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D. Bhati, P. Singh (Motilal Nehru Nat. Inst. Technology, Allahabad, India)

MULTI-OBJECTIVE NONLINEAR SUM OF FRACTIONAL OPTIMIZATION PROBLEMS WITH NON-CONVEX CONSTRAINTS USING DUALITY BASED BRANCH AND BOUND ALGORITHM*

МУЛЬТІ-ОБ'ЄКТНА НЕЛІНІЙНА СУМА ДРОБОВИХ ОПТИМІЗАЦІЙНИХ ПРОБЛЕМ З НЕОПУКЛИМИ ОБМЕЖЕННЯМИ З ВИКОРИСТАННЯМ ДУАЛЬНОГО АЛГОРИТМУ ГІЛОК ТА ГРАНИЦЬ

The present paper investigates the solution of multiobjective nonlinear sum of fractional optimization problems. A duality based branch and bound cut method is developed to obtain efficient solution and the methodology is validate by proving the theoretical assertions for the solution. The present method is the extension of the work P. P. Shen, Y. P. Duan and Y. G. Pei[19] which developed for single objective sum of ratios nonlinear optimization problem. The proposed method is coded in matlab (version 2014b). Two numerical problems are considered for solving by using the proposed method and global optimal solution is obtained.

В роботі вивчається розв'язок мульті-об'єктної нелінійної суми дробових оптимізаційних проблем. Для ефективного розв'язування таких проблем в роботі розроблено дуальний алгоритм гілок та границь. Запропонована методологія обгрунтована доведенням необхідних теоретичних тверджень. Метод, що застосовано, є узагальненням роботи П. П. Шена, І. П. Дуана та І. Г. Пея (повне посилання) для однооб'єктної нелінійної оптимізаційної задачі про суми відношень. Цей метод реалізовано в MatLab (версія 2014b). Дві числові проблеми розглянуто і розв'язано за допомогою цього методу та отримано їх глобальні оптимальні розв'язки.

1. Introduction. In this paper, we consider a multiobjective nonlinear sum of fractional(MONSOF) optimization problem in the following form

$$\max\left\{\sum\nolimits_j^p \frac{N_{1j}(x)}{D_{1j}(x)}, \sum\nolimits_j^p \frac{N_{2j}(x)}{D_{2j}(x)}, \ldots, \sum\nolimits_j^p \frac{N_{kj}(x)}{D_{kj}(x)}\right\}$$

subject to $\qquad (P)$

$$x \in S = \{x \in \mathbb{R}^n : Ax \leq b, Y_{\bar{l}}(x) \leq 0 \ \ x \geq 0\}, \quad p, k \geq 2.$$

$N_{ij}(x)$ and $D_{ij}(x)$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, p$, are all real-valued nonlinear convex functions over R^n, with $D_{ij} > 0$ and $Y_{\bar{l}}(x)'s$, $\bar{l} = 1, 2, \ldots, m$, are all non-convex functions on R^q, S denotes the set of all feasible solutions and $A \in R^{q \times n}$, $b \in R^q$.

The sum of fractional optimization problem is one of the most difficult problem in the field of fractional optimization. The sum of fractional optimization arises naturally in decision making when several fractional are to be optimized simultaneously and a compromise is sought which optimizes a weight sum of these ratios. The application of sum of fractional optimization can be described in the situation where a compromise is needed between absolute and relative terms profit and return on investment (or return/risk). Mathis and Mathis [1] identified the application of sum of fractional optimization problem and formulate a fee optimization model. The model is used by hospital administrators in the state of taxes to decide on relative increase of charges for different medical procedure in various department. Since the understanding of sum of fractional multiobjective problem until now is quite limited. This is because of the special structure of objective functions. It is very difficult and

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challenging problem corresponding to general fractional optimization. The studies of sum of fractional optimization problem is restricted to single objective only and it not surprising that theoretical and algorithmic development for this problem is very limited too. However, recently some single objective sum of fractional optimization problem have been made. Benson [3 – 9] proposed various branch and bound techniques for solving the different variants of sum of ratios optimization problems. The main work of developed methods are involvement of a sequence of convex programming problems that differ only their objective function coefficients. These developed methods are applied to solve numerical problems and global optimal solutions are obtained in methods. The convergence studies are discussed for the proposed methods. These method can be used to find global optimal solution for single objective sum of ratio programming problem [10] Shen and Jin [11] proposed a global optimization for maximization sum of concave-convex ration within the convex feasible region, they used branch and bound scheme to develop algorithm. Wang and Zhang [12] presented a branch and bound algorithm for globally solving the nonlinear sum of ratios problem on nonconvex feasible region. They claimed that the proposed algorithm is convergent to the global minimum through the refinement of the solution of series of linear programming problems. Shen and Wang [13] proposed a branch and bound algorithm for solving the sum of linear ratios problem. They also proved the proposed algorithm is convergent to the global optimal solution by means of subsequent solutions of a series of linear programming problems. Jiao and Shen [14] gave short extension of the work of Wang and Zhang for nonlinear sum of ratio problem. They proposed more general results and used different equivalent problem. Qu, Zhang and Zhao [15] proposed a new branch and bound algorithm based on rectangle partition and the Lagrangian relaxation for solving sum of quadratic ratios problem with non-convex quadratic convergency of the algorithm. Shen, Chen and Yuan [16] proposed a branch reduced-bound algorithm for solving sum of quadratic ratios with non-convex constraints. They modi?ed the problem as monotonic optimization problem and find the globally optimal solution. Sheri. Li and Bai [17] gave a method to solve the problem of minimization of sum of convex-convex ratios problem with a convex feasible region and used branch and bound algorithm to propose new algorithm. They also established the global convergence of the method. Shen and Wang [18] developed an algorithm for sum of general fractional functions using linearization of method and branch-bound method. Shen, Duan and Pei [19] proposed branch and bound type algorithm to solve the sum of convex-convex ratio with non convex constraints. They used branch and bound method and Lagrange duality to develop the proposed scheme. Jaberipour and Khorram [20] developed a harmony search algorithm to find the solution of sum of ratios programming problems. This method is based on probability based search algorithm which was motivated by musical perfognfance. Jin and Hou [21] proposed a branch and bound algorithm to find global optimum solution for sum of ratio problem with ratio of the absolute value of affine functions with coefficients. To develop this algorithm they used rectangular partition and used the space of small dimension. Gao and Jin [22] proposed branch and bound algorithm by transformation of sum ratios programming problem into bilinear programming problem. To develop this algorithm, they used linear characteristic of convex envelope and concave envelop of double variables production function. They also used linear relaxation programming of the bilinear programming problem. Freund and Jarre [23] proposed an algorithm for global minimum solution for the sum of quasi convex ratio. They used interior point method to compute global minimum solution. Shaible and Shi [24] gave the review published the sum-of-ratios program. They gave survey of applications, theoretical results and various algorithmic approaches for this dif?cult fractional programming. Carlsson and Shi [25] proposed a linear relaxation technique to solve sum of linear ratios problem. They concentrated on lower dimensional problem for particular application and used transformation to developed the method. Recently, Ashtiani and Ferreira [26] developed a new branch bound algorithm for sum of ratios programming problem. They used decomposition approach to develop the algorithm. The literature evident that several branch and bound techniques are developed for solving the sum of ratio optimization problem. But most of the studies are restricted to single objective optimization problems only.

In comparison of single objective sum of fractional programming (SOFP), a solution to multi-objective sum of fractional programming (MOSOFP) is a concept which satisfies all the objectives simultaneously. There is no single global solution, and it is often necessary to determine a set of points that all fit a predetermined criteria of optimality. The concept of Pareto optimal solution can be defined as.

Definition 1 (Pareto optimal solution (maximization case)). *A solution $x^0 \in S$ is said to be a Pareto optimal solution for MONSOF optimization problem, if and only if there is no other solution $x \in S$ such that $\sum_j^p \dfrac{N_{ij}(x)}{D_{ij}(x)} \geq \sum_j^p \dfrac{N_{ij}(x^0)}{D_{ij}(x^0)}$ $\forall i = 1, 2, \ldots, k$ and $\sum_j^p \dfrac{N_{sj}(x)}{D_{sj}(x)} > \sum_j^p \dfrac{N_{sj}(x^0)}{D_{sj}(x^0)}$ for at least one s.*

Definition 2 (Pareto optimal Solution (minimization case)). *A solution $x^0 \in S$ is said to be a Pareto optimal solution for MONSOF optimization problem, if and only if there is no other solution $x \in S$ such that $\sum_j^p \dfrac{N_{ij}(x)}{D_{ij}(x)} \leq \sum_j^p \dfrac{N_{ij}(x^0)}{D_{ij}(x^0)}$ $\forall i = 1, 2, \ldots, k$ and $\sum_j^p \dfrac{N_{sj}(x)}{D_{sj}(x)} < \sum_j^p \dfrac{N_{sj}(x^0)}{D_{sj}(x^0)}$ for at least one s.*

In this paper, we proposed a method for solving multiobjective nonlinear sum of fractional (MONSOF) optimization problems. The organization of the remaining part of the paper is as follows. In Section 2, an equivalent problem is discussed. Next in Section 3, the proposed methodology and theoretical results are developed for the problem. The branch and bound cut method and its convergence is presented in Section 4. In Section 5, numerical experiment are given to demonstrate the proposed method. Finally conclusion are drawn in Section 6.

2. Equivalent and transformation. In order to solve non-convex problem, we can take $N_{ij} \leq 0$ for all $x \in S$ and therefore the problem (P) takes the form

$$\min \left\{ \sum_j^p \frac{-N_{1j}(x)}{D_{1j}(x)}, \sum_j^p \frac{-N_{2j}(x)}{D_{2j}(x)}, \ldots, \sum_j^p \frac{-N_{kj}(x)}{D_{kj}(x)} \right\}$$
$$\text{subject to} \qquad (M)$$
$$x \in S = \{x \in \mathbb{R}^n : Ax \leq b, Y_{\bar{l}}(x) \leq 0 \ \ x \geq 0\}, \quad p, k \geq 2.$$

Since D_{ij} are all convex functions on R^n, then D_{ij} are continuous on a compact set $S \subseteq R^n$ for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$ which means that there exists l_{ij} and L_{ij} such that $l_{ij} = \min D_{ij}(x) > 0$, $L_{ij} = \max D_{ij}(x) > 0$. Thus a set $D = \left\{ Z_{ij} \in R^{k\times p} \;\middle|\; 0 < \dfrac{1}{L_{ij}} \leq Z_{ij} \leq \dfrac{1}{l_{ij}},\ i = 1, 2, \ldots, k \text{ and } j = 1, 2, \ldots, p \right\} \subset R_+^{k\times p}$ can be constructed.

Now by introducing an additional variable vector $Z = (Z_{1j}, Z_{2j}, \ldots, Z_{kj}) \in R^{k\times p}$, we use the transformation $Z_{ij} = \dfrac{1}{D_{ij}}$ to convert the problem (M) into the following equivalent problem $(\widehat{M})$ and the equivalence may be established by Theorem 1:

$$\min\left\{\sum\nolimits_{j=1}^{p} -\big(N_{1j}(x)Z_{1j}(x)\big),\ \sum\nolimits_{j=1}^{p} -\big(N_{2j}(x)Z_{2j}(x)\big), \ldots,\ \sum\nolimits_{j=1}^{p} -\big(N_{kj}(x)Z_{kj}(x)\big)\right\}$$

subject to

$$Y_{\bar{l}}(x) \le 0 \qquad (\widehat{M})$$

$$-Z_{ij}(x)D_{ij}(x) + 1 \le 0,$$

$$Ax - b \le 0.$$

Theorem 1. *If x^* is a Pareto efficient solution for problem (P), then (x^*, Z^*) is a Pareto efficient solution for the problem $\widehat{M}$. If (x^*, Z^*) be a Pareto optimal solution of the problem $(\widehat{M})$, then x^* is a Pareto efficient solution of the problem (P), where $Z^*_{ij} = \dfrac{1}{D_{ij}(x^*)}$, $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, k$.*

Proof. Assume that x^* is a Pareto efficient solution of the problem (P). It follows that

$$\sum_{j}^{p} -\frac{N_{ij}(x)}{D_{ij}(x)} \le \sum_{j}^{p} -\frac{N_{ij}(x^*)}{D_{ij}(x^*)} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_{j}^{p} -\frac{N_{sj}(x)}{D_{sj}(x)} < \sum_{j}^{p} -\frac{N_{sj}(x^*)}{D_{sj}(x^*)} \quad \text{for at least one } s, \quad \forall x \in S.$$

where S denote the feasible space of the problem (M).

Let us suppose that (x^*, Z^*) is not Pareto efficient solution of the problem $(\widehat{M})$. Then there exists some another solution $(x', Z') \in S$ of the problem $(\widehat{M})$ such that

$$\sum_{j}^{p} -N_{ij}(x')Z'_{ij} \le \sum_{j}^{p} -N_{ij}(x^*)Z^*_{ij} \quad \forall i = 1, 2, \ldots, k,$$

and

$$\sum_{j}^{p} -N_{sj}(x')Z'_{sj} < \sum_{j}^{p} -N_{sj}(x^*)Z^*_{sj} \quad \text{for at least one } s, \quad \forall x \in S.$$

Since $Z^*_{ij} = \dfrac{1}{D_{ij}(x^*)}$ and $Z'_{ij} = \dfrac{1}{D_{ij}(x')}$, we get

$$\sum_{j}^{p} \frac{-N_{ij}(x')}{D_{ij}(x')} \le \sum_{j}^{p} \frac{-N_{ij}(x^*)}{D_{ij}(x^*)} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_{j}^{p} \frac{-N_{sj}(x')}{D_{sj}(x')} < \sum_{j}^{p} \frac{-N_{sj}(x^*)}{D_{sj}(x^*)} \quad \text{for at least one } s, \quad \forall x \in S,$$

i.e.,

$$\sum_j^p \frac{N_{ij}(x')}{D_{ij}(x')} \leq \sum_j^p \frac{N_{ij}(x^*)}{D_{ij}(x^*)} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_j^p \frac{N_{sj}(x')}{D_{sj}(x')} > \sum_j^p \frac{N_{sj}(x^*)}{D_{sj}(x^*)} \quad \text{for at least one } s, \quad \forall x \in S.$$

This means that x' is a Pareto efficient solution of the problem (P). This is the contradiction of the assumption that (x^*) is Pareto efficient solution.

Conversely: Suppose (x^*, Z^*) is the Pareto optimal solution of the problem $(\widehat{M})$, it follows that

$$\sum_j^p -N_{ij}(x^*)Z_{ij}^* \leq \sum_j^p -N_{ij}(x)Z_{ij} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_j^p -N_{sj}(x^*)Z_{sj}^* < \sum_j^p -N_{sj}(x)Z_{sj} \quad \text{for at least one } s, \quad \forall x \in S.$$

Now let us assume that (x^*) is not a Pareto efficient solution of the problem (P). Hence there exist some another $x' \in S$ such that

$$\sum_j^p \frac{N_{ij}(x')}{D_{ij}(x')} \geq \sum_j^p \frac{N_{ij}(x^*)}{D_{ij}(x^*)} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_j^p \frac{N_{sj}(x')}{D_{sj}(x')} > \sum_j^p \frac{N_{sj}(x^*)}{D_{sj}(x^*)} \quad \text{for at least one } s, \quad \forall x \in S.$$

Since $Z'_{ij} = \dfrac{1}{D_{ij}(x')}$ and $Z^*_{ij} = \dfrac{1}{D_{ij}(x^*)}$, where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$.

This implies that

$$\sum_j^p N_{ij}(x')Z'_{ij} \geq \sum_j^p N_{ij}(x^*)Z^*_{ij} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_j^p N_{sj}(x')Z'_{sj} > \sum_j^p N_{sj}(x^*)Z^*_{sj} \quad \text{for at least one } s,$$

i.e.,

$$\sum_j^p -N_{ij}(x')Z'_{ij} \leq \sum_j^p -N_{ij}(x^*)Z^*_{ij} \quad \forall i = 1, 2, \ldots, k$$

and

$$\sum_j^p -N_{sj}(x')Z'_{sj} < \sum_j^p -N_{sj}(x^*)Z^*_{sj} \quad \text{for at least one } \ s.$$

Thus, (x^*, Z^*) is not a Pareto efficient solution of the problem $(\widehat{M})$. This is the contradiction of the assumption that (x^*, Z^*) is a Pareto optimal solution of the problem $(\widehat{M})$.

Theorem 1 is proved.

3. Key algorithmic process. It is established from the Theorem 1 that any technique to find Pareto optimal solution is also applicable for the original problem P. To develop a branch and bound method for MONSOF (P) optimization problem (P), we first explain five key process: transform the MONSOF (M) optimization problem into multiobjective nonlinear optimization problem; convert multiobjective optimization problem into single objective optimization problem; successively refine partition of feasible space along longest edge of the constructed simplicial computation of lower and upper bounds for the optimal value of the objective function; finally deleting technique over each subspace generated by the partitions.

The partition method is successive triangular partition of the initial simplex S_0 with the rule of partition. This means that any infinite nested sequence of partition space constructed through the developed method shrinks to single point. The commonly used partition rule is the bisection.

The process to find upper bound for the objective function works in two ways. First, each simplex S constructed through the branching process, the process to find upper bound seeks a upper bound for the maximum of the objective function taken over $X \cap S$. Second for any step of the method, the upper bound process finds an upper bound for the Pareto optimal solution of the problem $(\widehat{M})$. Hence for MONSOF (M) optimization problem.

The process to estimate lower bound for the objective function is the consideration of all feasible points found in the process of finding upper bounds for the Pareto optimal solution of the problem $(\widehat{M})$. Hence for MONSOF (P) optimization problem.

The process of deletion is consisted by deleting each partition of subspace in which there is no solution exists for further processing. In the next subsections, we are giving the detail of processes, respectively.

3.1. Primary simplex and partition. The n-dimensional simplex S_0 is constructed on the basis of methodology as developed by Horst and Tuy in [27] that S_0 containing S into n-dimensional subsimplices. This process help to find a location of Pareto optimal solution for MONSOF (P). In the whole process, n-dimensional simplex will be called n-simplex. An initial simplex S_0 which tightly enclosed the given feasible space S can be constructed in the following manner:

$$S_0 = \left\{ x \in R^n \mid x_\alpha \geq \mho_\alpha, \ \ \alpha = 1, 2, \ldots, n, \ \ \sum_{\alpha=1}^{n} x_\alpha \leq \mho \right\},$$

where $\mho = \max \left\{ \sum x_\alpha \,\middle|\, x \in \Lambda \subseteq S_0 \right\}$, and $\mho_\alpha = \min \left\{ x_\alpha | x \in \Lambda \subseteq S_0, \alpha = 1, 2, \ldots, n \right.$, the vertex set S_0 is $\widetilde{V}_0, \widetilde{V}_1, \ldots, \widetilde{V}_n \}$, where $\widetilde{V}_0 = (\mho_1, \mho_2, \ldots, \mho_n)$ and $\widetilde{V}_\alpha = (\mho_1, \ldots, \mho_{\alpha-1}, \gamma_\alpha, \mho_{\alpha+1}, \ldots$ $\ldots, \mho_n)$ and here $\gamma_\alpha = \mho - \sum_{\alpha \neq \bar{\alpha}} \mho_{\bar{\alpha}}$. Further, the subdivision of simplices can be performed in the following manner:

Using the technique of branch and bound method, the constructed n-simplex S_0 may be split into two parts. These parts are known as simplices of S. Suppose a subsimplex S_0 having the vertex $\left\{ \widetilde{V}_0, \widetilde{V}_1, \ldots, \widetilde{V}_n \right\}$ is required to be split into two parts and let c be any mid point of the longest edge $\left[\widetilde{V}_s, \widetilde{V}_{\bar{s}}\right]$ of S_0. This means that $\left[\widetilde{V}_s, \widetilde{V}_{\bar{s}}\right]\left\|\widetilde{V}_s - \widetilde{V}_{\bar{s}}\right\| = \max_{\bar{\theta}<\theta}\left\|\widetilde{V}_{\bar{\theta}} - \widetilde{V}_\theta\right\|$, where $\|\cdot\|$ is any prescribed norm in R^n and $\bar{\theta}$, $\theta = 0, 1, 2, \ldots, n$. Then S_1 and S_2 are known as simplicial bisection of S. Thus the vertex set of S_1 can be represented as $\widetilde{V}_0, \widetilde{V}_1, \ldots, \widetilde{V}_{s-1},\ c,\ \widetilde{V}_{s+1}, \ldots, \widetilde{V}_n$ and the vertex set of S_2 is $\widetilde{V}_0, \widetilde{V}_1, \ldots, \widetilde{V}_{\bar{s}-1},\ c,\ \widetilde{V}_{\bar{s}+1}, \ldots, \widetilde{V}_n$. In the process of branch and bound, if $S_{\bar{\alpha}}$ is a nested subsequence of simplices $S_{\bar{\alpha}+1} \subset S_{\bar{\alpha}}$, $\forall\bar{\alpha}$, then for some unique point $x \in R^n$, we get $\cap_{\bar{\alpha}} S_{\bar{\alpha}} = x$.

3.2. Lower bound. For all simplex $S_{\bar{\alpha}}$, $(S_{\bar{\alpha}} \subseteq S_0)$ by the branching method, the lower bound process is used to compute a lower bound $lb(S)$ for the optimal value of $\eta(S)$ for the problem

$$\eta(S) = \begin{cases} \min \sum_{j=1}^{p} -Z_{ij}(x)N_{ij}(x), \quad i = 1, 2, \ldots, k, \\ \text{subject to} \\ Y_{\bar{l}}(x) \le 0, \quad \bar{l} = 1, 2, \ldots, m, \\ -Z_{ij}D_{ij}(x) + 1 \le 0, \quad i = 1, 2, \ldots, k \quad \text{and} \quad j = 1, 2, \ldots, p, \\ Ax - b \le 0 \quad \forall x \in S \quad \text{and} \quad Z \in D. \end{cases}$$

Furthermore, the weighted sum method converts the multiobjective fractional programming into the following single objective function nonlinear programming:

$$h(x)\eta(S) = \begin{cases} \min \sum_{i=1}^{k} \sum_{j=1}^{p} -w_i Z_{ij}(x)N_{ij}(x) \\ \text{subject to} \\ Y_{\bar{l}}(x) \le 0, \quad \bar{l} = 1, 2, \ldots, m, \\ -Z_{ij}D_{ij}(x) + 1 \le 0, \quad i = 1, 2, \ldots, k \quad \text{and} \quad j = 1, 2, \ldots, p, \\ Ax - b \le 0 \quad \forall x \in S \quad \text{and} \quad Z \in D, \end{cases}$$

where $\left\{-N_{ij}(x)\right\} \in R^{k\times p}$, $\left\{Z_{ij}(x)\right\} \in R^{k\times p}$, and $\left\{D_{ij}(x)\right\} \in R^{k\times p}$ and the weights $[w_i] \in (0, 1)$ such that $\sum_i^k w_i = 1$.

Theorem 2. *Suppose the feasible space S of MONSOF optimization problem is a subsimplex of S_0 obtained by the branch and bound technique having the vertices $\widetilde{V_0}, \widetilde{V_1}, \ldots, \widetilde{V_n}$. Then $lb(S) \le \eta(S)$, where $lb(S)$ is the optimal value of the linear programming problem in variable β_{ij}, $i = 1, 2, \ldots k$, $j = 1, 2, \ldots, p$, t and $\lambda_{\bar{l}}$, $\bar{l} = 1, 2, \ldots, m$, and u_l, $l = 1, 2, \ldots, q$, given by*

$$\begin{gathered} lb(S) = \max \sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} + t \\ \textit{subject to} \\ \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(\widetilde{V_r}) + \sum_{l=1}^{q} u_l(A_l\widetilde{V_r} - b_l) - t \ge 0, \quad r = 0, 1, \ldots, n, \\ -w_i N_{ij}(\widetilde{V_r}) - \beta_{ij} D_{ij}(\widetilde{V_r}) \ge 0, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, p, \\ \beta \ge 0, \quad \lambda \ge 0, \quad u \ge 0, \quad t - \textit{free}, \end{gathered} \tag{1}$$

where A_l denotes the l^{th} row of A_l, b_l denotes the components of b, $l = 1, \ldots, q$.

Proof. According to the definition of $\eta(S)$ and the Lagrangian weak duality theorem of nonlinear programming, we have $\eta(S) \geq lb(S)$,

$$lb(S) = \max_{\beta \geq 0, u \geq 0} \left[\left(\min_{x \in S, Z \in D} \sum_{i=1}^{k} \sum_{j=1}^{p} -w_i Z_{ij} N_{ij}(x) + \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) \ + \right.\right.$$

$$\left.\left. + \sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij}(-Z_{ij} D_{ij}(x) + 1) + \sum_{l=1}^{q} u_l (A_l x - b_l) \right) \right],$$

$$lb(S) = \max_{\beta \geq 0, u \geq 0} \left[\sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} + \min_{x \in S, z \in D} \left[\sum_{i=1}^{k} \sum_{j=1}^{p} -w_i Z_{ij} N_{ij}(x) \ - \right.\right.$$

$$\left.\left. - \ \beta_{ij} D_{ij} Z_{ij}(x) + \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) + \sum_{l=1}^{q} u_l (A_l x - b_l) \right] \right] =$$

$$= \max_{\beta \geq 0, u \geq 0} \left[\sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} + \min_{x \in S, Z \in D} \left[\big(\langle W^T A, B^T \rangle + \langle \beta^T I, B^T \rangle \big) + \right.\right.$$

$$\left.\left. + \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) + \sum_{l=1}^{q} u_l (A_l x - b_l) \right] \right] =$$

$$= \max_{\beta \geq 0, u \geq 0} \left[\sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} + \min_{x \in S, Z \in D} \left[\big(\big\langle W^T A + \beta^T I, B^T \big\rangle \big) + \right.\right.$$

$$\left.\left. + \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) + \sum_{l=1}^{q} u_l (A_l x - b_l) \right] \right],$$

where $A(x) = \big\{-N_{ij}(x)\big\} \in R^{k \times p}$, $B(x) = \big\{Z_{ij}(x)\big\} \in R^{k \times p}$, $I(x) = \big\{D_{ij}(x)\big\} \in R^{k \times p}$ and $W = [w_{ij}] \in \mathbf{R}^{k \times p}$ can be taken as

$$w_{ij} = \begin{cases} 0, & \text{if} \quad i \neq j, \\ w_i(x), & \text{if} \quad i = j. \end{cases}$$

Since

$$\min_{z \in D} \Big[\langle W^T A + \beta^T I, B^T \rangle \Big] = \begin{cases} 0, & \text{if} \quad W^T A + \beta^T I \geq 0 \quad \forall x \in S, \\ -\infty, & \text{e.w.} \end{cases}$$

It follows that

$$lb(S) = \max_{\beta \geq 0, u \geq 0} \left[\sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} + \min_{x \in \beta} \left(\sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}} + \sum_{l=1}^{q} u_l (A_l x - b_l) \right) \right] \tag{2}$$

subject to

$$W^T A + \beta^T I \geq 0 \quad \forall x \in S, \quad \beta \geq 0, \quad \lambda \geq 0, \quad u \geq 0.$$

For each $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, k$, from the problem (2), we get $-w_i N_{ij} - \beta_{ij} D_{ij} \geq 0$ for all $x \in S$. Since N_{ij} and D_{ij} are convex functions and for $\beta_{ij} \geq 0$, then $-w_i N_{ij} - \beta_{ij} D_{ij}$ is convex function for all $j = 1, 2, \ldots, p$ and $i = 1, 2, \ldots, k$. As we know S is simplex with extreme points $\widetilde{V_0}, \widetilde{V_1}, \ldots, \widetilde{V_n}$, we have problem (2) hold if and only if $-w_i N_{ij} - \beta_{ij} D_{ij} \geq 0$, where $r = 0, 1, \ldots, n$.

This implies that lower bound $lb(S)$ is given by

$$lb(S) = \max \sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} + t$$

subject to

$$\begin{aligned}
&t \leq \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) + \sum_{l=1}^{q} u_l (A_l x - b_l) \quad \forall x \in S, \\
&-w_i N_{ij}(\widetilde{V_r}) - \beta_{ij} D_{ij}(\widetilde{V_r}) \geq 0 \quad \forall \beta \geq 0, \quad \lambda \geq 0, \quad u \geq 0, \\
&j = 1, 2, \ldots, p, \\
&r = 0, 1, \ldots, n, \\
&i = 1, 2, \ldots, k.
\end{aligned} \tag{3}$$

For each $\lambda \geq 0$ and $u \geq 0$ and since $Y_{\bar{l}}(x)$ and $(A_l x - b_l)$ are non-convex functions for each $\bar{l} = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, q$ we can get $\sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) + \sum_{l=1}^{q} u_l (A_l x - b_l)$ is a non-convex function of x. Additionally, since S is a simplex with vertices $\widetilde{V_0}, \widetilde{V_1}, \ldots, \widetilde{V_n}$. It can be seen that for each $\widetilde{V_r} \geq 0$ hold if and only if

$$\sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(\widetilde{V_r}) + \sum_{l=1}^{q} u_l (A_l \widetilde{V_r} - b_l) - t \geq 0, \quad r = 0, 1, \ldots, n.$$

Theorem 2 is proved.

Theorem 3. *Let S_1 and S_2 be subsimplex of S_0 formed by the branching process such that $S_2 \supseteq S_1 \supseteq S_0$. Then*

(i) $lb(S_2) \geq lb(S_1)$,

(ii) $lb(S_1) > -\infty$.

Proof. The first part of the theorem can be proved directly from the definition of $lb(S)$ given in the proof of Theorem 2 for an arbitrary simplex S.

To prove the (ii), using the part (i), we require only to show that $lb(S_0) > -\infty$. Now from the proof of Theorem 2, we have

$$lb(S_0) = \max_{\beta \geq 0, u \geq 0} \left[\left(\min_{x \in S, z \in D} \sum_{i=1}^{k} \sum_{j=1}^{p} -w_i Z_{ij} N_{ij}(x) + \right.\right.$$
$$\left.\left. + \sum_{i=1}^{k} \sum_{j=1}^{p} \beta_{ij} (-Z_{ij} D_{ij}(x) + 1) + \sum_{\bar{l}=1}^{m} \lambda_{\bar{l}} Y_{\bar{l}}(x) + \sum_{l=1}^{q} u_l (A_l x - b_l) \right) \right].$$

Assuming $\beta = 0,\ \lambda = 0,\ u = 0$, then

$$lb(S^0) \geq \min_{x \in S, z \in D} \sum_{i=1}^{k} \sum_{j=1}^{p} -w_i Z_{ij} N_{ij}(x).$$

For $w_i > 0$, and since $N_{ij}(x)$ is convex on $\mathbf{R}^n$, then $N_{ij}(x)$ is continuous for each $j = 1, 2, \ldots, p$, $i = 1, 2, \ldots k$, and hance the function $F(x, z) = \sum_{i=1}^{k} \sum_{j=1}^{p} -w_i Z_{ij} N_{ij}(x)$ is a continuous function of (x, Z) on $\Psi = S_0 \times D$. By the compactness of Ψ, it mean that $\min_{x \in S, Z \in D} F(x, Z)$ is finite, therefore $lb(S_0) > -\infty$.

Theorem 3 is proved.

3.3. Upper bound. This section describes the process to determine the upper bound for the Pareto optimal solution of the problem M for each simplex S generated by the process such that $lb(S)$ is finite. To find upper bound in each iteration in the algorithmic process, check all the feasible solutions contained in S and we get more feasible solution and the value of upper bound may be improved iteratively.

Theorem 4. *Suppose S be a subsimplex of S_0 with vertices $\widetilde{V_0}, \widetilde{V_1}, \ldots, \widetilde{V_n}$ and suppose that $lb(S) \neq +\infty$. Let $(Z^0, Z^1, Z^2, \ldots, Z^n)$ be optimal dual variable corresponding to the first $(n+1)$ constraints of linear program* (1) *and let $\omega = \sum_{r=0}^{n} Z^r \widetilde{V_r}$. If $Y_{\bar{l}}(\omega) \leq 0$ for each $\bar{l} = 1, 2, \ldots, m$, then ω is a feasible solution of the problem M.*

Proof. The dual linear program of problem (1) is

$$DLP(S) \quad lb(S) = \min \sum_{j=1}^{P} \sum_{i=1}^{k} \sum_{r=0}^{n} -Z_{ij}^r N_{ij}(\widetilde{V_r})$$

subject to

$$\sum_{r=0}^{n} Z^r = 1, \tag{4}$$

$$\sum_{r=0}^{n} -Z^r Y_{\bar{l}}(\widetilde{V_r}) \geq 0,$$

$$\sum_{r=0}^{n} Z_{ij}^r D_j(\widetilde{V_r}) \geq 1, \quad j = 1, 2, \ldots, p,$$

$$\sum_{r=0}^{n} Z^r (b_l - A_l \widetilde{V_r}) \geq 0, \quad l = 1, 2, \ldots, q, \tag{5}$$

$$Z_{ij}^r \geq 0, \quad i = 1, \ldots, p \quad \text{and} \quad r = 0, 1, \ldots, n.$$

From constraint (5) of the problem $DLP(S)$, we have

$$\sum_{r=0}^{n} Z^r (b_l - A_l \widetilde{V_r}) \geq 0, \quad l = 1, 2, \ldots, q.$$

Using the constraint (4) of the problem $DLP(S)$, we get

$$A_l \sum_{r=0}^{n} Z^r \widetilde{V_r} \leq \sum_{r=0}^{n} Z^r b_l = b_l, \quad l = 1, 2, \ldots, q.$$

This implies that

$$A_l \omega \leq b_l, \quad l = 1, 2, \ldots, q.$$

Therefore, $Y_{\bar{l}}(\omega) \leq 0$, $\bar{l} = 1, 2, \ldots, m$, then ω is a feasible solution of the problem (M).

3.4. Process of deletion. In the branch and bound search proceeds, finite simplices are generated by the developed algorithm. Among them, certain simplices are eliminated from further consideration. At the beginning of each steps, simplex $S_{\bar{\alpha}}$ is created by simplicial bisection in the branching process and it is subjected to the detection by infeasibility test. Suppose $\widetilde{V_0}, \widetilde{V_1}, \ldots, \widetilde{V_n}$ denote the vertices of such a simplex S. If for any $l \in 1, 2, \ldots, q$ such that

$$\min \left\{ A_l \widetilde{V_0} - b_l, A_l \widetilde{V_1} - b_l, \ldots, A_l \widetilde{V_n} - b_l > 0 \right\}$$

or, for some $\bar{l} \in 1, 2, \ldots, m$, such that

$$\min \left\{ Y_{\bar{l}}(\widetilde{V_0}), Y_{\bar{l}}(\widetilde{V_1}), \ldots, Y_{\bar{l}}(\widetilde{V_n}) > 0 \right\},$$

then the simplex S is said to pass the deletion by infeasibility test.

4. Proposed method and its convergence. In this section, based upon the results and branch and bound process discussed in Section 3, the basic outlines of the developed branch and bound method are given to solve MONSOF optimization problem (P). The method is coded in Matlab (Ver. 2014(b)) and run on computer machine 64 bit, RAM 4GB and the whole process is summarized in the following manner.

Step 1. *Initial Setting*:

(i) First we choose a suitable weight vector $w = (w_1, w_2, \ldots, w_k)$ where, each $w_i \in (0, 1)$ such that $\sum_{i=1}^{k} w_i = 1$ and the MONSOF optimization problem $(\eta(S))$ converts into single objective $(h(x)\ \eta(S))$.

(ii) Solve the linear programming problem (1) for the initial simplex S_0 to find the optimal value $lb(S_0)$ and find the feasible set $F(S_0)$.

(iii) Setting, $\mu_0 = lb(S_0)$. If $F(S_0) \neq \phi$, then compute $\nu_0 = \min\{h(x) : x \in F(S_0)\}$ and choosing $x^0 \in F(S_0)$ such that $h(x^0) = \nu_0$ otherwise, setting $\nu_0 = +\infty$ and setting $G_0 = \{S_0\}$, $\sigma = 1$ (iteration counter).

Step 2. *Stopping criteria*: If $\nu_\sigma = \mu_\sigma$, then STOP and x^σ is a Pareto efficient solution for problem (P).

Step 3. (*Iterations*:) Setting the iteration counter, $\sigma \geq 1$ and execute the process from (a) to (h).

(a) Set $x^\sigma = x^{\sigma-1}, \mu_\sigma = \mu_{\sigma-1}$, $\nu_\sigma = \nu_{\sigma-1}$ if $S^{\sigma-1}$ exist, set $S^\sigma = S^{\sigma-1}$. If $\nu_\sigma = \mu_\sigma$, then STOP and x^σ is a Pareto efficient solution for problem (P) otherwise continue as follows.

(b) Divide S^σ into two parts S_1^σ and S_2^σ using the bisection along the longest edge of simplicial. Suppose $T' = \{S_1^\sigma, S_2^\sigma\}$.

(c) Test the infeasibility conditions in deletion technique for each simplex of T' and delete the each simplex which does not pass the infeasibility test of deletion technique. Now we obtained T which represent the subset of T'.

(d) For each $S_{\bar{\alpha}}^\sigma \subset T$, execute the process from (d1) to (d3).

(d1) find the optimal value lb(S) of the linear programming problem (1), (d2) determine the set $F(S)$, of feasible solutions contained in S. If $F(S) \neq \phi$ then compute $h(\widehat{x}) = \min\{h(x) : x \in F(S)\}$ and choose $\widehat{x} \in F(S)$ otherwise, set $h(\widehat{x}) = +\infty$,

(d3) if $h(\widehat{x}) < h(x^\sigma)$, set $x^\sigma = \widehat{x}$ and set $\nu_\sigma = h(\widehat{x})$.

(e) Set $G_\sigma = \{G_{\sigma-1} \setminus S^\sigma\} \cup T$.

(f) Delete from G_σ all $S \in G_\sigma$ such that $lb(S) > \nu_\sigma$.

(g) If $G_\sigma \neq \infty$, then set $\mu_\sigma = \min\{lb(S) : S \in G_\sigma\}$ and choose $S^\sigma \in G_\sigma$ such that $lb(S^\sigma) = \mu_\sigma$ otherwise set $\mu_\sigma = \nu_\sigma$.

(h) Set $\sigma = \sigma + 1$ and go to iteration σ.

4.1. Convergence. The convergence of the propose method depends upon the quality of ν_σ. Therefore, the method may be benefited from a fast local search. This method either finite or infinite. If it is finite, then it terminates a iteration σ. The feasible point x^σ is a Pareto optimal solution of original problem P. If the method does not terminates at iteration σ, then it is easy to show that it generates at least one infinite nested subsequence $\{S_{\bar{\alpha}}\}$ of simplices $S_{\bar{\alpha}+1} \subseteq S_{\bar{\alpha}}$ for all $\bar{\alpha}$. In this situation, following result is a key to the convergence of the algorithm for the sake of convenience. Suppose, $F(x, Z) : S_0 \times D \to \mathbf{R}$ be the objective function of problem $(\widehat{M})$ with $F(x, Z) = \sum_{i=1}^{k} \sum_{j=1}^{p} -w_{ij} Z_{ij} N_{ij}(x)$ and let $H(x, Z) : S_0 \times D \to \mathbf{R}^{k\times p} \times \mathbf{R}^q$ be a vector function formed by the constraints of $(\widehat{M})$ with $H(x, Z) = (H_1(x, Z), \ldots, H_{k\times p+q+m}(x, Z)) \in \in R^{k\times p} \times R^q \times R^m$ where each components of $H(x, Z)$ is given by

$$H_{\bar{k}}(x, Z) = \begin{cases} Y_{\bar{k}}(x), & \text{if} \quad \bar{k} = 1, 2, \ldots, m, \\ -Z_{\bar{k}}.D_{\bar{k}}(x) + 1, & \text{if} \quad \bar{k} = m + i, j, \quad \text{where} \quad i = 1, 2, \ldots k, \quad j = 1, 2, \ldots, p, \\ A_{\bar{k}-i,j} x - b_{\bar{k}-i,j}, & \text{if} \quad \bar{k} = i, j + 1, \ldots, m + i, j + q. \end{cases}$$

Theorem 5. *Suppose that the proposed method is infinite, and assume $\{S_{\bar{\alpha}}\}$ be an infinite nested subsequence of simplices generated by the method. Let $\cap_{\bar{\alpha}} S_{\bar{\alpha}} = \{x^*\}$. Then $\{x^*\}$ is an optimal solution of the problem (M). Hence the solution $\{x^*\}$ is a Pareto efficient solution of the problem (P).*

Proof. Assume that the method is infinite and the simplicial $\{S_{\bar{\alpha}}\}$ be chosen in the theorem. So, from R. Host and H. Tuy [27], $\cap_{\bar{\alpha}} S_{\bar{\alpha}} = \{x^*\}$ for some point $x^* \in \mathbf{R}^n$. Then $x^* \in \mathbf{S}_0$ since for all $\bar{\alpha}, S_{\bar{\alpha}} \subseteq S_0$. By the definition of $F(x, Z)$ and $H(x, Z)$, we can rewrite the problem $(\widehat{M})$ as follows:

$$\min\left\{F(x, Z) H(x, Z) \leq 0, \ x \in S_0, \ Z \in D\right\}$$

from Theorem 3, the sequence $lb(S_{\bar{\alpha}})$ is nondecreasing, therefore, the limit $u^* = \lim_{\bar{\alpha}\to\infty} lb(S_{\bar{\alpha}})$ exist. Suppose

$$\eta(S_0) = \min\left\{F(x, Z) : H(x, Z) \leq 0, \ x \in S_0, \ Z \in D\right\}$$

then, it is obviously that,

$$u^* \leq \eta(S_0). \tag{6}$$

In the next, we can show that

$$u^* \geq \eta(S_0). \tag{7}$$

It is noticed that

$$\min\big\{F(x,Z)\colon H(x,Z)\leq 0,\ Z\in D\big\}\geq \eta(S^0). \tag{8}$$

Therefore, we only require to prove

$$u^*\geq \min\big\{F(x,Z)\colon H(x,Z)\leq 0,\ Z\in D\big\}. \tag{9}$$

Contrary, suppose that (6) does not holds. This means that

$$\min\big\{F(x,Z)\colon H(x,Z)\leq 0,\ Z\in D\big\}>u^*. \tag{10}$$

From the problem, we know that fractional functions $N_{ij}(x)$ and $D_{ij}(x)$ are convex function on $\mathbf{R}^n$ for each $i=1,2,\ldots,k,\ j=1,2,\ldots,p$. The nonlinear constraint's $Y_{\bar{l}}(x)$ are the concave functions on R^n for all $\bar{l}=1,2,\ldots,m$. This implies that $N_{ij}(x),\ D_{ij}(x)$ and $Y_{\bar{l}}(x)$ are continuous function. Therefore, the functions $F(x,Z)$ and $H_{\bar{k}}(x,Z)$, where, $\bar{k}=1,2,\ldots,k\times p+q+m)$ are continuous on $\Psi=S_0\times D$ and linear in Z for every fixed value of x. Therefore, for the fixed value of $x\in S_0$, the function $(Z,\bar{\beta})\to F(x,Z)+\langle\bar{\beta},H(x,Z)\rangle$ is linear in Z and $\bar{\beta}$ respectively, where

$$\bar{\beta}=\big(\beta_{1j},\beta_{2j},\ldots,\beta_{ij},\lambda_1,\lambda_2,\ldots,\lambda_m,u_1,u_2,\ldots,u_q\big),$$

$$\min_{Z\in D}\max_{\bar{\beta}\in\mathbf{R}_+^{k\times p}\times\mathbf{R}_+^{q}\times\mathbf{R}_+^{m}}\big\{F(x,Z)+\langle\bar{\beta},H(x,Z)\rangle\big\}=$$

$$=\max_{\bar{\beta}\in\mathbf{R}_+^{k\times p}\times\mathbf{R}_+^{q}\times\mathbf{R}_+^{m}}\min_{Z\in D}\to\big\{F(x,Z)+\langle\bar{\beta},H(x,Z)\rangle\big\}. \tag{11}$$

Form the above, we can write clearly

$$\max_{\bar{\beta}\in\mathbf{R}_+^{k\times p}\times\mathbf{R}_+^{q}\times\mathbf{R}_+^{m}}\big\{F(x,Z)+\langle\bar{\beta},H(x,Z)\rangle\big\}=\begin{cases}F(x,Z), & \text{if}\quad H(x,Z)\leq 0,\\ +\infty, & \text{otherwise.}\end{cases}$$

From (10), for every $x\in S_0$, we get the following equation:

$$\max\big\{F(x,Z)+\langle\bar{\beta},H(x,Z)\rangle\leq 0,\ Z\in D\big\}=$$

$$=\max_{\bar{\beta}\in\mathbf{R}_+^{k\times p}\times\mathbf{R}_+^{q}\times\mathbf{R}_+^{m}}\min_{Z\in D}\big\{F(x,Z)+\langle\bar{\beta},H(x,Z)\rangle\big\}. \tag{12}$$

Using the equation (9), we can write

$$\max_{\bar{\beta}\in\mathbf{R}_+^{k\times p}\times\mathbf{R}_+^{q}\times\mathbf{R}_+^{m}}\min\big\{F(x^*,Z)+\langle\bar{\beta},H(x^*,Z)\rangle\big\}>u^*$$

so there exist $\bar{\beta}^*$ satisfying

$$\min_{Z\in D}\big\{F(x^*,Z)+\langle\bar{\beta},H(x^*,Z)\rangle\big\}>u^*$$

using the continuity of function $(Z,\bar{\beta}^*)\to\big\{F(x,Z)+\langle\bar{\beta}^*,H(x^*,Z)\rangle\big\}$, we can then find, for every fixed $Z\in D$ an open ball U_Z in $\mathbf{R}^n$ around x^* and an open ball V_Z in $\mathbf{R}^p$ around Z such that

$$F(\widehat{x},\widehat{Z})+\big\{\big\langle\bar{\beta}^*,H(\widehat{x},\widehat{Z})\big\rangle\big\}>u^*\quad\forall\widehat{x}\in U_Z,\quad \widehat{Z}\in V_Z.$$

Since the ball $V_Z,\ Z \in D$, form a covering of the compact set D. There is a finite set $E \subseteq D$ such that the balls $V_Z,\ Z \in E$, still form a covering of D. Let $U \subseteq_{Z \in E} U_Z$, then for every $Z \in D$, we have $Z \in V_{\widehat{Z}}$ for some $\widehat{Z} \in E$. Therefore,

$$F(x^*, Z) + \left\{\left\langle \bar{\beta}, H(x^*, Z)\right\rangle\right\} > u^* \quad \forall \widehat{x} \in U_Z, \quad \widehat{Z} \in V_Z.$$

But $S_{\bar{\alpha}} \subseteq U$ for all sufficient large $\bar{\alpha}$, because $\cap_{\bar{\alpha}} S_{\bar{\alpha}} = \{x^*\}$. Then

$$\max_{\bar{\beta} \in \mathbf{R}_+^{k \times p} \times \mathbf{R}_+^q \times \mathbf{R}_+^m} \min\left\{F(x, Z) + \left\langle \bar{\beta}, H(x, Z)\right\rangle : x \in S_{\bar{\alpha}},\ Z \in D\right\} > u^*.$$

Hence $lb(S_{\bar{\alpha}}) > u^*$.

This is a contradiction of our assumption. Therefore we can obtain 7 combining (6)–(9), we can obtain $u^* = V(S_0) = \min\left\{F(x^*, z^*) \colon H(x^*, Z) \le 0,\ Z \in D\right\}$. This implies that

$$\min\left\{F(x^*, Z) \colon H(x^*, Z) \le 0,\ Z \in D\right\} = \min\{F(x, Z) \colon H(x, Z) \le 0,\ x \in S_0,\ Z \in D\}.$$

An optimal solution Z^* of the problem (13) is also an optimal solution (x^*, Z^*) for problem $(\widehat{M})$.

Therefore, by Theorem 1, x^* is a Pareto optimal solution of the problem (P).

5. Computational experiments. In this section, we are given two numerical problem for the demonstration of proposed methods. These problems are made by combining the objective functions of the numerical problems of Shen, Duan and Pei [19] in the sense of multiobjective optimization problem.

5.1. Numerical problem 1. Consider a multiobjective nonlinear sum of fractional optimization problem

$$\text{Max}\Bigg(F_1 = \frac{-2x_1 - x_2}{x_1 + 10} + \frac{-2}{x_2 + 10},$$

$$F_2 = \frac{x_1^2 - 3x_1 + x_2^2 - 3x_2 - 3.5}{x_1 + 1} + \frac{-x_2}{x_1^2 - 2x_1 + x_2^2 - 8x_2 + 20}\Bigg)$$

subject to

$$-x_1^2 - x_2^2 + 3 \le 0,$$

$$-x_1^2 - x_2^2 + 8x_2 - 14 \le 0,$$

$$2x_1 + x_2 \le 6,$$

$$3x_1 + x_2 \le 8,$$

$$x_1 - x_2 \le 1,$$

$$x_1, x_2 \ge 1.$$

Using the method proposed in Section 4, first we find the simplicial S_0. The vertices of S_0 are obtained as $(1.0000, 1.0000)$, $(4.0000, 1.0000)$, $(1.0000, 4.0000)$. Choosing the suitable weights for each objective functions and converts MONSOF optimization problem into single objective optimization problem. After applying the proposed method we achieve the efficient solution which is given in Table 1.

By examining the table 1, the efficient solution for the problem is $x_1 = 1.8295$, $x_2 = 1.8295$ and the value of the objective functions at obtained points are $F_1 = -0.6329$, $F_2 = -2.9683$.

Table 1. Solution of MONSOF optimization Problem 1 using proposed method

S. No.	w_1	w_2	x_1	x_2	F_1	F_2	No. of Iteration	CPU Time Sec.
1	0.10	0.90	1.5147	1.5147	−0.5682	−3.3414	31	1.632
2	0.20	0.80	1.5330	1.5330	−0.5721	−3.3210	32	1.664
3	0.30	0.70	1.5574	1.5574	−0.5772	−3.2933	30	1.668
4	0.40	0.60	1.5899	1.5899	−0.5830	−3.2562	30	1.593
5	0.50	0.50	1.6363	1.6363	−0.5936	−3.2019	30	1.640
6	0.60	0.40	1.6221	1.6221	−0.5907	−3.2164	30	1.647
7	**0.70**	**0.30**	**1.8295**	**1.8295**	**−0.6329**	**−2.9683**	**28**	**1.603**
8	0.80	0.20	1.8925	1.8295	−0.6329	−2.9683	28	1.601
9	0.90	0.10	2.1685	1.4945	−0.6531	−2.9741	39	2.011

5.2. Numerical problem 2. Consider a multiobjective nonlinear sum of ratios programming problem as:

$$\text{Max}\left(\frac{f_{11}}{g_{11}}+\frac{f_{12}}{g_{12}},\ \frac{f_{21}}{g_{21}}+\frac{f_{22}}{g_{22}}+\frac{f_{23}}{g_{23}}\right)$$

subject to

$$x_1+x_2+x_3+x_4\le 24,$$

$$6\le x_1\le 10,$$

$$4\le x_2\le 6,$$

$$8\le x_3\le 12,$$

$$6\le x_4\le 8,$$

where

$$f_{11}=x_1^2-4x_1+2x_2^2-8x_2+3x_3^3-12x_3+4x_4^2-16x_4-65,$$

$$f_{12}=2x_1^2-16x_1+x_2^2-8x_2+0.5x_3^2-4x_3+0.25x_4-2x_4-15,$$

$$g_{11}=x_1^2-2x_1+x_2^2-2x_2+x_3^2-3x_3+x_4+28,$$

$$g_{12}=2x_2+4x_2+6x_3+8x_4,$$

$$f_{21}=\sum_{j=1}^{4}\left(x_j^2-16x_j\right)+214,$$

$$f_{22}=x_1^2-16x_1+2x_2^2-20x_2+3x_3^2-60x_3+4x_4^2-56x_4+586,$$

$$f_{23}=\sum_{j=1}^{4}\left(x_j^2-20x_j\right)+324,$$

Table 2. Solution of MONSOF optimization problem 2 using proposed method

S. No.	w_1	w_2	x_1	x_2	x_3	x_4	F_1	F_2	No. of Iteration	CPU Time Sec.
1	0.10	0.90	6.0624	4.0001	8.0000	7.9375	1.1610	−5.7985	60	3.193
2	0.20	0.80	6.2499	4.0000	8.0001	7.7500	1.0770	−4.9110	61	3.196
3	0.30	0.70	6.2500	4.0000	8.0000	7.7500	1.0770	−4.9106	62	3.273
4	**0.40**	**0.60**	**6.0000**	**4.0000**	**8.0000**	**6.0000**	**0.3907**	−2.7833	**61**	**3.236**
5	0.50	0.50	6.0000	4.0000	8.0000	6.0000	0.3907	−2.7833	64	3.319
6	0.60	0.40	6.0001	4.0000	8.0000	6.0000	0.3907	−2.7834	61	3.178
7	0.70	0.30	6.0000	4.0000	8.0000	6.0000	0.3907	−2.7833	63	3.272
8	0.80	0.20	6.0000	4.0000	8.0000	6.0000	0.3907	−2.7833	61	3.251
9	0.90	0.10	6.0000	4.0000	8.0000	6.0000	0.3907	−2.7833	63	3.250

$$g_{21} = 2x_1 - x_2 - x_3 + x_4 + 2,$$

$$g_{22} = -x_1 + x_2 + x_2 - x_4 + 10,$$

$$g_{23} = x_1^2 - 4x_4.$$

Using the method proposed in Section 4, the simplex S_0 containing β. The vertices of S_0 are given by $(6.0000,\ 4.0000,\ 8.0000,\ 6.0000)$, $(8.0000,\ 4.0000,\ 8.0000,\ 6.0000)$, $(6.0000,\ 6.0000,\ 8.0000,\ 6.0000)$, $(6.0000,\ 4.0000,\ 10.0000,\ 6.0000)$, $(6.0000,\ 4.0000,\ 8.0000,\ 8.0000)$.

After choosing the suitable weights for each objective functions and apply the method then we get the solution of the results which are given in Table 2.

By examining the Table 2, the efficient solution of the problem is $x_1 = 6.0000$, $x_2 = 4.0000$, $x_3 = 8.0000$, $x_4 = 6.0000$ and the value of the objective function at achieved point are $F_1 = 0.3907$, $F_2 = -2.7833$.

6. Conclusions. Sum of fractional optimization problems are considered to be very difficult even for single objectives only. In this paper, we proposed a weighted sum based branch and bound method for solving multiobjective nonlinear sum of fractional optimization problems. The method is the extension of the method presented by Shen, Duan and Pei [19]. We also designed two numerical problems by combining the numerical problem which are available in the literature and these problems are solved by proposed method which is coded in Matlab (Ver 2014(b)). From the above computational results, we obtain the Pareto optimal solution for the MONSOF (P) optimization problem. The computational results are summarized in Table 1 and Table 2.

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