper* (*resp. lower*) *quasicontinuous at* (x_0,y_0) *with respect to the second variable if for each nonempty open subset* W *of* Z *with* $(x_0,y_0)\in F^+(W)$ (*resp.* $(x_0,y_0)\in F^-(W)$) *and a neighborhood* U *of* x_0, *there are a nonempty open subset* U' *of* U *and a neighborhood* V *of* y_0 *such that* $U'\times V\subseteq F^+(W)$ (*resp.* $U'\times V\subseteq F^-(W)$);

(b) *upper* (*resp. lower*) *horizontally quasicontinuous at* (x_0,y_0) *if whenever* U, V *and* W *are open subsets* X, Y *and* Z *respectively with* $(x_0,y_0)\in U\times V$ *and* $(x_0,y_0)\in F^+(W)$ (*resp.* $(x_0,y_0)\in F^-(W)$), *there are a nonempty open subset* U' *of* U *and* $y'\in V$ *such that* $(x,y')\in F^+(W)$ (*resp.* $(x,y')\in F^-(W)$) *for each* $x\in U'$.

Now, we are ready to state one of the main results of this section.

Theorem 2.1. *Let* X *be a Baire space,* Y *a space and* Z *a space with a weakly* k*-development. Let* $F\colon X\times Y\to\mathcal{K}(Z)$ *be symmetrically upper quasicontinuous with respect to the second variable at each point of* $X\times\{y_0\}$ *for some* $y_0\in Y$. *Then there is a dense* G_δ *subset* D *of* X *such that* F *is jointly upper semicontinuous at each point of* $D\times\{y_0\}$.

Proof. Let $\{\mathcal{G}_n : n\in\mathbb{N}\}$ be a k-weak development for Z. For each natural number n, let D_n be the set of all $x\in X$ such that for some finite subset $\mathcal{H}$ of $\mathcal{G}_n$ and neighborhoods U of x, V of y_0 respectively, we have $U\times V\subseteq F^+(\bigcup\mathcal{H})$ and $F(t,y_0)\cap H\neq\varnothing$ for each $H\in\mathcal{H}$ and $t\in U$.

By the definition, each D_n is open in X. We will show that each D_n is dense in X. Let n be a fixed natural number and U be a nonempty open subset of X.

We say that the property $p(k)$, $k\in\mathbb{N}$, holds if the following statement is true.

If there is a nonempty open subset U' of U such that $F(U',y_0)\subseteq\bigcup\mathcal{H}$, where $\mathcal{H}\subseteq\mathcal{G}_n$ with $|\mathcal{H}|\le k$, then D_n contains a nonempty open subset of U.

Let U' be a nonempty open subset of U such that $F(U',y_0)\subseteq H$ for some $H\in\mathcal{G}_n$. By upper symmetrical quasicontinuity of F on $U'\times\{y_0\}$, we can find a nonempty open subset U'' of U' and a neighborhood V of y_0 such that $F(x,y)\subseteq H$ for each $(x,y)\in U''\times V$. Therefore $U''\subseteq D_n$. Thus $p(1)$ is true.

Let $p(k)$ hold for some k and U' be a nonempty open subset of U such that $F(U',y_0)\subseteq\bigcup\mathcal{H}$ for some subset $\mathcal{H}$ of $\mathcal{G}_n$ with $|\mathcal{H}|=k+1$. If for some $x_0\in U'$, there is a proper subset of $\mathcal{H}$, say $\mathcal{H}'$, such that $F(x_0,y_0)\subseteq\bigcup\mathcal{H}'$, then $|\mathcal{H}'|\le k$ and by symmetrically upper quasicontinuity of F, we can find a nonempty open subset U'' of U' such that $F(U'',y_0)\subseteq\bigcup\mathcal{H}'$. Therefore, by our hypothesis, D_n contains a nonempty open subset of U. So that we may assume that $\mathcal{H}$ has exactly $k+1$ elements say $H_1,\ldots,H_{k+1}$ and $F(x,y_0)\cap H_i\neq\varnothing$ for each $x\in U'$ and $1\le i\le k+1$. By symmetrical upper quasicontinuity of F on $U'\times\{y_0\}$, we can find a nonempty open subset U'' of U' and a neighborhoods V of y_0 such that $F(U'',V)\subseteq\bigcup\mathcal{H}$. Then

$$F(U''\times V)\subseteq\bigcup\mathcal{H},\qquad F(x,y_0)\cap H_i\neq\varnothing\quad\forall x\in U''.$$

Therefore $U''\subseteq D_n$. Hence $p(k)$ holds for all $k\in\mathbb{N}$.

Now, if $x\in U$ and $F(x,y_0)\subseteq\bigcup\mathcal{H}$, where $\mathcal{H}$ is a finite subset of $\mathcal{G}_n$. By symmetrically upper quasicontinuity of F at (x,y_0), there is a nonempty open subset U' of U and a neighborhood V of y_0 such that $F(U'\times V)\subseteq\bigcup\mathcal{H}$. According to $p(k)$, where $k=|\mathcal{H}|$, the set D_n contains a nonempty subset of U. This shows that D_n is dense in X. Let $D=\bigcap_{n=1}^\infty D_n$. Let $x\in D$ and W be an open subset of Z with $F(x,y_0)\subseteq W$. By the definition of the sets D_n, for each n, there are neighborhoods U_n of x and V_n of y_0 and a finite subset $\mathcal{H}_n$ of $\mathcal{G}_n$ such that $U_n\times V_n\subseteq F^+(\bigcup\mathcal{H}_n)$

and $F(x, y_0) \cap H \neq \varnothing$ for each $H \in \mathcal{H}_n$. Since Z is weakly k-developable, $\{\bigcap_{i \leq n}(\bigcup \mathcal{H}_i)\}$ is a base at $F(x, y_0)$. Hence there is some $n_0 \in \mathbb{N}$ such that $\bigcap_{i \leq n_0}(\bigcup \mathcal{H}_i) \subseteq W$. Therefore $\big(\bigcap_{i \leq n_0} U_i\big) \times \big(\bigcap_{i \leq n_0} V_i\big) \subseteq F^+(W)$. This proves upper semicontinuity of F at each point of $D \times \{y_0\}$.

In order to obtain an application for Theorem 2.1, we need to the following axillary result.

Lemma 2.1. *Let $F : X \times Y \to \mathcal{P}(Z)$ be a lower horizontally quasicontinuous function. If U and V are open subsets of X and Y respectively such that $U \subseteq \overline{A}$ for some subset A of X. Then $F(U \times V) \subseteq \overline{F(A \times V)}$.*

Proof. Suppose that z is an arbitrary element of $F(U \times V)$ and W is a neighborhood of z. Take some $a \in U$ and $b \in V$ such that $z \in F(a, b)$. By lower horizontal quasicontinuity of F, we can find some $(a_1, b_1) \in U \times V$ and a neighborhood U_1 of a such that $U_1 \subseteq U$ and $F(x, b_1) \cap W \neq \varnothing$ for all $x \in U_1$. Since $U_1 \subseteq U \subseteq \overline{A}$, we have $U_1 \cap A \neq \varnothing$. Let a_0 be an element of $U_1 \cap A$, then

$$\varnothing \neq F(a_0, b_1) \cap W \subseteq F(A \times V) \cap W.$$

This means that $z \in \overline{F(A \times V)}$.

The following result gives a sufficient condition for symmetrically upper quasicontinuity of a lower horizontally quasicontinuous function.

Theorem 2.2. *Let X be a Baire space, Y a space with a countable base at $y_0 \in Y$ and Z a normal space. Let $F : X \times Y \to \mathcal{C}(Z)$ is a lower horizontally quasicontinuous function such that $F_x : Y \to \mathcal{C}(Z)$ is upper semicontinuous for each $x \in X$ and $F^{y_0} : X \to \mathcal{C}(Z)$ is categorically upper quasicontinuous. Then F is symmetrically upper quasicontinuous with respect to the second variable at each point of $X \times \{y_0\}$.*

Proof. Let $x \in X$ and G be a neighborhood of x. Let $F(x, y_0) \subseteq W$ for some open subset W of Z. By normality of Z, there is an open subset W' of W such that

$$F(x, y_0) \subseteq W' \subseteq \overline{W'} \subseteq W.$$

Since $F^{y_0} : X \to \mathcal{C}(Z)$ is categorically upper quasicontinuous at x, there is a subset A of G such that A is of the second category in X and $F(a, y_0) \subseteq W'$ for all $a \in A$.

Let $\{V_n\}$ be a base of neighborhoods of y_0. Define

$$A_n = \{a \in A : F_a(V_n) \subseteq W'\}, \quad n \in \mathbb{N}.$$

It follows from upper semicontinuity of $F_a : Y \to \mathcal{C}(Z)$ for each $a \in A$ that $A = \bigcup_{n=1}^{\infty} A_n$. Since A is of the second category in X, there is some $m \in \mathbb{N}$ such that $U_m = (\overline{A_m})^\circ \neq \varnothing$. Let $U = G \cap U_m$, $V = V_m$ and $A_0 = A_m \cap U$. Since $U_m \subseteq \overline{A_m \cap U_m}$, we have $A_m \cap U_m \neq \varnothing$. The relation

$$\varnothing \neq A_m \cap U_m \subseteq G \cap U_m = U$$

implies that U is a nonempty open subset of G. Since $U \subseteq U \cap \overline{A_m} \subseteq \overline{A_0}$, by Lemma 2.1,

$$F(U \times V) \subseteq \overline{F(A_0 \times V)} \subseteq \overline{W'} \subseteq W.$$

This proves that F is symmetrically upper quasicontinuous at (x, y_0).

The following result follows immediately from Theorems 2.1 and 2.2.

Theorem 2.3. *Let X be a Baire space, Y a space with countable base at $y_0 \in Y$ and Z a normal weakly k-developable space. Let $F: X \times Y \to \mathcal{K}(Z)$ is a lower horizontally quasicontinuous function such that $F_x: Y \to \mathcal{K}(Z)$ is upper semicontinuous for each $x \in X$ and $F^{y_0}: X \to \mathcal{C}(Z)$ is categorically upper quasicontinuous. Then there is a dense G_δ subset D of X such that F is jointly upper semicontinuous at each point of $D \times \{y_0\}$.*

The following example shows that lower horizontal quasicontinuity of F in Theorem 2.3 is necessary.

***Example* 2.1** ([19], Example 1). Let $E = \{(p_n, q_n): n \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{R}^2$ such that if (p_n, q_n) and (p_m, q_m) are two distinct elements of E, then $p_n \neq p_m$ and $q_n \neq q_m$. Define $F: \mathbb{R}^2 \to \mathcal{K}(\mathbb{R})$ by

$$F(x,y) = \begin{cases} \{0\}, & (x,y) \notin E, \\ [0,n], & (x,y) = (p_n, q_n) \in E. \end{cases}$$

It is easy to see that all selection F_x and F^y are upper semicontinuous but F has no point of joint upper semicontinuity.

3. Lower semicontinuity of two variable set-valued functions. In this section, we discuss about conditions which imply joint lower semicontinuity of a set-valued function $F: X \times Y \to \mathcal{C}(Z)$ on a set $D \times \{y_0\}$, where D dense subset of X and $y_0 \in Y$.

Theorem 3.1. *Let X be a Baire space, Y a space and Z be second countable. Let $F: X \times Y \to \mathcal{P}(Z)$ is symmetrically lower quasicontinuous with respect to the second variable at each point of $X \times \{y_0\}$ for some $y_0 \in Y$. Then there is a dense G_δ subset D of X such that F is lower semicontinuous at each point of $D \times \{y_0\}$.*

Proof. Let $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ be a countable base for Z. Define

$$A_n = \{x \in X: (x, y_0) \in F^-(W_n) \setminus (F^-(W_n))^o\}, \quad n \in \mathbb{N}.$$

Let $D_n = X \setminus \overline{A}_n$ for each n and $D = \bigcap_{n=1}^\infty D_n$. Since each D_n is an open subset of X, D is a G_δ subset of X. We will show that D_n is dense in X for each $n \in \mathbb{N}$. Take some fixed $n \in \mathbb{N}$ and a nonempty open subset U of X with $\overline{A_n} \cap U \neq \varnothing$. Let $x \in A_n \cap U$. By symmetrical lower quasicontinuity of F at (x, y_0), there is a nonempty open subset U' of U and a neighborhood V of y_0 such that $U' \times V \subseteq F^-(W_n)$. Since U' is open $\overline{A_n} \cap U' = \varnothing$. Hence $U' \subseteq U \cap D_n \neq \varnothing$. Clearly F is jointly lower semicontinuous at each point of $D \times \{y_0\}$. Let $x \in D$ and W be an open subset of Z with $F(x, y_0) \cap W \neq \varnothing$. Then there is some natural number n_0 such that $W_{n_0} \subset W$ and $F(x, y_0) \cap W_{n_0} \neq \varnothing$. Since $x \notin A_{n_0}$, we have $(x, y_0) \in (F^-(W_n))^o \subseteq (F^-(W))^o$. This proves lower semicontinuity of F on $D \times \{y_0\}$.

It is natural to ask when a set-valued two variable function is symmetrically lower semicontinuous? The following result gives a partial answer to this question.

Theorem 3.2. *Let X be a Baire space, Y be a space with a countable base at $y_0 \in Y$ and let Z be a regular space. Let $F: X \times Y \to \mathcal{C}(Z)$ is an upper horizontally quasicontinuous function such that $F_x: Y \to \mathcal{C}(Z)$ is lower semicontinuous at y_0 for each $x \in X$ and $F^{y_0}: X \to \mathcal{C}(Z)$ is categorically lower quasicontinuous. Then F is symmetrically lower quasicontinuous with respect to the second variable at each point of $X \times \{y_0\}$.*

Proof. Let for some $x_0 \in X$, F is not symmetrically lower quasicontinuous with respect to the second variable at (x_0, y_0). Then there are open sets $U \subset X$, $V \subset Y$ and $W \subset Z$ such that $F(x_0, y_0) \cap W \neq \varnothing$ but for every nonempty open subset $U' \subseteq U$ and neighborhood $V' \subseteq V$ of y_0, there is some $(x', y') \in U' \times V'$ such that $F(x', y') \cap W = \varnothing$. Let $z \in F(x_0, y_0) \cap W$. Using the regularity of Z, we can choose some open subset W' of W such that

$$z \in W' \subseteq \overline{W'} \subseteq W.$$

Since F^{y_0} is lower categorically quasicontinuous at x_0, there is a subset A of U of the second category such that

$$F(a, y_0) \cap W' \neq \varnothing, \quad a \in A.$$

Let $\{V_n : n \in \mathbb{N}\}$ be a countable base at y_0, define

$$A_n = \{a \in A : V_n \subseteq F_a^-(W')\}, \quad n \in \mathbb{N}.$$

Since for every $x \in X$, the function F_x is lower quasicontinuous at $y_0 \in Y$, we have $A = \bigcup_{n=1}^{\infty} A_n$.

We will get into a contradiction by showing that each A_n has empty interior. Let n be a fixed positive integer and G be an arbitrary nonempty open subset of X. By our assumption, $F(x_1, y_1) \cap W = \varnothing$ for some $(x_1, y_1) \in G \times V_n$. Therefore $F(x_1, y_1) \subseteq Z \setminus \overline{W'}$. Since F^{y_1} is upper horizontally quasicontinuous, there is some nonempty open subset G' of G and $y_2 \in V_n$ such that $F(G' \times \{y_2\}) \subseteq Z \setminus \overline{W'}$. It follows that $G' \cap A_n = \varnothing$.

The following result follows immediately from Theorems 3.1 and 3.2.

Theorem 3.3. *Let X be a Baire space, Y a space with countable base at $y_0 \in Y$ and Z a second countable regular space. Let $F : X \times Y \to \mathcal{C}(Z)$ is an upper horizontally quasicontinuous function such that $F_x : Y \to \mathcal{C}(Z)$ is lower semicontinuous and $F^y : X \to \mathcal{C}(Z)$ is categorically lower quasicontinuous for each $(x, y) \in X \times Y$. Then there is a dense G_δ subset D of X such that F is jointly lower semicontinuous at each point of $D \times \{y_0\}$.*

***Example* 3.1** ([19], Example 2)**.** Let E be the set that was defined in Example 2.1. Define $F : \mathbb{R}^2 \to \mathcal{C}(\mathbb{R})$ by

$$F(x, y) = \begin{cases} \{n\}, & (x, y) = (p_n, q_n) \in E, \\ [0, \infty), & (x, y) \notin E. \end{cases}$$

One can easily check that all selection F_x and F^y are lower semicontinuous but F has no point of joint lower semicontinuity. Therefore the assumption of upper horizontal quasicontinuity of F in Theorem 3.3 is necessary.

Theorem 3.4. *Let X be a Baire space, Y a topological space, $y_0 \in Y$ and Z a developable space. Let $F : X \times Y \to \mathcal{K}(Z)$ be symmetrically lower quasicontinuous with respect to the second variable and F^{y_0} be upper quasicontinuous. Then there is a dense G_δ subset D of X such that F is jointly lower semicontinuous at each point of $D \times \{y_0\}$.*

Proof. Let $\{\mathcal{G}_n\}$ be a development for Z. For each $n \in \mathbb{N}$, let A_n be the set of all $x \in X$ such that for some $\mathcal{H}_n \subseteq \mathcal{G}_n$, $F(x, y_0) \subseteq \bigcup \mathcal{H}_n$ and $(x, y_0) \in \big(F^-(W)\big)^o$ for each $W \in \mathcal{H}_n$ with $(x, y_0) \in F^-(W)$.

We will show that $D_n = (A_n)^o$ is dense in X for each $n \in \mathbb{N}$. Take some $n \in \mathbb{N}$ and let U be an arbitrary nonempty open subset of X. Let $x \in U$ and $\mathcal{H}_n$ be a finite subset of $\mathcal{G}_n$ with $F(x, y_0) \subseteq \bigcup \mathcal{H}_n$. Since F^{y_0} is upper quasicontinuous at x, there is a nonempty open subset U' of U such that $F(t, y_0) \subseteq \bigcup \mathcal{H}_n$ for each $t \in U'$. Let $\mathcal{H}_n = \{W_1, \ldots, W_k\}$. If $U' \times \{y_0\} \cap F^-(W_1) = \varnothing$ put $U_1 = U'$ and $V_1 = Y$. Otherwise, by symmetrical lower semicontinuity of F on $U' \times \{y_0\}$, we can choose a nonempty open subset U_1 of U' and a neighborhood V_1 of y_0 such that $U_1 \times V_1 \subseteq F^-(W)$.

By applying this method k-times, we can find a nonempty open subsets U_i of U and a neighborhood V_i of y_0 such that either $U_i \times \{y_0\} \cap F^-(W_i) = \varnothing$ or $U_i \times V_i \subseteq F^-(W_i)$, where $U_i \subseteq U_{i-1}$ and $V_i \subseteq V_{i-1}$ for each $1 < i \leq k$. Thus $U_k \subseteq A_n$. Since U_k is open, $U_k \subseteq D_n$. Therefore, each D_n is dense in X. Let $D = \bigcap_{n=1}^{\infty} D_n$. Since X is a Baire space, D is dense in X. We will show that F is lower semicontinuous at each point of $D \times \{y_0\}$.

Let $x \in D$ and W be an open subset of Z with $F(x, y_0) \cap W \neq \varnothing$. Take some $z \in F(x, y_0) \cap W$. It follows from the definition that for each $n \in \mathbb{N}$, there is some $W_n \in \mathcal{H}_n \subseteq \mathcal{G}_n$ such that $F(x, y_0) \subseteq \bigcup \mathcal{H}_n$ and $z \in W_n$. Since $\{W_i\}$ is a base at z, there is some $n_0 \in \mathbb{N}$ such that $W_{n_0} \subseteq W$. Therefore

$$(x, y) \in \big(F^-(W_{n_0})\big)^o \subseteq \big(F^-(W)\big)^o.$$

Hence F is jointly lower semicontinuous at (x, y_0).

***Corollary* 3.1.** *Let X be a Baire space, Y a space with countable base at $y_0 \in Y$ and let Z be a metric space. Let*

(a) $F\colon X \times Y \to \mathcal{K}(Z)$ *is upper and lower horizontally quasicontinuous function*,

(b) $F_x\colon Y \to \mathcal{K}(Z)$ *is upper and lower semicontinuous and*

(c) $F^y\colon X \to \mathcal{K}(Z)$ *is categorically upper and lower quasicontinuous for each $(x, y) \in X \times Y$.*

Then there is a dense G_δ subset D of X such that F is jointly upper and lower semicontinuous at each point of $D \times \{y_0\}$.

Proof. According to Theorems 2.1 and 3.4, there are dense G_δ subsets D_1 and D_2 of X such that $F|_{D_1 \times \{y_0\}}$ is jointly lower semicontinuous and $F|_{D_2 \times \{y_0\}}$ is jointly upper semicontinuous. Since X is Baire, $D = D_1 \cap D_2$ is a dense G_δ subset of X and $F|_{D \times \{y_0\}}$ is jointly upper and lower semicontinuous.

The following result follows immediately from Theorem 3.4.

***Corollary* 3.2** ([21], Theorem 2)**.** *Let X be a Baire space, Y a topological space and Z a developable space. If $f\colon X \times Y \to Z$ is symmetrically quasicontinuous with respect to the second variable, then, for each $y_0 \in Y$, there is a dense G_δ subset D of X such that f is jointly continuous at each point of $D \times \{y_0\}$.*

***Corollary* 3.3** ([15], Theorem 2)**.** *Let X be a Baire space, Y a topological space and Z a Moore space. If Y has a countable base in $y_0 \in Y$ and $f\colon X \times Y \to Z$ is a horizontally quasicontinuous function such that f^{y_0} is categorically quasicontinuous and f_x is continuous for each $x \in X$, then there is a dense G_δ subset D of X such that f is jointly continuous at each point of $D \times \{y_0\}$.*

Proof. By Theorem 3.2, f is symmetrically quasicontinuous with respect to the second variable at each point of $X \times \{0\}$. So that the result follows from Corollary 3.2.

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