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TWO-WEIGHTED INEQUALITIES FOR RIESZ POTENTIAL IN p-CONVEX WEIGHTED MODULAR BANACH FUNCTION SPACES

TWO-WEIGHTED INEQUALITIES FOR RIESZ POTENTIAL IN p-CONVEX WEIGHTED MODULAR BANACH FUNCTION SPACES

The main goal of this paper is to prove a two-weight boundedness for Riesz potential from one weighted Banach function space to another weighted Banach function space. In particular, we obtain a two-weight boundedness for Riesz potential and find sufficient conditions on the weights for boundedness of Riesz potential in weighted Musielak – Orlicz spaces.

Основна мета роботи — встановити двохвагову обмеженість потенціалу Ріса з одного вагового Банахового простору в інший ваговий Банахів простір. Зокрема, встановлено двохвагову обмеженість потенціалу Ріса та отримано достатні умови, що треба накласти на вагу з метою гарантувати обмеженість потенціалу Ріса у вагових просторах Мусіляка – Орліча.

1. Introduction. The investigation of Riesz operator in weighted Banach function spaces (BFS) have recent history. The goal of this investigations were closely connected with founding the criterion on the geometry and on the weights of BFS for validity of boundedness of Riesz operator in BFS. Characterization of the mapping properties such as boundedness and compactness was considered in the papers [9, 10, 14, 33] and etc. More precisely, in [9, 10] were considered the boundedness of certain integral operator in ideal Banach spaces. In [14] the boundedness of Hardy operator was proved in Orlicz spaces. Also, in [33] the compactness and measure of noncompactness of Hardy type operator in BFS was proved. But in this paper we used the boundedness of Hardy operator in p-convex BFS. Note that the notion of BFS was introduced in [35]. In particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm, Musielak – Orlicz spaces, etc are BFS.

In this paper, we establish an integral-type sufficient condition on weights, which provides the boundedness of the Riesz operator from one weighted BFS to another weighted BFS.

2. Preliminaries. Let (Ω, μ) be a complete σ-finite measure space. By $L_0 = L_0(\Omega, \mu)$ we denote the collection of all real-valued μ-measurable functions on Ω.

Definition 2.1 [20]. *Let L be a real vector space. A function $\rho\colon L \mapsto [0, \infty]$ is called a semimodular on L if the following properties hold*:

(a) $\rho(0) = 0$.

(b) $\rho(\lambda x) = \rho(x)$ *for all* $x \in L$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| = 1$.

(c) ρ *is convex.*

(d) ρ *is left-continuous.*

(e) $\rho(\lambda x) = 0$ *for all* $\lambda > 0$ *implies* $x = 0$.

A semimodular ρ is called modular if

(f) $\rho(x) = 0$ *implies* $x = 0$.

A semimodular ρ is called continuous if

(g) *the mapping* $\lambda \mapsto \rho(\lambda x)$ *is continuous on* $[0, \infty)$ *for every* $x \in L$.

If ρ is semimodular or modular on L, then

$$L_\rho := \left\{ x \in L\colon \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}$$

is called a semimodular space or modular space, respectively. The limit $\lambda \to 0$ takes place in $\mathbb{R}$.

Theorem 2.1 [20]. *Let ρ be semimodular on L. Then L_ρ is a normed real vector space. The norm called the Luxemburg norm, is defined by*

$$\|x\|_\rho := \inf \left\{ \lambda > 0\colon \rho\left(\frac{1}{\lambda} x\right) \le 1 \right\}.$$

Definition 2.2 [8, 32, 35]. *We say that real normed space X is a Banach function space* (*BFS*) *if*:

(P$_1$) *the norm $\|f\|_X$ is defined for every μ-measurable function f, and $f \in X$ if and only if $\|f\|_X < \infty$; $\|f\|_X = 0$ if and only if $f = 0$ a.e.*;

(P$_2$) $\|f\|_X = \||f|\|_X$ *for all* $f \in X$;

(P$_3$) *if* $0 \le f_n \uparrow f \le g$ *a.e.*, *then* $\|f_n\|_X \uparrow \|f\|_X$ (*Fatou property*);

(P$_4$) *if E is a measurable subset of Ω such that $\mu(E) < \infty$, then $\|\chi_E\|_X < \infty$, where χ_E is the characteristic function of the set E*;

(P$_5$) *for every measurable set $E \subset \Omega$ with $\mu(E) < \infty$, there is a constant $C_E > 0$ such that* $\displaystyle\int_E f(x)\,dx \le C_E \|f\|_X$.

Recall that condition (P$_3$) immediately yields the following property:

$$\text{if} \quad 0 \le f \le g, \qquad \text{then} \quad \|f\|_X \le \|g\|_X.$$

Given a BFS X we can always consider its associate space X' consisting of those $g \in L_0$ that $f \cdot g \in L_1$ for every $f \in X$ with usual order and the norm $\|g\|_{X'} = \sup\left\{\|f \cdot g\|_{L_1} : \|g\|_{X'} \le 1\right\}$. Note that X' is a BFS in (Ω, μ) and a closed norming subspace.

Let X be a BFS and ω be a weight, that is, positive Lebesgue measurable and a.e. finite function on Ω. Let $X_\omega = \{f \in L_0\colon f\omega \in X\}$. This space is a weighted BFS equipped with the norm $\|f\|_{X_\omega} = \|f\,\omega\|_X$. (For more detail and proofs of results about BFS we refer the reader to [8, 32].)

Let us recall the notion of p-convexity and p-concavity of BFS.

Definition 2.3 [42]. *Let X be a BFS. Then X is called p-convex for $1 \le p \le \infty$ if there exists a constant $M > 0$ such that for all $f_1, \ldots, f_n \in X$*

$$\left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X \le M \left(\sum_{k=1}^n \|f_k\|_X^p \right)^{1/p} \qquad \textit{if} \quad 1 \le p < \infty,$$

or $\left\|\sup_{1\le k\le n} |f_k|\right\|_X \le M \max_{1\le k\le n} \|f_k\|_X$ *if* $p = \infty$. *Similarly, X is called p-concave for $1 \le \le p \le \infty$ if there exists a constant $M > 0$ such that for all $f_1, \ldots, f_n \in X$*

$$\left(\sum_{k=1}^n \|f_k\|_X^p \right)^{1/p} \le M \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X \qquad \textit{if} \quad 1 \le p < \infty,$$

or $\displaystyle\max_{1\le k\le n} \|f_k\|_X \le M \left\|\sup_{1\le k\le n} |f_k|\right\|_X$ *if* $p = \infty$.

***Remark* 2.1.** Note that the notion of p-convexity, respectively p-concavity are closely related to the notion of upper p-estimate (strong ℓ_p-composition property), respectively lower p-estimate (strong ℓ_p-decomposition property) as can be found in [32].

Now we show some examples of p-convex and respectively p-concave BFS.

Let $\mathbb{R}^n$ be n-dimensional Euclidean space of the points $x = \big(x_1, \ldots, x_n\big)$ and let Ω be a Lebesgue measurable subset in $\mathbb{R}^n$ and $|x| = \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}$. The Lebesgue measure of a set Ω will be denoted by $|\Omega|$. It is well known that $|B(0,1)| = \dfrac{\pi^{n/2}}{\Gamma\left(\dfrac{n}{2}+1\right)}$, where $B(0,1) = \{x\colon x \in \mathbb{R}^n\ |x| < 1\}$. Assume that $\delta\colon \Omega \to [1,\infty)$. Throughout this paper, assume $\underline{\delta} = \operatorname{ess\,inf}_{x\in\Omega} \delta(x)$ and $\overline{\delta} = \operatorname{ess\,sup}_{x\in\Omega} \delta(x)$ and $p' = \dfrac{p}{p-1}$ be conjugate exponent of $p > 1$.

***Example* 2.1.** Let $1 \le q \le \infty$ and $X = L_q$. Then the space L_q is p-convex (p-concave) modular BFS if and only if $1 \le p \le q \le \infty$ $(1 \le q \le p \le \infty.)$

The proof implies from Minkowski inequality in Lebesgue spaces.

***Example* 2.2.** The following lemma shows that the variable Lebesgue space $L_{q(\cdot)}(\Omega)$ is a p-convex modular BFS.

Lemma 2.1 [1]. *Let $1 \le p \le q(x) \le \overline{q} < \infty$ for all $x \in \Omega_2 \subset \mathbb{R}^m$. Then the inequality*

$$\Big\| \|f\|_{L_p(\Omega_1)} \Big\|_{L_{q(\cdot)}(\Omega_2)} \le C_{p,q}^{2/p} \Big\| \|f\|_{L_{q(\cdot)}(\Omega_2)} \Big\|_{L_p(\Omega_1)}$$

is valid, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + p\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right)\big(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty \big),$$

$\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2 \colon q(y) = p\}$, $\Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1$ *and* $f\colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is any measurable function such that*

$$\Big\| \|f\|_{L_p(\Omega_1)} \Big\|_{L_{q(\cdot)}(\Omega_2)} = \inf\left\{ \mu > 0 \colon \int\limits_{\Omega_2} \left(\frac{\|f(\cdot,y)\|_{L_p(\Omega_1)}}{\mu} \right)^{q(y)} dy \le 1 \right\} < \infty$$

and $\|f(\cdot,y)\|_{L_p(\Omega_1)} = \left(\displaystyle\int_{\Omega_1} |f(x,y)|^p\, dx \right)^{1/p}$.

Analogously, if $1 \le q(x) \le p < \infty$, then $L_{q(x)}(\Omega)$ is a p-concave BFS.

Definition 2.4 [20, 40]. *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. A real function $\varphi\colon \Omega \times \times [0,\infty) \mapsto [0,\infty)$ is called a generalized φ-function if it satisfies*:

(a) $\varphi(x,\cdot)$ *is a φ-function for all $x \in \Omega$, i.e., $\varphi(x,\cdot)\colon [0,\infty) \mapsto [0,\infty)$ is convex and satisfies* $\varphi(x,0) = 0$, $\lim_{t\to+0} \varphi(x,t) = 0$;

(b) $\psi\colon x \mapsto \varphi(x,t)$ *is measurable for all $t \ge 0$.*

If φ is a generalized φ-function on Ω, we briefly write $\varphi \in \Phi$.

Definition 2.5 [20, 40]. *Let $\varphi \in \Phi$ and be ρ_φ defined by the expression*

$$\rho_\varphi(f) := \int\limits_\Omega \varphi(x, |f(x)|)\, dx \qquad \text{for all} \quad f \in L_0(\Omega).$$

We put $L_\varphi = \{f \in L_0(\Omega) \colon \rho_\varphi(\lambda_0 f) < \infty$ *for some* $\lambda_0 > 0\}$ *and*

$$\|f\|_{L_\varphi} = \inf\left\{\lambda > 0 \colon \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The space L_φ *is called Musielak–Orlicz space.*

Let ω be a weight function on Ω, i.e., ω be a nonnegative, almost everywhere positive function on Ω. We denote

$$L_{\varphi,\omega} = \{f \in L_0(\Omega) \colon f\omega \in L_\varphi\}.$$

It is obvious that the norm in this space is given by

$$\|f\|_{L_{\varphi,\omega}} = \|f\omega\|_{L_\varphi}.$$

Remark **2.2.** Let $\varphi(x,t) = t^{q(x)}$ in Definition 2.4, where $1 \le q(x) < \infty$ and $x \in \Omega$. Then we have the definition of variable exponent weighted Lebesgue spaces $L_{q(x)}(\Omega)$. About detail information on variable exponent Lebesgue spaces we refer to [18].

Example **2.3.** The following lemma shows that the Musielak–Orlicz space L_φ is a p-convex modular BFS.

Lemma 2.2 [6]. *Let* $\Omega_1 \subset \mathbb{R}^n$ *and* $\Omega_2 \subset R^m$. *Let* $(x,t) \in \Omega_1 \times [0,\infty)$ *and* $\varphi\left(x, t^{1/p}\right) \in \Phi$ *for some* $1 \le p < \infty$. *Suppose* $f \colon \Omega_1 \times \Omega_2 \mapsto R$. *Then the inequality*

$$\left\|\|f(x,\cdot)\|_{L_p(\Omega_2)}\right\|_{L_\varphi} \le 2^{1/p}\left\|\|f(\cdot,y)\|_{L_\varphi}\right\|_{L_p(\Omega_2)}$$

is valid.

We note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces, etc are p-convex (p-concave) modular BFS. Now we reduce a more general result connected with Minkowski's integral inequality.

Let X and Y be BFS on (Ω_1,μ) and (Ω_2,ν), respectively. By $X[Y]$ and $Y[X]$ we denote the spaces with mixed norm and consisting of all functions $g \in L_0(\Omega_1 \times \Omega_2, \mu \times \nu)$ such that $\left\|g(x,\cdot)\right\|_Y \in X$ and $\left\|g(\cdot,y)\right\|_X \in Y$. The norms in this spaces define are as follows:

$$\|g\|_{X[Y]} = \left\|\|g(x,\cdot)\|_Y\right\|_X, \qquad \|g\|_{Y[X]} = \left\|\|g(\cdot,y)\|_X\right\|_Y.$$

It is known that $X[Y]$ and $Y[X]$ are BFS on $\Omega_1 \times \Omega_2$ (see [32].)

Definition 2.6 [40]. *We say that modular BFS* X *satisfies the* Δ_2*-condition if there exists* $K \ge 2$ *such that*

$$\rho(2f) \le K\,\rho(f)$$

for all $f \in X$ *and all* $t > 0$. *The smallest such* K *is called the* Δ_2*-constant of* ρ.

Lemma 2.3. *Let* X *modular BFS,* $\gamma \ge 1$ *and* $1 \le q(x) \le \overline{q} < \infty$. *Further, let*

$$\min_{s>0}\{s, s^\gamma\}\,\rho(f) \le \rho(sf) \le \max_{s>0}\left\{s, s^{q(x)}\right\}\rho(f) \tag{2.1}$$

for almost all $x \in \Omega$ *and all* $f \in X_\rho$. *Then* $\rho\left(\dfrac{f}{\|f\|_\rho}\right) = 1$ *and*

$$\min_{\|f\|_\rho}\left\{\|f\|_\rho, \|f\|_\rho^\gamma\right\} \le \rho(f) \le \max_{\|f\|_\rho}\left\{\|f\|_\rho, \|f\|_\rho^{q(x)}\right\} \qquad \textit{for any} \quad x \in \Omega.$$

Proof. Let $0 < \|f\|_\rho < \infty$ and $\rho\left(\frac{f}{\|f\|_\rho}\right) < 1$. We choose a positive number $\lambda \leq \|f\|_\rho$ such that $\rho\left(\frac{f}{\lambda}\right) < 1$. Indeed, we put $\lambda = \|f\|_\rho \rho^{1/\overline{q}}\left(\frac{f}{\|f\|_\rho}\right)$. Then $\lambda < \|f\|_\rho$ and by virtue of condition (2.1) for $s > 1$ we have

$$\rho\left(\frac{f}{\lambda}\right) = \rho\left(\frac{f}{\|f\|_\rho \rho^{1/\overline{q}}\left(\frac{f}{\|f\|_\rho}\right)}\right) \leq \rho^{-q(x)/\overline{q}}\left(\frac{f}{\|f\|_\rho}\right)\rho\left(\frac{f}{\|f\|_\rho}\right) \leq$$

$$\leq \rho^{-1}\left(\frac{f}{\|f\|_\rho}\right)\rho\left(\frac{f}{\|f\|_\rho}\right) = 1.$$

Lemma 2.3 is proved.

We consider the multidimensional Hardy type operator and its dual operator

$$Hf(x) = \int\limits_{|y|<|x|} f(y)\,dy \qquad \text{and} \qquad H^*f(x) = \int\limits_{|y|>|x|} f(y)\,dy,$$

where $f \geq 0$ and $x \in \mathbb{R}^n$.

Now we reduce a two-weight criterion for multidimensional Hardy type operator acting from the p-concave weighted BFS to weighted Lebesgue spaces. Suppose that $M > 0$ the constant in Definition 2.3.

Theorem 2.2 [7]. *Let $v(x)$ and $w(x)$ be weights on $\mathbb{R}^n$. Suppose that X_w is a p-convex weighted BFSs for $1 \leq p < \infty$ on $\mathbb{R}^n$. Then the inequality*

$$\|Hf\|_{X_w} \leq C\|f\|_{L_{p,\,v}} \tag{2.2}$$

holds for every $f \geq 0$ if and only if there is a $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0}\left(\int\limits_{|y|<t} v(y)^{-p'}\,dy\right)^{\alpha/p'}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|} v(y)^{-p'}\,dy\right)^{(1-\alpha)/p'}\right\|_{X_w} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (2.2), *then*

$$\sup_{0<\alpha<1}\frac{p'\,A(\alpha)}{(1-\alpha)\left(\left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha\,(p-1)}\right)^{1/p}} \leq C \leq M \inf_{0<\alpha<1}\frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

For the dual operator, the below stated theorem is proved analogously.

Theorem 2.3 [7]. *Let $v(x)$ and $w(x)$ be weights on $\mathbb{R}^n$. Suppose that X_w is a p-convex weighted BFS for $1 \leq p < \infty$ on $\mathbb{R}^n$. Then the inequality*

$$\|H^*f\|_{X_w} \leq C\,\|f\|_{L_{p,\,v}} \tag{2.3}$$

holds for every $f \geq 0$ if and only if there is a $\gamma \in (0,1)$

$$B(\gamma)=\sup_{t>0}\left(\int\limits_{|y|>t} v(y)^{-p'}\,dy\right)^{\gamma/p'}\left\|\chi_{\{|z|<t\}}(\cdot)\left(\int\limits_{|y|>|\cdot|} v(y)^{-p'}\,dy\right)^{(1-\gamma)/p'}\right\|_{X_w}<\infty.$$

Moreover, if $C>0$ *is the best possible constant in* (2.3), *then*

$$\sup_{0<\gamma<1}\frac{p'\,B(\gamma)}{(1-\gamma)\left(\left(\dfrac{p'}{1-\gamma}\right)^{p}+\dfrac{1}{\gamma\,(p-1)}\right)^{1/p}}\le C\le M\inf_{0<\gamma<1}\frac{B(\gamma)}{(1-\gamma)^{1/p'}}.$$

***Corollary* 2.1.** *Note that Theorems* 2.2 *and* 2.3 *in the case* $X_w=L_{\varphi,w}$, $\varphi\left(x,t^{1/p}\right)\in\Phi$ *for some* $1\le p<\infty$, $x\in\mathbb{R}^n$ *were proved in* [6]. *In the case* $X_w=L_{q,w}$, $1<p\le q<\infty$, *for* $x\in(0,\infty)$, $\alpha=\dfrac{s-1}{p-1}$ *and* $s\in(1,p)$ *Theorems* 2.2 *and* 2.3 *were proved in* [44]. *For* $x\in\mathbb{R}^n$ *in the case* $X_w=L_{q(x),w}$ *and* $1<p\le q(x)\le \operatorname{ess\,sup}_{x\in\mathbb{R}^n} q(x)<\infty$ *Theorems* 2.2 *and* 2.3 *were proved in* [3] (*see also* [2]).

***Remark* 2.3.** In the case $n=1$, $X_w=L_{q,w}$, $1<p\le q\le\infty$, at $x\in(0,\infty)$, for classical Lebesgue spaces the various variants of Theorems 2.2 and 2.3 were proved in [12, 23, 25–28, 30, 31, 34, 38, 39, 43] etc. In particular, in the Lebesgue spaces with variable exponent the boundedness of Hardy type operator was proved in [15–17, 19, 21, 24, 29, 36, 37] etc. For $X_w=L_{q(x),w}$, $1< < p\le q(x)\le \operatorname{ess\,sup}_{x\in[0,1]} q(x)<\infty$ and $x\in[0,1]$ the two-weighted criterion for one-dimensional Hardy operator was proved in [29]. Also, other type two-weighted criterion for multidimensional Hardy type operator in the case $X_w=L_{q(x),w}$, $1<p\le q(x)\le \operatorname{ess\,sup}_{x\in\mathbb{R}^n} q(x)<\infty$ and $x\in\mathbb{R}^n$ was proved in [36] (see also [37] and [17]). In the case $L_{q(x),w}$ for $0<\underline{q}\le\overline{q}<1$ the boundedness of classical Hardy operator was proved in [5]. In the papers [11] and [41] the inequalities of modular type for more general operators were proved. Also, in [13] the Hardy type inequalities with special power-type weights in Orlicz spaces were proved.

3. Main result. Now we consider the Riesz potential $\mathcal{R}^s f(x)=\displaystyle\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-s}}\,dy$, where $0<s<n$.

The sufficient conditions for general weights ensuring the validity of the two-weight strong type inequalities for the Riesz potential in BFS are given in the following theorem.

Theorem 3.1. *Suppose that* $v(x)$ *and* $w(x)$ *be weight functions on* $\mathbb{R}^n$. *Let* Y_w *be a modular* p*-convex weighted BFS for* $1\le p<\infty$ *and* $x\in\mathbb{R}^n$. *Let* $0<s<n$, $\mathcal{R}^s$ *is bounded from* X *into* Y *and let* $L_{p,v}(\mathbb{R}^n)\hookrightarrow X_v$. *Let there exists* $r(x)\colon 1<p\le r(x)\le\overline{r}<\infty$ *such that, for all* $C>0$ $\rho(Cf)\le C_1(r)\,\rho(f)$, *where* $C_1(r)=\max\left\{C^{\underline{r}},C^{\overline{r}}\right\}$.

Moreover, *let* $v(x)$ *and* $w(x)$ *satisfy the following three conditions*:

$$1)\quad A=\sup_{t>0}\left(\int_{|y|<t} v(y)^{-p'}\,dy\right)^{\alpha/p'}\left\|\frac{\chi_{\{|x|>t\}}}{|x|^{n-s}}\left(\int_{|y|<|x|} v(y)^{-p'}\,dy\right)^{(1-\alpha)/p'}\right\|_{Y_w}<\infty; \qquad (3.1)$$

$$2)\quad B=\sup_{t>0}\left(\int_{|y|>t}\left(v(y)|y|^{n-s}\right)^{-p'}\,dy\right)^{\beta/p'}\times$$

$$\times \left\| \chi_{\{|x|<t\}} \left(\int_{|y|>|x|} \big(v(y)|y|^{n-s}\big)^{-p'}\, dy \right)^{(1-\beta)/p'} \right\|_{Y_w} < \infty, \tag{3.2}$$

where $0 < \alpha, \beta < 1$;

3) *there exists* $M > 0$ *such that*

$$\sup_{|x|/2<|y|\le 4\,|x|} w(y) \le M \inf_{|x|/2<|y|\le 4\,|x|} v(y). \tag{3.3}$$

Then there exists a positive constant C, *independent of* f, *such that for all* $f \in X_v$

$$\|\mathcal{R}^s f\|_{Y_w} \le C\|f\|_{X_v}.$$

Proof. Let $Z = \{0, \pm1, \pm2, \ldots\}$. For $k \in Z$ we define $E_k = \{x \in \mathbb{R}^n : 2^k < |x| \le 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n : |x| \le 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^{k+2}\}$, $E_{k,3} = \{x \in \mathbb{R}^n : |x| > 2^{k-1}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,v}(\mathbb{R}^n)$, we write

$$\big|\mathcal{R}^s f(x)\big| = \sum_{k\in Z} |\mathcal{R}^s f(x)|\, \chi_{E_k}(x) \le$$

$$\le \sum_{k\in Z} |\mathcal{R}^s f_{k,1}(x)|\, \chi_{E_k}(x) + \sum_{k\in Z} |\mathcal{R}^s f_{k,2}(x)|\, \chi_{E_k}(x) + \sum_{k\in Z} |\mathcal{R}^s f_{k,3}(x)|\, \chi_{E_k}(x) =$$

$$= \mathcal{R}^s_1 f(x) + \mathcal{R}^s_2 f(x) + \mathcal{R}^s_3 f(x),$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1, 2, 3$.

First we shall estimate $\|\mathcal{R}^s_1 f\|_{Y_w}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $|y| < 2^{k-1} \le |x|/2$. Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \varnothing$ and $|x-y| \ge |x| - |y| \ge |x| - |x|/2 = |x|/2$. Hence we have

$$\big|\mathcal{R}^s_1 f(x)\big| \le C \sum_{k\in Z} \left(\int_{\mathbb{R}^n} \frac{|f_{k,1}(y)|}{|x-y|^{n-s}}\, dy \right) \chi_{E_k} \le C \int_{|y|<|x|/2} \frac{|f(y)|}{|x-y|^{n-s}}\, dy \le$$

$$\le C \int_{|y|<|x|} \frac{|f(y)|}{|x-y|^{n-s}}\, dy \le 2^n\, C\, \frac{1}{|x|^{n-s}} \int_{|y|<|x|} |f(y)|\, dy$$

for any $x \in E_k$. Hence we get

$$\|\mathcal{R}^s_1 f\|_{Y_w} \le 2^n\, C \left\| \frac{1}{|x|^{n-s}} \int_{|y|<|x|} |f(y)|\, dy \right\|_{Y_w} = \left\| \int_{|y|<|x|} |f(y)|\, dy \right\|_{Y_{w/|x|^{n-s}}}.$$

By condition (3.1) and Theorem 2.2, we obtain

$$\|\mathcal{R}^s_1 f\|_{Y_w} \le C_1 \|f\|_{L_{p,v}(\mathbb{R}^n)} \le C_2 \|f\|_{X_v}, \tag{3.4}$$

where $C_1 > 0$ is independent of f and $x \in \mathbb{R}^n$.

Next we estimate $\|\mathcal{R}_3^s f\|_{Y_w}$. It is obvious that, for $x \in E_k,\ y \in E_{k,3}$ we have $|y| > 2\,|x|$ and $|x-y| \ge |y| - |x| \ge |y| - |y|/2 = |y|/2$. Since $E_k \cap \operatorname{supp} f_{k,3} = \varnothing$ for $x \in E_k$ we have

$$|\mathcal{R}_3^s f(x)| \le C \int\limits_{|y|>2|x|} \frac{|f(y)|}{|x-y|^{n-s}}\,dy \le 2^n\,C \int\limits_{|y|>2|x|} \frac{|f(y)|}{|y|^{n-s}}\,dy.$$

Hence we get

$$\|\mathcal{R}_3^s f\|_{Y_w} \le 2^n\,C \left\| \int\limits_{|y|>2|x|} \frac{|f(y)|}{|y|^{n-s}}\,dy \right\|_{Y_w} \le$$

$$\le 2^n\,C \left\| \int\limits_{|y|>|x|} \frac{|f(y)|}{|y|^{n-s}}\,dy \right\|_{Y_w}.$$

By condition (3.2) and Theorem 2.3, we obtain

$$\left\|\mathcal{R}_3^s f\right\|_{Y_w} \le C_2\,\|f\|_{L_{p,v}(\mathbb{R}^n)} \le C_3 \|f\|_{X_v}, \tag{3.5}$$

where $C_2 > 0$ is independent of f and $x \in \mathbb{R}^n$.

Finally we estimate $\|\mathcal{R}^s f_{k,2}\|_{Y_w}$, where

$$\|\mathcal{R}^s f_{k,2}\|_{Y_w} = \left\| \sum_{k\in Z} |\mathcal{R}^s f_{k,2}|\ \chi_{E_k} \right\|_{Y_w}.$$

By virtue of Lemma 2.3 it suffices to prove that from $\|f\|_{X_v} \le 1$ implies

$$\rho\left(w \sum_{k\in Z} |\mathcal{R}^s f_{k,2}|\, \chi_{E_k} \right) \le C,$$

where $C > 0$ is independent of $k \in Z$.

By the boundedness of $\mathcal{R}^s$ from X to Y and condition (3.3), we have

$$\rho\left(w(y) \sum_{k\in Z} |\mathcal{R}^s f_{k,2}(y)|\ \chi_{E_k}(y) \right) = \sum_{m\in Z} \rho\left(w(y) \sum_{k\in Z} |\mathcal{R}^s f_{k,2}(y)|\ \chi_{E_k}(y) \right) =$$

$$\sum_{k\in Z} \rho\left(w(y)\ |\mathcal{R}^s f_{k,2}(y)| \right) = \sum_{k\in Z} \rho\left(C\,w(y)\,\|f_{k,2}\|_X\ \frac{|\mathcal{R}^s f_{k,2}|}{C\,\|f_{k,2}\|_X} \right) \le$$

$$\le \sum_{k\in Z} \left(C\,w(y)\,\|f_{k,2}\|_X \right)^{r(y)} \rho\left(\frac{|\mathcal{R}^s f_{k,2}|}{C\,\|f_{k,2}\|_X} \right) \le$$

$$\le C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(w(y) \|f\|_{X(E_{k,2})} \right)^{r(y)} \rho\left(\frac{|\mathcal{R}^s f_{k,2}|}{C\,\|f_{k,2}\|_X} \right) \le$$

$$\le C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(w(y) \|f\|_X \right)^{r(y)} = C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(\|f\,w\|_{X(E_{k,2})} \right)^{r(y)} \le$$

$$\leq C_3 \sum_{k\in Z} \sup_{y\in E_k} \left(\|f \inf_{y\in E_{k,2}} v(y)\|_{X\left(E_{k,2}\right)} \right)^{r(y)} \leq C_3 \sum_{k\in Z} \sup_{y\in E_k} \left(\|f\, v\|_{X\left(E_{k,2}\right)} \right)^{r(y)} =$$

$$= C_3 \sum_{k\in Z} \left(\|f\|_{X_v\left(E_{k,2}\right)} \right)^{\inf\limits_{y\in E_k} r(y)} \leq C_3 \sum_{k\in Z} \left(\|f\|_{X_v\left(E_{k,2}\right)} \right)^{\underline{r}} \leq$$

$$\leq C_3 \sum_{k\in Z} \rho \left(|f(y)| v(y) \chi_{E_{k,2}} \right)^{\underline{r}/\gamma} =$$

$$= C_3 \sum_{k\in Z} \left[\rho \left(|f(y)| v(y) \left(\chi_{E_{k-1}} + \chi_{E_k} + \chi_{E_{k+1}} \right) \right) \right]^{\underline{r}/\gamma} \leq$$

$$\leq C_3 \left[\rho \left(|f(y)| v(y) \right) \right]^{\underline{r}/\gamma} \left(\sum_{k\in Z} \chi_{E_{k-1}} + \sum_{k\in Z} \chi_{E_k} + \sum_{k\in Z} \chi_{E_{k+1}} \right)^{\underline{r}/\gamma} =$$

$$= C_3 \left(3\, \rho \left(|f(y)| v(y) \right) \right)^{\underline{r}/\gamma} \leq 3^{\underline{r}/\gamma} C_3 \leq C_4.$$

Thus

$$\|\mathcal{R}_2^s f\|_{Y_w} \leq C_5, \tag{3.6}$$

where $C > 0$ is independent of f and $x \in \mathbb{R}^n$.

Combining the inequalities (3.4), (3.5) and (3.6), we obtain the proof of Theorem 3.4.

Theorem 3.2 [40]. *Let $\psi \in \Phi$ and $\delta \geq 1$. Then $L_\psi(\mathbb{R}^n) \hookrightarrow L_\delta(\mathbb{R}^n)$ if and only if there exists $C > 0$ and $h \in L_1(\mathbb{R}^n)$ with $\|h\|_{L_1(\mathbb{R}^n)} \leq 1$ such that*

$$\left(\frac{t}{C} \right)^\delta \leq \psi(x,t) + h(x) \tag{3.7}$$

for almost all $x \in \mathbb{R}^n$ and all $t \geq 0$.

Lemma 3.1. *Let $\psi \in \Phi$, $\gamma \geq 1$ and $1 \leq q(x) \leq \overline{q} < \infty$. Further, let*

$$\min_{s>0} \{s,\, s^\gamma\}\, \psi(x,t) \leq \psi(x,st) \leq \max_{s>0} \left\{s,\, s^{q(x)}\right\} \psi(x,t) \tag{3.8}$$

for almost all $x \in \Omega$ and all $t \geq 0$. Then $\rho_\psi\left(\dfrac{f}{\|f\|_{L_\psi}} \right) = 1$ and

$$\min_{\|f\|_{L_\psi}} \left\{ \|f\|_{L_\psi},\, \|f\|_{L_\psi}^\gamma \right\} \leq \rho_\psi(f) \leq \max_{\|f\|_{L_\psi}} \left\{ \|f\|_{L_\psi},\, \|f\|_{L_\psi}^{q(x)} \right\}.$$

From Theorem 3.1 we have the following corollary.

***Corollary* 3.1.** *Let for some $1 < p < \infty$, $\varphi(x, t^{1/p}) \in \Phi$ and a function $\psi \in \Phi$ satisfy conditions* (3.7) *and* (3.8), *where $x \in \mathbb{R}^n$. Suppose that $v(x)$ and $w(x)$ be weight functions on $\mathbb{R}^n$. Let $\mathcal{R}^s$ is bounded from $L_\psi(\mathbb{R}^n)$ to $L_\varphi(\mathbb{R}^n)$. Let there exists $r(x)\colon 1 < \theta \leq r(x) \leq \overline{r} < \infty$ such that, for all $C > 0$ $\varphi(x, Ct) \leq C^{r(x)}\, \varphi(x,t)$.*

Moreover, let $v(x)$ and $w(x)$ satisfy the following three conditions:

$$1)\ \sup_{t>0} \left(\int_{|y|<t} v(y)^{-p'}\, dy \right)^{\alpha/p'} \left\| \frac{w(\cdot)}{|\cdot|^{n-s}} \left(\int_{|y|<|\cdot|} v(y)^{-p'}\, dy \right)^{(1-\alpha)/p'} \right\|_{L_\varphi(|\cdot|>t)} < \infty;$$

2) $\sup\limits_{t>0}\left(\int_{|y|>t}\left(v(y)|y|^{n-s}\right)^{-p'}dy\right)^{\beta/p'}\left\|w(\cdot)\left(\int_{|y|>|\cdot|}\left(v(y)|y|^{n-s}\right)^{-p'}dy\right)^{(1-\beta)/p'}\right\|_{L_\varphi(|\cdot|<t)}<\infty,$

where $0<\alpha,\beta<1$.

3) *there exists* $M>0$ *such that*

$$\sup_{|x|/2<|y|\le 4\,|x|} w(y)\le M\inf_{|x|/2<|y|\le 4\,|x|} v(y).$$

Then there exists a positive constant C, *independent of* f, *such that for all* $f\in L_{\psi,v}(\mathbb{R}^n)$

$$\|\mathcal{R}^s f\|_{L_{\varphi,w}(\mathbb{R}^n)}\le C\|f\|_{L_{\psi,v}(\mathbb{R}^n)}.$$

Further, we assume that the exponent $p(x)$ satisfies the standard conditions

$$\bigl|p(x)-p(y)\bigr|\le\frac{M_1}{-\ln|x-y|},\qquad 0<|x-y|\le\frac12,\quad x,y\in\mathbb{R}^n, \tag{3.9}$$

together with the following conditions at infinity:

$$|p(x)-p(y)|\le\frac{M_2}{\ln(e+|x|)},\qquad |x|\ge|y|,\quad x,y\in\mathbb{R}^n, \tag{3.10}$$

where the positive constants M_1 and M_2 are independent of x and y. Note that, from condition (3.10) implies that there is some number p_∞ such that $p(x)\to p_\infty$ as $|x|\to\infty$, and this limit holds uniformly in all directions. It is known that if $p(x)$ satisfies (3.10), $p_\infty=\underline{p}$ and $\dfrac{1}{r(x)}=\dfrac{1}{\underline{p}}-\dfrac{1}{p(x)}$, then $\dfrac{1}{r(x)}$ satisfies (3.10), $\lim_{|x|\to\infty}r(x)=\infty$ and $L_{p(x)}(\mathbb{R}^n)\hookrightarrow L_{\underline{p}}(\mathbb{R}^n)$. In particular, for $X_v= L_{p(x),v}(\mathbb{R}^n)$ and $Y_w=L_{q(x),w}(\mathbb{R}^n)$ from Theorem 3.1 we have the following corollary.

Corollary 3.2. *Let* $\dfrac{1}{p(x)}-\dfrac{1}{q(x)}=\dfrac{s}{n}$, $\underline{p}>1$, $\overline{p}<n/s$, $\underline{q}\ge\overline{p}$ *and* $p(x)$ *satisfy conditions* (3.9) *and* (3.10) *with* $p_\infty=\underline{p}$. *Moreover, let* $v(x)$ *and* $w(x)$ *be weight functions on* $\mathbb{R}^n$ *and satisfy the following three conditions*:

1) $\sup\limits_{t>0}\left(\int_{|y|<t}v(y)^{-\overline{p}'}\,dy\right)^{\alpha/\overline{p}'}\left\|\frac{w(\cdot)}{|\cdot|^{n-s}}\left(\int_{|y|<|\cdot|}v(y)^{-\overline{p}'}\,dy\right)^{(1-\alpha)/\overline{p}'}\right\|_{L_{q(\cdot)}(|\cdot|>t)}<\infty,$

2) $\sup\limits_{t>0}\left(\int_{|y|>t}\left(v(y)|y|^{n-s}\right)^{-\overline{p}'}\,dy\right)^{\beta/\overline{p}'}\left\|w(\cdot)\left(\int_{|y|>|\cdot|}\left(v(y)|y|^{n-s}\right)^{-\overline{p}'}dy\right)^{(1-\beta)/\overline{p}'}\right\|_{L_{q(\cdot)}(|\cdot|<t)}<\infty$, *where* $0<\alpha,\beta<1$;

3) *there exists a constant* $M>0$ *such that*

$$\sup_{|x|/4<|y|\le 4\,|x|} w(y)\le M\inf_{|x|/4<|y|\le 4\,|x|} v(y)\qquad \textit{for a.e.}\quad x\in\mathbb{R}^n.$$

Then there exists a positive constant C *independent of* f *such that for all* $f\in L_{p(x),v}(\mathbb{R}^n)$

$$\bigl\|\mathcal{R}^s f\bigr\|_{L_{q(\cdot),w}(\mathbb{R}^n)}\le C\|f\|_{L_{p(\cdot),v}(\mathbb{R}^n)}.$$

***Remark* 3.1.** In the case $X_v = L_{p,v}$, $Y_w = L_{q,w}$, $1 < p \leq q \leq \infty$ for classical Lebesgue spaces various variants of Theorem 3.1 were proved in [4, 22, 45] etc.

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