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GENERALIZED HIGHER DERIVATIONS ON ALGEBRAS

УЗАГАЛЬНЕНІ ВИЩІ ПОХІДНІ НА АЛГЕБРАХ

We study the structure of generalized higher derivations on an algebra $\mathcal{A}$ and show that there exists a one-to-one correspondence between the set of all generalized higher derivations $\{G_k\}_{k=0}^n$ on $\mathcal{A}$ with $G_0 = I$ and the set of all sequences $\{g_k\}_{k=0}^n$ of generalized derivations on $\mathcal{A}$ with $g_0 = 0$.

Вивчено структуру узагальнених вищих похідних на алгебрі $\mathcal{A}$ та показано, що існує взаємно однозначна відповідність між множиною всіх узагальнених вищих похідних $\{G_k\}_{k=0}^n$ на $\mathcal{A}$ з $G_0 = I$ та множиною всіх послідовностей $\{g_k\}_{k=0}^n$ узагальнених похідних на $\mathcal{A}$ з $g_0 = 0$.

1. Introduction. Let $\mathcal{A}$ be an associative algebra and I the identity map on $\mathcal{A}$. A linear mapping $d: \mathcal{A} \to \mathcal{A}$ is a derivation if

$$d(xy) = d(x)y + xd(y), \quad x, y \in \mathcal{A}.$$

A derivation d is called inner if there exists a fixed element $a \in \mathcal{A}$ such that $d(x) = [a, x] = ax - xa$ for all $x \in \mathcal{A}$. Now, let $n \in \mathbb{N} \cup \{0, \infty\}$ and let $\{D_k : \mathcal{A} \to \mathcal{A} \mid k = 0, 1, \ldots, n\}$ be a family of linear mappings with $D_0 = I$. The sequence $D = \left\{D_k\right\}_{k=0}^n$ is called a higher derivation of rank n if

$$D_k(xy) = \sum_{i=0}^{k} D_i(x)D_{k-i}(y), \quad x, y \in \mathcal{A}, \quad k = 0, 1, \ldots, n. \tag{1}$$

Obviously, D_1 satisfies $D_1(xy) = D_0(x)D_1(y) + D_1(x)D_0(y) = xD_1(y) + D_1(x)y$ for all $x, y \in \mathcal{A}$. Thus, D_1 is a derivation on $\mathcal{A}$. If $d: \mathcal{A} \to \mathcal{A}$ is a derivation, then $D = \left\{\dfrac{d^k}{k!}\right\}_{k=0}^n$ is a standard example of a higher derivation of rank n. This kind of higher derivation is called an ordinary higher derivation. Note also that this is not the only example of higher derivations.

Higher derivations have been considered for the first time by Hasse and Schmidt [4] and independently by Teichmüller [10], being difficult to affirm with certainty which of them was the pioneer in this study. The concept of higher derivations was originally introduced in order to remove some of the anomalies in the calculus of derivations on fields of characteristic $p \neq 0$ (cf. [3, 6]). Nevertheless, higher derivations have important applications in characteristic zero situations, as shown by Heerema [5]. In a celebrated paper [5], Heerema showed a review of the theory of higher derivations on fields which form a background for the study of the uses of higher derivations in automorphism theory of complete local rings. For more information about higher derivations, their chronological development, importance and applications we refer the reader to [1] and [2].

A linear mapping $g: \mathcal{A} \to \mathcal{A}$ is a generalized derivation if there exists a derivation $d: \mathcal{A} \to \mathcal{A}$ such that

$$g(xy) = g(x)y + xd(y), \quad x, y \in \mathcal{A}.$$

In this case we sometimes call g a generalized d-derivation (i.e., g is a generalized derivation associated with the derivation d). If $\mathcal{A}$ has a unit $1 \in \mathcal{A}$, then we can write the above equality as $g(xy) = g(x)y + xg(y) - xg(1)y,\ x, y \in \mathcal{A}$. A generalized derivation g is called inner if there exist elements $a, b \in \mathcal{A}$ such that $g(x) = ax + xb$ for all $x \in \mathcal{A}$. The sequence of linear mappings $G = \{G_k\}_{k=0}^n$ is called a generalized higher derivation of rank n if there exists a higher derivation $D = \{D_k\}_{k=0}^n$ such that

$$G_k(xy) = \sum_{i=0}^{k} G_i(x)D_{k-i}(y), \quad x, y \in \mathcal{A}, \quad k = 0, 1, \ldots, n. \tag{2}$$

In this case G is a generalized higher derivation associated with the higher derivation D. Thus, we sometimes call G simply a generalized higher D-derivation. Of course, if $D = G$, then G is just a higher derivation.

As far as we are aware, the above definition was first introduced by Ribenboim in [9]. Ribenboim gave some properties of higher derivations of modules. His higher derivation f from an $\mathcal{A}$-module $\mathcal{M}$ to $\mathcal{M}$ is defined by using a higher derivation $D = \{D_k\}_{k=0}^n \colon \mathcal{A} \to \mathcal{A}$. In 2000, Nakajima [8] (Definition 2.1) defined generalized higher derivations from an R-algebra $\mathcal{A}$ (here R is a commutative ring) to an $\mathcal{A}/R$-bimodule $\mathcal{M}$ and gave some of their categorical properties. He also treated generalized higher Jordan and Lie derivations. For more results on generalized higher derivations we refer the reader to the survey [2].

Let us continue with a standard example of generalized higher derivations which is, of course, not the only example of generalized higher derivations (see Section 2).

***Example* 1.** Let $g\colon \mathcal{A} \to \mathcal{A}$ be a generalized d-derivation. Then it is easy to see that $G = \left\{\frac{g^k}{k!}\right\}_{k=0}^n$ is a generalized higher derivation associated with the ordinary higher derivation $D = \left\{\frac{d^k}{k!}\right\}_{k=0}^n$.

The concept of generalized higher derivations is rather new and relatively few results have been obtained so far concerning it. This motivated us to study the structure of generalized higher derivations and to generalize some known results. We show that there exists a one to one correspondence between the set of all generalized higher derivations $\{G_k\}_{k=0}^n$ on $\mathcal{A}$ with $G_0 = I$ and the set of all sequences $\{g_k\}_{k=0}^n$ of generalized derivations on $\mathcal{A}$ with $g_0 = 0$. The importance of our work is to transfer the problems such as innerness and automatic continuity of generalized higher derivations into the same problems concerning generalized derivations. Namely, we will prove that each component of a generalized higher derivation is a combination of compositions of generalized derivations.

2. Generalized higher derivations. Throughout the paper, $\mathcal{A}$ denotes an associative algebra over a field of characteristic zero. Recently, Mirzavaziri [7] characterized all higher derivations on an algebra $\mathcal{A}$ in terms of derivations on $\mathcal{A}$. In particular, he proved that each component of a higher derivation is a combination of compositions of derivations. The natural question here is whether the analogue result holds true for generalized higher derivations. Theorem 1 answers this question in the affirmative.

Theorem 1. *Let $G = \{G_k\}_{k=0}^n$ be a generalized higher derivation on $\mathcal{A}$ with $G_0 = I$. Then there exists a sequence $\{g_k\}_{k=1}^n$ of generalized derivations on $\mathcal{A}$ such that*

$$G_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) g_{r_1} \ldots g_{r_i} \right), \tag{3}$$

where the inner summation is taken over all positive integers r_j such that $\sum_{j=1}^{i} r_j = k$.

Below we use the characterization of higher derivations proved by Mirzavaziri [7]. Recall that higher derivations in the sense of Mirzavaziri are infinite sequences of linear maps satisfying certain conditions while (generalized) higher derivations in this paper can be finite or infinite sequences of linear maps satisfying certain conditions. Nevertheless, it is easy to see (following the proofs in [7]) that the results in [7] hold true in the finite case as well.

***Proof of Theorem* 1.** Suppose that $G = \{G_k\}_{k=0}^n$ is a generalized higher derivation associated with the higher derivation $D = \{D_k\}_{k=0}^n$. Then, by Theorem 2.3 in [7], there exists a sequence $\{d_k\}_{k=1}^n$ of derivations on $\mathcal{A}$ such that

$$D_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) d_{r_1} \ldots d_{r_i} \right). \tag{4}$$

Here, always, the inner summation is taken over all positive integers r_j such that $\sum_{j=1}^{i} r_j = k$. We divide the proof into two steps.

Step 1. Using the induction on k, we first show that there exists a sequence $\{g_k\}_{k=1}^n$ of generalized d_k-derivations on $\mathcal{A}$ such that

$$(k+1)G_{k+1} = \sum_{i=0}^{k} g_{i+1} G_{k-i} \tag{5}$$

for each $k = 0, 1, \ldots, n-1$.

First of all, it is easy to see that G_1 is a generalized derivation. Namely, for all $x, y \in \mathcal{A}$, we have

$$G_1(xy) = G_0(x)D_1(y) + G_1(x)D_0(y) = xD_1(y) + G_1(x)y.$$

Recall that, according to (4), $D_1 = d_1$ is a derivation and, thus, $G_1 = g_1$ is a generalized d_1-derivation. In particular, we have proved (5) for $k = 0$.

Now, suppose that $k \geq 1$ and that there exist appropriate generalized derivations $g_1, \ldots, g_k$ on $\mathcal{A}$ such that (5) holds true. Let us define a mapping $g_{k+1} : \mathcal{A} \to \mathcal{A}$ by

$$g_{k+1} = (k+1)G_{k+1} - \sum_{i=0}^{k-1} g_{i+1} G_{k-i}.$$

Of course, g_{k+1} is linear. Now, let $x, y \in \mathcal{A}$ be arbitrary elements. Then

$$g_{k+1}(xy) = (k+1)G_{k+1}(xy) - \sum_{i=0}^{k-1} g_{i+1} G_{k-i}(xy) =$$

$$= (k+1) \sum_{i=0}^{k+1} G_i(x) D_{k+1-i}(y) - \sum_{i=0}^{k-1} g_{i+1} \left(\sum_{j=0}^{k-i} G_j(x) D_{k-i-j}(y) \right).$$

Furthermore, since $g_1,\dots,g_k$ are generalized derivations associated with derivations $d_1,\dots,d_k$, we have

$$g_{k+1}(xy)=\sum_{i=0}^{k+1} iG_i(x)D_{k+1-i}(y)+\sum_{i=0}^{k+1}(k+1-i)G_i(x)D_{k+1-i}(y)=$$

$$-\sum_{i=0}^{k-1}\sum_{j=0}^{k-i}\left(g_{i+1}\big(G_j(x)\big)D_{k-i-j}(y)+G_j(x)d_{i+1}\big(D_{k-i-j}(y)\big)\right).$$

Let us divide the above sum into two parts

$$S_1=\sum_{i=0}^{k+1} iG_i(x)D_{k+1-i}(y)-\sum_{i=0}^{k-1}\sum_{j=0}^{k-i} g_{i+1}\big(G_j(x)\big)D_{k-i-j}(y),$$

$$S_2=\sum_{i=0}^{k+1}(k+1-i)G_i(x)D_{k+1-i}(y)-\sum_{i=0}^{k-1}\sum_{j=0}^{k-i} G_j(x)d_{i+1}\big(D_{k-i-j}(y)\big).$$

Of course, $g_{k+1}(xy)=S_1+S_2$.

Let us first compute S_1. Recall that in the summation $\sum_{i=0}^{k-1}\sum_{j=0}^{k-i}$ we have $0\le i+j\le k$ and $i\ne k$. Thus, if we write $l=i+j$, we get

$$S_1=\sum_{i=0}^{k+1} iG_i(x)D_{k+1-i}(y)-\sum_{l=0}^{k}\sum_{0\le i\le l, i\ne k} g_{i+1}\big(G_{l-i}(x)\big)D_{k-l}(y)=$$

$$=\sum_{i=0}^{k+1} iG_i(x)D_{k+1-i}(y)-\sum_{l=0}^{k-1}\sum_{i=0}^{l} g_{i+1}\big(G_{l-i}(x)\big)D_{k-l}(y)-$$

$$-\sum_{i=0}^{k-1} g_{i+1}\big(G_{k-i}(x)\big)y.$$

Therefore, writing $l+1$ instead of i in the first summation, we obtain

$$S_1=\sum_{l=0}^{k}(l+1)G_{l+1}(x)D_{k-l}(y)-\sum_{l=0}^{k-1}\sum_{i=0}^{l} g_{i+1}\big(G_{l-i}(x)\big)D_{k-l}(y)-$$

$$-\sum_{i=0}^{k-1} g_{i+1}\big(G_{k-i}(x)\big)y=$$

$$=\sum_{l=0}^{k-1}\left((l+1)G_{l+1}(x)D_{k-l}(y)-\sum_{i=0}^{l} g_{i+1}\big(G_{l-i}(x)\big)D_{k-l}(y)\right)+$$

$$+(k+1)G_{k+1}(x)y-\sum_{i=0}^{k-1} g_{i+1}\big(G_{k-i}(x)\big)y.$$

By our assumption,

$$(l+1)G_{l+1}(x) = \sum_{i=0}^{l} g_{i+1}\big(G_{l-i}(x)\big)$$

for $l = 0, 1, \ldots, k-1$. Thus,

$$S_1 = \left((k+1)G_{k+1}(x) - \sum_{i=0}^{k-1} g_{i+1}\big(G_{k-i}(x)\big)\right) y = g_{k+1}(x)y.$$

Using the fact that $d_{k+1} = (k+1)D_{k+1} - \sum_{i=0}^{k-1} d_{i+1}D_{k-i}$, we can similarly deduce that

$$S_2 = xd_{k+1}(y).$$

Therefore,

$$g_{k+1}(xy) = g_{k+1}(x)y + xd_{k+1}(y).$$

Since x and y were arbitrary elements from $\mathcal{A}$, this yields that g_{k+1} is a generalized d_{k+1}-derivation, as desired.

Step 2. Suppose that G_k is of the form (3) for $k \geq 1$. If we denote

$$\alpha_{r_1,\ldots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i},$$

then we have

$$(k+1)G_{k+1} = (k+1)\sum_{i=1}^{k+1}\left(\sum_{\sum_{j=1}^{i} r_j = k+1} \alpha_{r_1,\ldots,r_i} g_{r_1}\ldots g_{r_i}\right) =$$

$$= \sum_{i=2}^{k+1}\left(\sum_{\sum_{j=1}^{i} r_j = k+1} (k+1)\alpha_{r_1,\ldots,r_i} g_{r_1}\ldots g_{r_i}\right) + g_{k+1}.$$

Note that $(k+1)\alpha_{r_1,\ldots,r_i} = \alpha_{r_2,\ldots,r_i}$ since $r_1 + \ldots + r_i = k+1$. Therefore,

$$(k+1)G_{k+1} = \sum_{i=2}^{k+1}\left(\sum_{r_1=1}^{k+2-i} g_{r_1} \sum_{\sum_{j=2}^{i} r_j = k+1-r_1} \alpha_{r_2,\ldots,r_i} g_{r_2}\ldots g_{r_i}\right) + g_{k+1} =$$

$$= \sum_{r_1=1}^{k} g_{r_1} \sum_{i=1}^{k-(r_1-1)}\left(\sum_{\sum_{j=2}^{i} r_j = k-(r_1-1)} \alpha_{r_2,\ldots,r_i} g_{r_2}\ldots g_{r_i}\right) + g_{k+1} =$$

$$= \sum_{r_1=1}^{k} g_{r_1} G_{k-(r_1-1)} + g_{k+1}.$$

If we write $r_1 = i$, we get

$$(k+1)G_{k+1} = \sum_{i=0}^{k-1} g_{i+1}G_{k-i} + g_{k+1} = \sum_{i=0}^{k} g_{i+1}G_{k-i}.$$

Namely, $g_{k+1} = g_{k+1}I = g_{k+1}G_0$. Consequently, $G_1, \ldots, G_n$ satisfy the recursive relation (5). Since the solution of the recursive relation in Step 1 is unique, this completes the proof.

Theorem 2. *Let Φ be the set of all generalized higher derivations $\{G_k\}_{k=0}^n$ on $\mathcal{A}$ with $G_0 = I$ and Γ the set of all sequences $\{g_k\}_{k=0}^n$ of generalized derivations on $\mathcal{A}$ with $g_0 = 0$. Then there is a one-to-one correspondence between Φ and Γ.*

Proof. Let $G = \{G_k\}_{k=0}^n \in \Phi$ be a generalized higher derivation on $\mathcal{A}$. Then, by Theorem 1, there exists a sequence $g = \{g_k\}_{k=0}^n$ of generalized derivations on $\mathcal{A}$ with $g_0 = 0$ such that (5) holds true for each $k = 0, 1, \ldots, n-1$. Thus, we can define a mapping $\varphi\colon \Phi \to \Gamma$ by

$$\{G_k\}_{k=0}^n = G \quad \stackrel{\varphi}{\longrightarrow} \quad g = \{g_k\}_{k=0}^n .$$

Note that a map φ is injective since the solution of the recursive relation (5) is unique.

It remains to prove that φ is surjective. For a given sequence $g = \{g_k\}_{k=0}^n \in \Gamma$ of generalized d_k-derivations on $\mathcal{A}$ with $g_0 = 0$, one can define mappings $G_0, \ldots, G_n$ with $G_0 = I$ and

$$(k+1)G_{k+1} = \sum_{i=0}^{k} g_{i+1}G_{k-i}$$

for $k = 0, 1, \ldots, n-1$. We have to prove that $G = \{G_k\}_{k=0}^n$ is a generalized higher derivation. Obviously, G_k are linear mappings on $\mathcal{A}$. Furthermore, we know that there exists a higher derivation $D = \{D_k\}_{k=0}^n$ such that

$$(k+1)D_{k+1} = \sum_{i=0}^{k} d_{i+1}D_{k-i}$$

for $k = 0, 1, \ldots, n-1$.

Let $x, y \in \mathcal{A}$. Using the induction on k, we show that

$$G_k(xy) = \sum_{i=0}^{k} G_i(x)D_{k-i}(y), \quad k = 0, 1, \ldots, n. \tag{6}$$

For $k = 0$, we have $G_0(xy) = xy = G_0(x)D_0(y)$ and, for $k = 1$, we have

$$G_1(xy) = g_1G_0(xy) = g_1(xy) = g_1(x)y + xd_1(y) =$$
$$= G_1(x)D_0(y) + G_0(x)D_1(y).$$

Assume now that $G_l(xy) = \sum_{i=0}^{l} G_i(x)D_{l-i}(y)$ for $l = 0, 1, \ldots, k$. Then

$$(k+1)G_{k+1}(xy) = \sum_{i=0}^{k} g_{i+1}G_{k-i}(xy) =$$
$$= \sum_{i=0}^{k} g_{i+1} \sum_{j=0}^{k-i} G_j(x)D_{k-i-j}(y) =$$

$$= \sum_{i=0}^{k} \sum_{j=0}^{k-i} \Big(g_{i+1} G_j(x) D_{k-i-j}(y) + G_j(x) d_{i+1} D_{k-i-j}(y) \Big).$$

Therefore,

$$(k+1)G_{k+1}(xy) = \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} g_{i+1} G_{k-j-i}(x) \right) D_j(y) +$$

$$+ \sum_{j=0}^{k} G_j(x) \sum_{i=0}^{k-j} d_{i+1} D_{k-j-i}(y) = \sum_{j=0}^{k} (k-j+1) G_{k-j+1}(x) D_j(y) +$$

$$+ \sum_{j=0}^{k} G_j(x)(k-j+1) D_{k-j+1}(y) = \sum_{j=1}^{k+1} j G_j(x) D_{k+1-j}(y) +$$

$$+ \sum_{j=0}^{k} (k-j+1) G_j(x) D_{k+1-j}(y) = (k+1) \sum_{j=0}^{k+1} G_j(x) D_{k+1-j}(y).$$

Thus, we have proved (6).

Theorem 2 is proved.

3. Additional remarks. In this last section of the paper we write some additional observations about our results. The first remark is connected with Example 1.

***Remark* 1.** It is easy to see that a generalized higher derivation $G = \{G_k\}_{k=0}^n = \varphi^{-1}\big(\{g_k\}_{k=0}^n\big)$ is an ordinary generalized higher derivation if and only if $g_k = 0$ for $k \geq 2$. In this case

$$G_k = \frac{g_1^k}{k!}, \quad k = 0, 1, \ldots, n.$$

***Remark* 2.** In our definition of higher derivations we assumed that D_0 is the identity map on $\mathcal{A}$. But if D_0 is any automorphism of $\mathcal{A}$ and $D = \big\{D_k\big\}_{k=0}^n$ a sequence of linear mappings with the property (1), then it is easy to see that D_1 is a (D_0, D_0)-derivation, i.e.,

$$D_1(xy) = D_1(x) D_0(y) + D_0(x) D_1(y), \quad x, y \in \mathcal{A},$$

and (following the proof of Proposition 2.1 in [7]), we can deduce that the sequence $\{D_k\}_{k=0}^n$ corresponds to the sequence $\{d_k\}_{k=0}^n$ of (D_0, D_0)-derivations on $\mathcal{A}$ with $d_0 = 0$.

Similarly, we can define a generalized higher derivation as a sequence of linear mappings $G = \{G_k\}_{k=0}^n$, where G_0 is any automorphism of $\mathcal{A}$, such that (2) holds true for all $x, y \in \mathcal{A}$, $k = 0, 1, \ldots, n$, and some higher derivation $D = \{D_k\}_{k=0}^n$ with $D_0 = G_0$. Suppose that $\{D_k\}_{k=0}^n$ corresponds to the sequence $\{d_k\}_{k=0}^n$ of (G_0, G_0)-derivations on $\mathcal{A}$ with $d_0 = 0$. For all $k = 0, 1, \ldots, n$, let us denote $\hat{G}_k = G_k G_0^{-1}$, $\hat{D}_k = D_k G_0^{-1}$, and $\hat{d}_k = d_k G_0^{-1}$. Then it is easy to see that $\hat{G} = \{\hat{G}_k\}_{k=0}^n$ is a generalized higher derivation associated with a higher derivation $\hat{D} = \{\hat{D}_k\}_{k=0}^n$. Moreover, $\hat{G}_0 = \hat{D}_0 = I$. Thus, there exists a sequence $\{\hat{g}_k\}_{k=1}^n$ of generalized $\hat{d}_k$-derivations such that (3) holds true for $k = 1, \ldots, n$. Now, one can easily conclude that the sequence $\{G_k\}_{k=0}^n$ corresponds to the sequence $\{\hat{g}_k G_0\}_{k=0}^n$ of generalized (G_0, G_0)-derivations on $\mathcal{A}$ associated with (G_0, G_0)-derivations $\{\hat{d}_k G_0\}_{k=0}^n = \{d_k\}_{k=0}^n$. Here, of course, $\hat{g}_0 = 0 = \hat{d}_0$.

***Remark* 3.** Similarly as higher derivations and generalized higher derivations, we can define higher Jordan (Lie) derivations and generalized higher Jordan (Lie) derivations. Moreover, following the proofs in Section 2, we can deduce that there is a one to one correspondence between the set of all (generalized) higher Jordan derivations $\{J_k\}_{k=0}^n$ on $\mathcal{A}$ with $J_0 = I$ and the set of all sequences $\{j_k\}_{k=0}^n$ of (generalized) Jordan derivations on $\mathcal{A}$ with $j_0 = 0$. Similarly, there is a one-to-one correspondence between the set of all (generalized) higher Lie derivations $\{L_k\}_{k=0}^n$ on $\mathcal{A}$ with $L_0 = I$ and the set of all sequences $\{l_k\}_{k=0}^n$ of (generalized) Lie derivations on $\mathcal{A}$ with $l_0 = 0$.

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