UDC 512.5

P. V. Danchev (Plovdiv State Univ., Bulgaria)

A GENERALIZATION OF WUU RINGS

УЗАГАЛЬНЕННЯ WUU КІЛЕЦЬ

We define the class of UNI rings and present the results of their comprehensive investigations in connection with clean rings, namely, our main result describes commutative UNI clean rings up to an isomorphism. This new concept is a common generalization of the so-called UU rings examined by Danchev – Lam in (Publ. Math. Debrecen, 2016) and of the so-called WUU rings studied by the author in (Tsukuba J. Math., 2016).

Визначено клас UNI кілець та наведено результати їх детальних досліджень у зв'язку з чистими кільцями, тобто основний результат статті описує комутативні чисті UNI кільця з точністю до ізоморфізму. Це нове поняття є типовим узагальненням так званих UU кілець, що вивчалися Данчевим та Ламом в (Publ. Math. Debrecen, 2016), та так званих WUU кілець, що вивчалися Данчевим в (Tsukuba J. Math., 2016).

1. Introduction and background. Throughout the current article all rings into consideration shall be assumed to be associative, containing identity element 1 which differs from the zero element 0. Standardly, $U(R)$ denotes the set of all units of R, $\mathrm{Inv}(R)$ the set of all involutions of R, $\mathrm{Id}(R)$ the set of all idempotents of R and $\mathrm{Nil}(R)$ the set of all nilpotents of R. Moreover, $J(R)$ stands for the Jacobson radical of R and $Z(R)$ for the center of R. All other notions and notations, not explicitly introduced herein, may be found in [8]. For instance, accounting to [9], a ring R is called *clean* if $R = U(R) + \mathrm{Id}(R)$, and *nil-clean* if $R = \mathrm{Nil}(R) + \mathrm{Id}(R)$, accounting to [6]. Imitating [1], a ring R is called *weakly nil-clean*, provided that $R = \mathrm{Nil}(R) \pm \mathrm{Id}(R)$. Besides, a ring R is said to be exchange if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Clean rings are always exchange, while the converse is false – notice that for Abelian rings (that are rings whose idempotents are central) these two concepts are tantamount.

In [5] a ring R is called UU if all units are unipotents, that is, $U(R) = \mathrm{Nil}(R) + 1$. This is obviously equivalent to the equality $U(R) = \mathrm{Nil}(R) - 1$. On the other side, in [3] were defined the so-named WUU rings that are rings R whose $U(R) = \mathrm{Nil}(R) \pm 1$ and which are a common extension of UU rings.

The objective of this article is to generalize these WUU rings by using involutions that are units v whose square is 1, i.e., $v^2 = 1$; e.g., ± 1 is so. We call them UNI rings. Resultantly, we shall establish a complete description of nil-clean UNI rings (Proposition 2.4) and commutative clean UNI rings (Theorem 2.1). These two assertions are stated in the next section. We close the work with stating a series of unanswered problems.

2. UNI rings. The next notion is our starting point of view.

Definition 2.1. *A ring R is called* UNI *if, for each $u \in U(R)$, there are $n \in \mathrm{Nil}(R)$ and $i \in \mathrm{Inv}(R) \cap Z(R)$ such that $u = n + i$. This is obviously tantamount to $U(R) = \mathrm{Nil}(R) + \mathrm{Inv}(R) \cap Z(R) = \big(\mathrm{Inv}(R) \cap Z(R)\big)\big(1 + \mathrm{Nil}(R)\big)$.*

The following are non trivial examples of UNI rings and of non UNI rings, thus illustrating that there is a plenty of them:

***Example* 2.1.** (a) Any WUU ring, that is a ring R with $U(R) = \mathrm{Nil}(R)\pm 1$, is UNI. In particular, for any $s \in \mathbb{N}$, the upper triangular $s \times s$ matrix ring $\mathbb{T}_s(\mathbb{Z}_2)$ is UNI.

(b) For any ordinal μ the direct product $B \times \prod_\mu \mathbb{Z}_3$, where for instance $B \subseteq \prod_\lambda \mathbb{Z}_2$ is a Boolean ring for some ordinal λ, is UNI. In addition, the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is UNI but *not* WUU.

(c) Paralleling to [3], for any $s \in \mathbb{N} \setminus \{1\}$, the upper triangular $s \times s$ matrix ring $\mathbb{T}_s(\mathbb{Z}_3)$ is *not* UNI and hence *not* WUU. However, contrasting with that, the equality $U(\mathbb{T}_s(\mathbb{Z}_3)) = \mathrm{Nil}(\mathbb{T}_s(\mathbb{Z}_3)) + \mathrm{Inv}(\mathbb{T}_s(\mathbb{Z}_3))$ (eventually) holds, but the involutions are not necessarily central. In fact, if for instance $s = 2$, the matrices $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are both involutions satisfying $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where the first term $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent of index 2, and furthermore simple matrix computations show that these two involutions do not lie in $Z(\mathbb{T}_2(\mathbb{Z}_3))$, because $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as well as $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

(d) For any $s \in \mathbb{N} \setminus \{1\}$, the full $s \times s$ matrix rings $\mathbb{M}_s(\mathbb{Z}_2)$ and $\mathbb{M}_s(\mathbb{Z}_3)$ are *not* UNI.

In some cases the reverse of item (a) holds. Specifically, the following is true.

Proposition 2.1. *If R is a ring of characteristic* 2, *or R is a ring with all nilpotents central for which $2 \in \mathrm{Nil}(R)$, then the notions UNI*, *WUU and UU do coincide.*

Proof. Firstly, suppose $2 = 0$ in R. Write $u = n+i$ for some $n \in \mathrm{Nil}(R)$ and $i \in \mathrm{Inv}(R)\cap Z(R)$, and lifting by 2, we deduce that $u^2 = q+1$ for some $q \in \mathrm{Nil}(R)$. Consequently, $u^2 - 1 = (u-1)^2 \in \mathrm{Nil}(R)$ implies that $u \in 1 + \mathrm{Nil}(R)$, as needed.

Secondly, let R be a ring with $2 \in \mathrm{Nil}(R) \subseteq Z(R)$. Since for every $i \in \mathrm{Inv}(R)$ we have $(1-i)^2 = 2(1-i)$, we obtain by induction that $(1-i)^{m+1} = 2^m(1-i)$ for all $m \in \mathbb{N}$. Hence $1 - i \in \mathrm{Nil}(R)$ and thus $\mathrm{Inv}(R) \subseteq 1 + \mathrm{Nil}(R)$ which ensures that $U(R) \subseteq \mathrm{Nil}(R) + \mathrm{Inv}(R) \subseteq 1 + \mathrm{Nil}(R)$, as required.

***Remark* 2.1.** It is worth noticing that rings R whose $\mathrm{Nil}(R) \subseteq Z(R)$ are studied in details in [11].

Lemma 2.1. *Let R be a UNI ring. Then the following are true*:

(1) $2 \in U(R) \iff 3 \in \mathrm{Nil}(R)$;

(2) $3 \in U(R) \iff 2 \in \mathrm{Nil}(R)$.

Proof. For any $a \in U(R)$ one writes that $a = n+i$, where $n \in \mathrm{Nil}(R)$ and $i \in \mathrm{Inv}(R)\cap Z(R)$, and squaring this equality we see that $a^2 = 1 + (n^2 + 2ni) = 1 + q$, where $q = n^2 + 2ni \in \mathrm{Nil}(R)$, because n^2 and $2ni$ are commuting nilpotents.

(1) $\Rightarrow$. Since by what we have shown above $4 = 2^2 = 1 + q \in \mathrm{Nil}(R)$, it follows at once that $3 \in \mathrm{Nil}(R)$.

$\Leftarrow$. Since $1 + \mathrm{Nil}(R) \subseteq U(R)$, it follows that $2^2 = 4 = 1 + 3 \in U(R)$, i.e., $2 \in U(R)$.

(2) $\Rightarrow$. Since by what we have proven above $9 = 3^2 = 1 + q \in \mathrm{Nil}(R)$, it follows at once that $8 = 2^3 \in \mathrm{Nil}(R)$, that is, $2 \in \mathrm{Nil}(R)$.

$\Leftarrow$. Since $1 + \mathrm{Nil}(R) \subseteq U(R)$, as desired it follows that $1 + 2 = 3 \in U(R)$.

***Remark* 2.2.** If R is a commutative UNI ring with $1 \in U(R) + U(R)$, then $3 \in \mathrm{Nil}(R)$, i.e., $2 \in U(R)$.

In fact, writing $1 = n_1 + i_1 + n_2 + i_2$ for some $n_1, n_2 \in \mathrm{Nil}(R)$ and $i_1, i_2 \in \mathrm{Inv}(R) \cap Z(R)$, and squaring this, one deduces that $2i_1i_2 = -1 - q$ for some $q \in \mathrm{Nil}(R)$. Again lifting the last equality by 2, we infer that $4 = 1 + d$ for some $d \in \mathrm{Nil}(R)$ which gives $3 \in \mathrm{Nil}(R)$, as stated.

Proposition 2.2. *If R is a UNI clean ring, then* $6 \in \mathrm{Nil}(R)$.

Proof. Write $3 = u + e = n + i + e$ for some $u \in U(R)$, $e \in \mathrm{Id}(R)$, $n \in \mathrm{Nil}(R)$ and $i \in \mathrm{Inv}(R) \cap Z(R)$. Hence $3 - i = n + e$ and thus $ne = en$. Therefore, $3 - i$ is a strongly nil-clean element, so that $(3 - i)^2 - (3 - i) \in \mathrm{Nil}(R)$ and $7 - 5i \in \mathrm{Nil}(R)$. Further, $5i = 7 - q$ for some $q \in \mathrm{Nil}(R)$ and lifting that by 2 we routinely conclude that $24 \in \mathrm{Nil}(R)$ whence $6^3 = 24.9 \in \mathrm{Nil}(R)$ guarantees that $6 \in \mathrm{Nil}(R)$, as asserted.

Proposition 2.3. *The next three issues hold*:

(i) *Let R be a UNI ring. Then $J(R)$ is nil and $R/J(R)$ is a UNI ring.*

(ii) *Let R be a commutative ring of prime characteristic. Then R is a UNI ring if and only if $J(R)$ is nil and $R/J(R)$ is a UNI ring.*

(iii) *Let R be an Abelian ring with $3 \in J(R)$. Then R is a UNI ring if and only if $J(R)$ is nil and $R/J(R)$ is a UNI ring.*

Proof. (i) Given $j \in J(R)$ and since $1 + J(R) \subseteq U(R)$, one writes that $1 + j = n + i$ for $n \in \mathrm{Nil}(R)$ and $i \in \mathrm{Inv}(R) \cap Z(R)$. Lifting this equality by 2, we have $2j + j^2 \in \mathrm{Nil}(R)$. Similarly, $-2j + j^2 \in \mathrm{Nil}(R)$ by considering $1 - j \in U(R)$. Since $2j + j^2$ and $-2j + j^2$ commute, their sum $2j^2$ again lies in $\mathrm{Nil}(R)$. Since $j^2 \in J(R)$ and $1 + j^2 \in U(R)$, by continuing with the same trick we derive that $2j^2 + j^4 \in \mathrm{Nil}(R)$ whence $j^4 \in \mathrm{Nil}(R)$. Finally $j \in \mathrm{Nil}(R)$, as expected.

Dealing with the second half-part, it follows easily taking into account the well-known critical fact that the map $U(R) \to U(R/J(R))$, induced by the natural map $R \to R/J(R)$, is always surjective due to one of the arguments that either $1 + J(R) \leq U(R)$ or that $J(R)$ is nil. In addition, $U(R/J(R)) \cong U(R)/(1 + J(R))$.

(ii) Suppose now that R is commutative of prime characteristic, say p. One way follows from point (i) above, so we consider the reciprocal. Since the quotient $R/J(R)$ is commutative UNI semiprimitive, and hence reduced, each its unit must be an involution. Given $u \in U(R)$, it plainly follows that $u + J(R)$ is a unit in $R/J(R)$. Therefore, $(u + J(R))^2 = u^2 + J(R) = 1 + J(R)$ giving that $u^2 - 1 \in J(R)$. Firstly, if $\mathrm{char}(R) = 2$, we obtain that $(u - 1)^2 \in J(R) \subseteq \mathrm{Nil}(R)$. Thus $u - 1 \in \mathrm{Nil}(R)$, so that $u \in 1 + \mathrm{Nil}(R)$. That is why, R is UU whence UNI.

Assume now that $\mathrm{char}(R) = p$ is odd. As above, write $u^2 = 1 + z$ for some $z \in J(R)$. Since we have that $J(R)$ is nil, it must be that $z^k = 0$ and so $z^{p^k} = 0$ for some natural number k. So, $(u^2)^{p^k} = (1+z)^{p^k} = 1+z^{p^k} = 1$, i.e., $(u^{p^k})^2 = 1$. We furthermore write that $u = u^{p^k} u^{-(p^k+1)}(1+z)$. But $p^k + 1$ is always even, so that $u^{-(p^k+1)}(1 + z) \in 1 + J(R) \subseteq 1 + \mathrm{Nil}(R)$, because $u^{-2} = (1 + z)^{-1} \in 1 + J(R)$. Consequently, $U(R) = \mathrm{Inv}(R)(1 + \mathrm{Nil}(R))$, as required.

(iii) One way follows again from point (i) above, so we treat the converse. Now, supposing that $u \in U(R)$, it simple follows that $u + J(R)$ is a unit of $R/J(R)$, which factor-ring is Abelian as well, because $J(R)$ is nil. Since $U(R/J(R)) \subseteq \mathrm{Inv}(R/J(R)) + \mathrm{Nil}(R/J(R))$, we write $u + J(R) = (w + J(R)) + (q + J(R))$, where $(w + J(R))^2 = 1 + J(R)$ and $(q + J(R))^m = J(R)$ for some $m \in \mathbb{N}$. Since $J(R)$ is nil, it readily follows that $q \in \mathrm{Nil}(R)$. Moreover, $w^2 + J(R) = 1 + J(R)$ leads us to $(-w - 1 + J(R))^2 = (-w - 1)^2 + J(R) = w^2 + 2w + 1 + J(R) = 2w + 2 + J(R) = -w - 1 + 3w + 3 + J(R) = -w - 1 + J(R)$, that is, $(-w - 1)^2 - (-w - 1) \in J(R)$. Since $J(R)$ is nil, there exists $e \in \mathrm{Id}(R)$ such that $e + J(R) = -w - 1 + J(R)$, i.e., $w + J(R) = -1 - e + J(R)$. Therefore, one writes that $u + J(R) = -1 - e + q + J(R)$ and thus $u = (2e - 1) + (-3e + z) + q$ for some $z \in J(R)$. But $(2e - 1)^2 = 1$ and $z - 3e \in J(R)$, so that $q + (-3e + z) \in \mathrm{Nil}(R)$

because it is not too hard to verify that $J(R) + \mathrm{Nil}(R) = \mathrm{Nil}(R)$, provided $J(R) \subseteq \mathrm{Nil}(R)$. Finally, $u \in (\mathrm{Inv}(R) \cap Z(R)) + \mathrm{Nil}(R)$, because the idempotent e commutes with all elements of R.

We thus come to the following:

***Remark* 2.3.** In contrast to UU rings and same as WUU rings (see, e.g., [5] and [3], respectively) the reverse claim in Proposition 2.3 may fail in general: in fact, as in [3] and [5], there is a nil ideal $I = \{(a_{ij}) \in \mathbb{T}_2(\mathbb{Z}_3) : \forall a_{ii} = 0\}$ of $\mathbb{T}_2(\mathbb{Z}_3)$ such that $\mathbb{T}_2(\mathbb{Z}_3)/I \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. However, as pointed out in Example 2.1 (b) above, the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is UNI but not WUU (compare with Remark 2.5 below, too) and, as indicated in Example 2.1 (c), straightforward calculations show that the ring $\mathbb{T}_2(\mathbb{Z}_3)$ is not UNI. The reason for this is that although I equals to the radical of Jacobson for $\mathbb{T}_2(\mathbb{Z}_3)$, in accordance with Example 2.1 (c), $\mathbb{T}_2(\mathbb{Z}_3)$ need not be Abelian.

The following technicality is our critical tool (for a weaker variant of this claim the interested reader may see also [10]).

Lemma 2.2. *Suppose u is a unit and e is an idempotent of a ring R such that $u^2e = eu^2$ and $u = e + q$ or $u = -e + q$, where q is a nilpotent. Then $e = 1$.*

In particular, any involution of a weakly nil-clean ring is unipotent.

Proof. Letting $u = e+q$, for some $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$ with $q^t = 0, t \in \mathbb{N}$ say, we obtain that $u^2 = e + eq + qe + q^2$ and hence $u^2e = e + eqe + qe + q^2e$. Similarly, $eu^2 = e + eq + eqe + eq^2$ and thus $u^2e = eu^2$ insures that $(q + q^2)e = e(q + q^2)$. Thus e commutes with the nilpotent $(q + q^2)^n = [q(1 + q)]^n = q^n(1 + q)^n$ for all $n \in \mathbb{N}$, and therefore the same is valid for u. Furthermore, $u - (q + q^2) = e - q^2$ with $u - (q + q^2) = u^{(2)} = e - q^2$ being a unit, one sees that $u^{(2)} - (2q^3 + q^4) = e - (q^2 + 2q^3 + q^4) = e - (q + q^2)^2$. Putting $u^{(3)} = u^{(2)} + (q + q^2)^2$, we observe that $u^{(3)}$ is a unit since $u^{(2)}$ commutes with $(q + q^2)^2$ and that $u^{(3)} = e + q^3(2 + q)$. Since $u^{(3)}$ will commute with $2(q+q^2)^3$, it follows that $u^{(4)} = u^{(3)} - 2(q+q^2)^3 = e + f(q^4, q^5, q^6) = e + q^4 \cdot h(q, q^2)$, where f, h are functions of powers of q. Repeating the same procedure t-times, we will find a unit $u^{(t)}$ such that $u^{(t)} = e + q^t \cdot a = e$ for some element $a \in R$ depending on q; if $t = 2$ we just put $a = -1 = -q^0$. This yields that $e = 1$, which exhausts this case.

Analogously, $(q^2 - q)e = e(q^2 - q)$ and $(q^2 - q)u = u(q^2 - q)$. Hence $u - q^2 = (q - q^2) - e$. However, the same trick as above successfully works to conclude the claim after all.

The second part is now immediate.

We now have at our disposal the needed information for classifying (weakly) nil-clean UNI rings as follows:

Proposition 2.4. (1) *A ring R is UNI nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is a Boolean ring.*

(2) *A ring R is UNI weakly nil-clean if and only if $J(R)$ is nil and either $R/J(R) \cong B$ is a Boolean ring, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$.*

Proof. (1) $\Rightarrow$. With Proposition 2.3 at hand we deduce that $J(R)$ is nil. Furthermore, in accordance to Lemma 2.2, for any $u \in U(R)$, we write $u = n + i = 1 + n + q$ for some $i \in \mathrm{Inv}(R) \cap Z(R)$ and $n, q \in \mathrm{Nil}(R)$ such that $n + q \in \mathrm{Nil}(R)$, because $ni = in$ yields $nq = qn$. We consequently obtain that R is nil-clean UU and so the main result in [5] applies to get the pursued claim.

$\Leftarrow$. It follows from [5] that R is strongly nil-clean and hence it is UU nil-clean.

(2) $\Rightarrow$. Consulting with [3] (Theorem 2.17) (see also [1], Theorems 7(2) and 12(3) for the Abelian case) one may write that $R \cong R_1 \times R_2$, where R_1 is nil-clean, and R_2 is either $\{0\}$, or $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$. Since R_1 is nil-clean UNI, point (1) applies to get that $J(R_1)$ is nil and

$R_1/J(R_1)$ is Boolean. However, $J(R) \cong J(R_1) \times J(R_2)$ under the same isomorphism induced by that of R, and hence $R/J(R) \cong [R_1/J(R_1)] \times [R_2/J(R_2)]$ gives the wanted claim.

$\Leftarrow$. It follows from [2] (see also [3]) that R is weakly nil-clean with the strong property whence it is WUU weakly nil-clean.

Proposition 2.4 is proved.

***Remark* 2.4.** In terms of [5], a ring R is UNI nil-clean exactly when it is UU nil-clean, that is, R is strongly nil-clean. By the same token, in terms of [3], a ring R is UNI weakly nil-clean precisely when it is WUU weakly nil-clean ring, that is, R is weakly nil-clean with the strong property (cf. [2] as well).

Recall that a ring R is called *local* if $R/J(R)$ is a division ring, that is, every its element lies in either $U(R)$ or in $J(R)$. Note that a ring is local if and only if it is a clean ring with only trivial idempotents, and hence it is an Abelian clean ring.

The next assertion is important for our further successful presentation.

Proposition 2.5. *R is a local UNI ring if and only if $J(R)$ is nil and either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.*

Proof. The sufficiency being elementary, we concentrate on the necessity. Firstly, that $J(R)$ is nil and $R/J(R)$ is UNI follows from Proposition 2.3.

Next, if $2 \in J(R)$, then Proposition 2.1 is applicable to show that the quotient $R/J(R)$ is division UU and thus $R/J(R) \cong \mathbb{Z}_2$, which follows either directly or in view of [5].

If now $2 \notin J(R)$, then $2 \in U(R)$ whence by Lemma 2.1 (1) we have $3 \in J(R)$ and so the factor-ring $R/J(R)$ is UNI division of characteristic 3. Since in division rings we have no non-trivial nilpotent elements and all involutions are only $\{-1, 1\}$, we conclude that $R/J(R) \cong \mathbb{Z}_3$, as formulated.

The following decomposition is essential.

Lemma 2.3. *A ring R is UNI for which $6 \in J(R)$ if and only if $R \cong R_1 \times R_2$, where R_1 is a UU ring and R_2 is either $\{0\}$ or a UNI ring with $3 \in J(R_2)$.*

Proof. *Necessity*. Observe that for any $k \in \mathbb{N}$ we have $(2^k, 3^k) = 1$, i.e., there exist non-zero integers u, v such that $2^k u + 3^k v = 1$. This, consequently, allows us to write that $R = 2^k R + 3^k R$. Since $6 \in J(R)$, it follows in virtue of Proposition 2.3 that 6 must be nilpotent. This assures that $6^m = 0$ for some $m \in \mathbb{N}$, and so $2^m R \cap 3^m R = \{0\}$; in fact, if $x = 2^m a = 3^m b$ for some $a, b \in R$, then $2^m au = 3^m bu$. However, $(1 - 3^m v)a = 3^m bu$ whence $3^m(av + bu) = a$. Multiplying both sides by 2^m, we derive that $0 = 2^m a = x$, as needed. Finally, $R = 2^m R \oplus 3^m R$ and hence $R \cong (R/2^m R) \times (R/3^m R) = R_1 \times R_2$ with $R_1 = R/2^m R \cong 3^m R$ and $R_2 = R/3^m R \cong 2^m R$.

Note it is not too hard to check that R_1 and R_2 are both UNI rings as being direct factors of a ring isomorphic to the UNI ring R. Since $2 \in J(R_1)$, Proposition 2.3 is a guarantor that $R_1/J(R_1)$ is UNI of characteristic 2 as well as $J(R_1)$ is nil. Thus Proposition 2.1 gives that $R_1/J(R_1)$ is a UU ring and so an application of [5] insures that R_1 is UU as well. On the other hand, since $3 \in J(R_2)$, it follows again by Proposition 2.3 that $3 \in \mathrm{Nil}(R_2)$. Notice that with the aid of Lemma 2.1 (1) we also have $2 \in U(R_2)$.

Sufficiency. Since $2 \in J(R_1)$ and eventually $3 \in J(R_2)$, provided $R_2 \neq \{0\}$, it must be that $6 \in J(R)$, because $J(R) \cong J(R_1) \times J(R_2)$. Next, that R is UNI follows directly by observing that the direct product of two UNI rings is again a UNI ring.

***Remark* 2.5.** If the direct factor R_2 in Lemma 2.3 is indecomposable, things will be easy. Nevertheless, in contrast to WUU rings (cf. [3]), it cannot be expected that R_2 will be indecomposable. In fact, there exists a commutative UNI ring R with $2 \in U(R)$ such that $\mathrm{Id}(R) \neq \{0, 1\}$ and so

R is decomposable (compare with Example 2.1 (b) too). Indeed, consider $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence $2 = -1 \in U(R)$. Likewise, $U(\mathbb{Z}_3) = \{-1, 1\} = \mathrm{Inv}(\mathbb{Z}_3)$ with $U(R) = U(\mathbb{Z}_3) \times U(\mathbb{Z}_3)$ and $\mathrm{Nil}(R) = \{0\}$. Therefore, $U(R) = \mathrm{Inv}(R)$ which shows that R is UNI. Moreover, it is self-evident that R is decomposable having both $(0, 1)$ and $(1, 0)$ as nontrivial idempotents different to the trivial ones $(0, 0)$ and $(1, 1)$.

The next statement somewhat sheds light on the complicated structure of arbitrary UNI clean rings.

Theorem 2.1. *A ring R is clean UNI if and only if R can be decomposed as the direct product of a ring which is Boolean modulo its nil Jacobson radical and of a UNI clean ring whose nil Jacobson radical contains* 3.

In addition, if the latter ring modulo its Jacobson radical is commutative, then this factor-ring can be embedded in the direct product of the fields $\mathbb{Z}_3$.

In particular, a commutative ring R is UNI clean with $3 \in J(R)$ if and only if $J(R)$ is nil and $R/J(R) \subseteq \prod_\mu \mathbb{Z}_3$, where μ is an ordinal.

Proof. According to Proposition 2.3, $J(R)$ is nil. Now, knowing by Proposition 2.2 that $6 \in \mathrm{Nil}(R)$ and hence $6 \in J(R)$, we employ Lemma 2.3 to write that $R \cong R_1 \times R_2$, where R_1 is either $\{0\}$ or is UU clean and R_2 is either $\{0\}$ or is UNI clean with $3 \in J(R_2)$. As for R_1, we employ [5] to prove that $R_1/J(R_1)$ is Boolean, as stated.

Concerning the other part, assume now that $R_2/J(R_2)$ is commutative. Moreover, Proposition 2.3 together with [9] enable us that $R_2/J(R_2)$ is also clean and UNI. Since this quotient does not contain nontrivial nilpotents, any its element is the sum of an involution and an idempotent. Being of characteristic 3, it easily follows that any element in it is the sum of two idempotents, because for any element v in this factor-ring with $v^2 = 1$ it must be that $(v - 1)^2 = v^2 + 1 - 2v = v - 1$. We further employ [7] (see also [4]) to get the pursued claim.

Finally, it remains to show the reciprocal part that R is UNI clean, provided $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\prod_\mu \mathbb{Z}_3$. But one sees that $R/J(R)$ is UNI since any unit in $\prod_\mu \mathbb{Z}_3$ has to be an involution. We henceforth observe that issue (iii) in Proposition 2.3 can be applied to obtain that R is UNI, indeed. Moreover, since the elements x of every subring of $\prod_\mu \mathbb{Z}_3$ satisfies the equation $x^3 = x$ which is equivalent to $x = x^2(-x^2 + x + 1)$ where $-x^2 + x + 1$ is obviously an involution, it follows that $R/J(R)$ is unit-regular and hence clean. Thus [9] is workable to conclude the promised assertion.

Theorem 2.1 is proved.

In regard to the consideration above, we add the following comments: It is well known that any Boolean ring B is a subring of the direct product of copies of the field $\mathbb{Z}_2$, and vice versa. If B is finite, then $B \cong \prod_k \mathbb{Z}_2$ for some finite k; however, if B is infinite, analogous isomorphism is manifestly untrue as the following classical construction shows: Let X be a set of cardinal $\aleph_0$ and let R be the Boolean ring of all finite subsets of X with the usual operations given. Then R has cardinality $\aleph_0$. But the direct product of $\aleph_0$ copies of $\mathbb{Z}_2$ will have cardinality $2^{\aleph_0} > \aleph_0$. So, any isomorphism between them is impossible.

To finish, resuming, a commutative ring R is UNI clean if and only if $R \cong R_1 \times R_2$, where $J(R_1)$ is nil with $R_1/J(R_1) \subseteq \prod_\lambda \mathbb{Z}_2$ for some ordinal λ and $J(R_2)$ is nil with $R_2/J(R_2) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinal μ. So, $J(R) \cong J(R_1) \times J(R_2)$ is nil and $R/J(R) \cong [R_1/J(R_1)] \times [R_2/J(R_2)] \subseteq$ $\subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$.

3. Open questions. In ending, we pose the following four unsettled problems of interest:

Problem 1. Characterize exchange UNI rings. Are they clean?

In that aspect classify those rings R for which $J(R)$ is nil and $R/J(R)$ is embedding in $\prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$, where λ and μ are some ordinals. Since a subring of such a product $\prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ satisfies the equation $x^3 = x$ for all its elements x, it is a commutative von Neumman regular ring and thus clean. We further apply [9] to infer that R is clean as well. Nevertheless, it is not at all obvious whether R is UNI or not.

In other words, does it follow that R is UNI clean if and only if $J(R)$ is nil and $R/J(R)$ is embedding in $\prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$, where λ and μ are some ordinals? Notice that in virtue of Theorem 2.1, if the quotient $R/J(R)$ is commutative, the necessity of the problem is settled in the affirmative.

Problem 2. If R a UNI ring and $e \in \mathrm{Id}(R)$, does it follow that the corner ring eRe is also UNI?

Problem 3. Suppose that R is a ring and $s \in \mathbb{N}$. Find a criterion when the full $s \times s$ matrix ring $\mathbb{M}_s(R)$ is UNI.

Problem 4. Let R be a ring and G a multiplicative group. Find a necessary and sufficient condition when the group ring $R[G]$ is UNI.

References

1. *Breaz S., Danchev P., Zhou Y.* Rings in which every element is either a sum or a difference of a nilpotent and an idempotent // J. Algebra and Appl. – 2016. – **15**, № 8. – 11 p.
2. *Danchev P. V.* A class of weakly nil-clean rings // Irish Math. Soc. Bull. – 2016. – **77**. – P. 9 – 17.
3. *Danchev P. V.* Weakly UU rings // Tsukuba J. Math. – 2016. – **40**. – P. 101 – 118.
4. *Danchev P. V.* Invo-clean unital rings // Commun. Korean Math. Soc. – 2017. – **32**. – P. 19 – 27.
5. *Danchev P. V., Lam T. Y.* Rings with unipotent units // Publ. Math. Debrecen. – 2016. – **88**. – P. 449 – 466.
6. *Diesl A. J.* Nil clean rings // J. Algebra. – 2013. – **383**. – P. 197 – 211.
7. *Hirano Y., Tominaga H.* Rings in which every element is the sum of two idempotents // Bull. Aust. Math. Soc. – 1988. – **37**. – P. 161 – 164.
8. *Lam T. Y.* A first course in noncommutative rings. – Second Ed. // Grad. Texts Math. – Berlin etc.: Springer-Verlag, 2001.
9. *Nicholson W. K.* Lifting idempotents and exchange rings // Trans. Amer. Math. Soc. – 1977. – **229**. – P. 269 – 278.
10. *Šter J.* Nil clean involutions // arXiv preprint, 2015.
11. *Ungor B., Halicioglu S., Kose H., Harmanci A.* Rings in which every nilpotent is central // Algebras, Groups, and Geom. – 2013. – **30**. – P. 1 – 18.

Received 17.01.16,
after revision — 14.07.16