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APPROXIMATION BY NÖRLUND MEANS OF QUADRATICAL PARTIAL SUMS OF DOUBLE WALSH – KACZMARZ-FOURIER SERIES*

НАБЛИЖЕННЯ СЕРЕДНІМИ НОРЛУНДА КВАДРАТИЧНИХ ЧАСТКОВИХ СУМ ПОДВІЙНИХ РЯДІВ УОЛША – КАЧМАРЖА – ФУР'Є

We discuss the Nörlund means of quadratic partial sums of the Walsh – Kaczmarz – Fourier series of a function in L^p. We investigate the rate of approximation by this means, in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \le p \le \infty$. For $p = \infty$, by L^p, we mean C, i.e., the collection of continuous functions.

Our main theorem states that the approximation behavior of this two-dimensional Walsh – Kaczmarz – Nörlund means is as good as the approximation behavior of the one-dimensional Walsh– and Walsh – Kaczmarz – Nörlund means.

Earlier results for one-dimensional Nörlund means of the Walsh – Fourier series was given by Móricz and Siddiqi [J. Approxim. Theory. – 1992. – **70**, № 3. – P. 375 – 389] and Fridli, Manchanda and Siddiqi [Acta Sci. Math. (Szeged). – 2008. – **74**. – P. 593 – 608], for one-dimensional Walsh – Kaczmarz – Nörlund means by the author [Georg. Math. J. – 2011. – **18**. – P. 147 – 162] and for two-dimensional trigonometric system by Móricz and Rhoades [J. Approxim. Theory. – 1987. – **50**. – P. 341 – 358].

Розглядаються середні Норлунда для квадратичних часткових сум рядів Уолша – Качмаржа – Фур'є функції з простору L^p. Вивчено швидкість наближення цими середніми, зокрема, в $\mathrm{Lip}(\alpha, p)$, де $\alpha > 0$ та $1 \le p \le \infty$. Для $p = \infty$ під L^p ми розуміємо C, тобто набір всіх неперервних функцій.

Основна теорема у цій статті стверджує, що апроксимаційна поведінка таких двовимірних середніх Уолша – Качмаржа – Норлунда так само гарна, як і апроксимаційна поведінка одновимірних середніх Уолша та Уолша – Качмаржа – Норлунда.

Раніше результати для одновимірних середніх Норлунда рядів Уолша – Фур'є були отримані Морічем та Сіддікі [J. Approxim. Theory. – 1992. – **70**, № 3. – P. 375 – 389] та Фрідлі, Манчанда і Сіддікі [Acta Sci. Math. (Szeged). – 2008. – **74**. – P. 593 – 608]. Для одновимірних середніх Уолша – Качмаржа – Норлунда відповідні результати були отримані автором [Georg. Math. J. – 2011. – **18**. – P. 147 – 162]. Випадок двовимірних тригонометричних систем було розглянуто Морічем і Роадсом [J. Approxim Theory. – 1987. – **50**. – P. 341 – 358].

1. Nörlund means. Let $\{q_k : k \ge 1\}$ be a sequence of nonnegative numbers. The Nörlund means and kernels of the Walsh – (Kaczmarz) – Fourier series are defined by

$$t_n^\alpha(f,x) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} S_k^\alpha(f,x), \quad L_n^\alpha(x) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} D_k^\alpha(x),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$, $n \ge 1$, and α is the Walsh system in the Paley or Kaczmarz enumeration. We always assume that $q_1 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty.$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [17]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

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In particular case t_n^α are the Fejér means (for all k set $q_k = 1$) and t_n^α are the (C,β)-means $\left(q_k := A_k^\beta := \binom{\beta+k}{k} \text{ for } k \ge 1 \text{ and } \beta \ne -1,-2,\ldots\right)$.

In the paper [17] the rate of the approximation by Nörlund means for Walsh – Fourier series of a function in L^p (in particular, in $\mathrm{Lip}\,(\alpha,p)$, where $\alpha > 0$ and $1 \le p \le \infty$) was studied. In case $p=\infty$, by L^p we mean C, the collection of the continuous functions. As special cases Móricz and Siddiqi obtained the earlier results by Yano [32], Jastrebova [14] and Skvortsov [27] on the rate of the approximation by Cesàro means. The approximation properties of the Cesàro means of negative order was studied by Goginava in 2002 [10]. In 2008 Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [3]. Recently, Tephnadze discussed some new aspect of the Nörlund means [29, 30].

The case when $q_k = 1/k$ is not discussed in the paper of Móricz and Siddiqi, in this case t_n^α are called the Nörlund logarithmic means. It was studied for Walsh system by Gát, Goginava and Tkebuchava earlier [5, 9], for unbounded Vilenkin system by Blahota and Gát [2].

The Nörlund means and kernels of cubical partial sums of the two-dimensional Walsh – (Kaczmarz) – Fourier series are defined by

$$\mathbf{t}_n^\alpha(f,x^1,x^2) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} S_{k,k}^\alpha(f,x^1,x^2),\ \mathcal{L}_n^\alpha(x^1,x^2) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} D_k^\alpha(x^1)D_k^\alpha(x^2).$$

$\mathbf{t}_n^\alpha$ is called the nth Nörlund mean of quadratical partial sums or the nth Nörlund mean of Marcinkiewicz type. The approximation behaviour of this Nörlund means of Marcinkiewicz type of Walsh – Fourier series was treated by the author [19] in 2010. We mention that the case that $q_k := 1/k$ was not included in that paper. For Walsh system this case is discussed by Gát and Goginava in [6], they investigated the uniform and L-convergence of the Nörlund logarithmic means of Marcinkiewicz type. If we choose $q_k := A_k^\beta = \binom{\beta+k}{k}$ (for $k \ge 1$ and $\beta \ne -1,-2,\ldots$), then we get the (C,β)-means of Marcinkiewicz type which was discussed by Goginava [11, 13] with respect to double Walsh – Fourier series, for $\beta = 1$ we get the Marcinkiewicz means [20].

In 1948 Šneider [28] introduced the Walsh – Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty}\frac{D_n^\kappa(x)}{\log n} \ge C > 0$$

holds a.e. In 1974 Schipp [24] and Young [33] proved that the Walsh – Kaczmarz system is a convergence system. Skvortsov in 1981 [26] showed that the Fejér means with respect to the Walsh – Kaczmarz system converge uniformly to f for any continuous functions f. Gát [4] proved, for any integrable functions, that the Fejér means with respect to the Walsh – Kaczmarz system converge almost everywhere to the function. Gát's result was generalized by Simon [25] in 2004. Recently, the approximation behavior of the Walsh – Kaczmarz – Nörlund means in L^p, $1 \le p \le \infty$, [18] and the rate of the approximation of the Cesàro means of negative order in L^p was discussed by the author [21, 22].

In 2003 the uniform and L-convergence of double Walsh – Kaczmarz – Fourier series was discussed by Goginava [12]. In 2006 the almost everywhere convergence of the Walsh – Kaczmarz – Marcinkiewicz means of integrable functions was proved by the author [20] (see also [7]).

2. Walsh system. Now, we give a brief introduction to the Walsh – Fourier analysis [1, 23]. Let us denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and the topology is the discrete topology. The normalized Haar measure on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let $G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_2$, which is called the Walsh group. The elements of G are sequences $x = (x_0, x_1, \ldots, x_k, \ldots)$ with coordinates $x_k \in \{0, 1\}$, $k \in \mathbb{N}$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) is the product measure, the topology is the product topology. A base for the neighbourhoods are given by

$$I_0(x) := G, \quad I_n(x) := \{y \in G \colon y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$

for $x \in G, n \in \mathbb{P}$ ($\mathbb{P} := \mathbb{N}\backslash\{0\}$), they are called dyadic intervals. Let $0 = (0 \colon i \in \mathbb{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$. Set $e_i := (0, \ldots, 0, 1, 0, \ldots)$, where the ith coordinate is 1 the rest are 0.

Let L_p denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C and set $\|f\|_\infty := \sup\{|f(x)| \colon x \in G\}$.

For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$, for $x = (x^1, x^2) \in G^2$ by $|x|^2 := |x^1|^2 + |x^2|^2$.

Next, we define the modulus of continuity of a function $f \in L^p$, $1 \le p \le \infty$, by

$$\omega_p(\delta, f) := \sup_{|t|<\delta} \|f(. + t) - f(.)\|_p, \quad \delta > 0.$$

We define the mixed modulus of continuity as follows:

$$\omega_{1,2}^p(\delta_1, \delta_2, f) :=$$

$$:= \sup\{\|f(. + x^1, . + x^2) - f(. + x^1, .) - f(., . + x^2) + f(., .)\|_p \colon |x^1| \le \delta_1, |x^2| \le \delta_2\},$$

where $\delta_1, \delta_2 > 0$.

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\mathrm{Lip}(\alpha, p) := \{f \in L^p \colon \omega_p(\delta, f) = O(\delta^\alpha) \ \text{ as } \ \delta \to 0\}.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k}, \quad x \in G, \quad k \in \mathbb{N}.$$

Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0, 1\}$, $i \in \mathbb{N}$, where only a finite number of n_i's different from zero. Let us define the order $|n|$ of $n > 0$ by $|n| := \max\{j \in \mathbb{N} \colon n_j \neq 0\}$. The Walsh functions can be enumerated in Paley enumeration as follows, $w_0 = 1$ and for $n \ge 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

(Simply we say Walsh – Paley functions, Walsh – Paley system.) The Walsh functions can be given in other enumerations, the most investigated is the Kaczmarz rearrengement. The Walsh – Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \ge 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh – Kaczmarz functions (denoted by κ) and the set of Walsh – Paley functions (denoted by w) are equal in each dyadic block. Skvortsov [26] gave a relation between the Walsh – Kaczmarz functions and the Walsh – Paley functions by the transformation $\tau_A \colon G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots)$$

for $A \in \mathbb{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)), \quad n \in \mathbb{N}, \quad x \in G.$$

Moreover, it is showed that the transformation τ_A is measure-preserving. The Dirichlet kernels are defined by

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n (n \in \mathbb{P})$ or κ_n $(n \in \mathbb{P})$, $D_0^\alpha := 0$. The 2^n th Dirichlet kernels have a closed form (see, e. g., [23])

$$D_{2^n}^w = D_{2^n}^\kappa = D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbb{N}). \end{cases} \tag{1}$$

The nth Fejér mean and the nth Fejér kernel of the Fourier series of a function f is defined by

$$\sigma_n^\alpha(f; x) := \frac{1}{n} \sum_{k=0}^{n} S_k^\alpha(f; x), \quad K_n^\alpha(x) := \frac{1}{n} \sum_{k=0}^{n} D_k^\alpha(x), \quad x \in G,$$

where $\alpha := w$ or κ and $K_0^\alpha = 0$.

On G^2 we consider the two-dimensional system as $\{\alpha_{n^1}(x^1) \times \alpha_{n^2}(x^2) \colon n := (n^1, n^2) \in \mathbb{N}^2\}$. The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series and Dirichlet kernels are defined in the usual way. Let us define the nth Marcinkiewicz kernel $\mathcal{K}_n^\alpha$ by

$$\mathcal{K}_n^\alpha(x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n} D_k^\alpha(x^1) D_k^\alpha(x^2) \quad (x = (x^1, x^2) \in G^2,\ \alpha = w \ \text{ or } \ \kappa).$$

Recently, the almost everywhere convergence of the Walsh – Kaczmarz – Marcinkiewicz means of integrable functions was discussed by the author [20] and later by Gát, Goginava and the author [7].

3. The rate of the approximation. Now, we decompose the Walsh – Kaczmarz – Nörlund kernels $\mathcal{L}_n^\kappa$. The following lemma is the two-dimensional analogue of the decomposition lemmas in [17, 18].

Lemma 1. *Let* $|n| = A \geq 1$, *then*

$$Q_n \mathcal{L}_n^\kappa(x^1, x^2) = Q_{n-2^{A-1}+1} D_{2^A}(x^1) D_{2^A}(x^2) -$$

$$-D_{2^A}(x^1)w_{2^A-1}(x^2)\sum_{j=1}^{2^{A-1}-1}(q_{n-2^A+j}-q_{n-2^A+j+1})jK_j^w(\tau_{A-1}(x^2))-$$

$$-D_{2^A}(x^2)w_{2^A-1}(x^1)\sum_{j=1}^{2^{A-1}-1}(q_{n-2^A+j}-q_{n-2^A+j+1})jK_j^w(\tau_{A-1}(x^1))-$$

$$-q_{n-2^{A-1}}2^{A-1}D_{2^A}(x^1)w_{2^A-1}(x^2)K_{2^{A-1}}^w(\tau_{A-1}(x^2))-$$

$$-q_{n-2^{A-1}}2^{A-1}D_{2^A}(x^2)w_{2^A-1}(x^1)K_{2^{A-1}}^w(\tau_{A-1}(x^1))+$$

$$+w_{2^A-1}(x^1)w_{2^A-1}(x^2)\sum_{j=1}^{2^{A-1}-1}(q_{n-2^A+j}-q_{n-2^A+j+1})j\mathcal{K}_j^w(\tau_{A-1}(x^1),\tau_{A-1}(x^2))+$$

$$+q_{n-2^{A-1}}2^{A-1}w_{2^A-1}(x^1)w_{2^A-1}(x^2)\mathcal{K}_{2^{A-1}}^w(\tau_{A-1}(x^1),\tau_{A-1}(x^2)))+$$

$$+\sum_{j=1}^{2^{A-1}-2}(q_{n-j}-q_{n-j-1})j\mathcal{K}_j^\kappa(x^1,x^2)+$$

$$+q_{n-2^{A-1}+1}(2^{A-1}-1)\mathcal{K}_{2^{A-1}-1}^\kappa(x^1,x^2)+$$

$$+Q_{n-2^A}D_{2^A}(x^1)r_A(x^2)L_{n-2^A}^w(\tau_A(x^2))+$$

$$+Q_{n-2^A}D_{2^A}(x^2)r_A(x^1)L_{n-2^A}^w(\tau_A(x^1))+$$

$$+Q_{n-2^A}r_A(x^1)r_A(x^2)\mathcal{L}_{n-2^A}^w(\tau_A(x^1),\tau_A(x^2)).$$

Proof. During the proof of Lemma 1 we use the following equations:

$$D_{2^A+j}^\kappa(x)=D_{2^A}(x)+r_A(x)D_j^w(\tau_A(x)),\quad j=0,1,\ldots,2^A-1,\tag{2}$$

$$D_{2^A-j}^\kappa(x)=D_{2^A}(x)-w_{2^A-1}(x)D_j^w(\tau_{A-1}(x)),\quad j=0,1,\ldots,2^{A-1}.\tag{3}$$

Let $|n|=A$, then we write

$$Q_n\mathcal{L}_n^\kappa(x^1,x^2)=\sum_{k=1}^{2^A}q_{n-k}D_k^\kappa(x^1)D_k^\kappa(x^2)+\sum_{k=2^A+1}^{n-1}q_{n-k}D_k^\kappa(x^1)D_k^\kappa(x^2)=:I+II.$$

By the help of (2), we decompose II:

$$II=\sum_{j=1}^{n-2^A-1}q_{n-2^A-j}D_{2^A+j}^\kappa(x^1)D_{2^A+j}^\kappa(x^2)=$$

$$=D_{2^A}(x^1)D_{2^A}(x^2)\sum_{j=1}^{n-2^A-1}q_{n-2^A-j}+D_{2^A}(x^1)r_A(x^2)\sum_{j=1}^{n-2^A-1}q_{n-2^A-j}D_j^w(\tau_A(x^2))+$$

$$+D_{2^A}(x^2)r_A(x^1)\sum_{j=1}^{n-2^A-1}q_{n-2^A-j}D_j^w(\tau_A(x^1))+r_A(x^1)r_A(x^2)Q_{n-2^A}\mathcal{L}_{n-2^A}^w(\tau_A(x^1),\tau_A(x^2))=$$

$$=Q_{n-2^A}D_{2^A}(x^1)D_{2^A}(x^2)+D_{2^A}(x^1)r_A(x^2)Q_{n-2^A}L_{n-2^A}^w(\tau_A(x^2))+$$

$$+D_{2^A}(x^2)r_A(x^1)Q_{n-2^A}L_{n-2^A}^w(\tau_A(x^1))+r_A(x^1)r_A(x^2)Q_{n-2^A}\mathcal{L}_{n-2^A}^w(\tau_A(x^1),\tau_A(x^2)).$$

We write for I that

$$I=\sum_{j=0}^{2^A-1}q_{n-2^A+j}D_{2^A-j}^\kappa(x^1)D_{2^A-j}^\kappa(x^2)=$$

$$=\sum_{j=0}^{2^{A-1}}q_{n-2^A+j}D_{2^A-j}^\kappa(x^1)D_{2^A-j}^\kappa(x^2)+\sum_{j=2^{A-1}+1}^{2^A-1}q_{n-2^A+j}D_{2^A-j}^\kappa(x^1)D_{2^A-j}^\kappa(x^2)=:I_1+I_2.$$

We use (3) and Abel's transformation for the term I_1:

$$I_1=D_{2^A}(x^1)D_{2^A}(x^2)(Q_{n-2^{A-1}+1}-Q_{n-2^A})-$$

$$-D_{2^A}(x^1)w_{2^A-1}(x^2)\left(\sum_{j=1}^{2^{A-1}-1}(q_{n-2^A+j}-q_{n-2^A+j+1})jK_j^w(\tau_{A-1}(x^2))+\right.$$

$$\left.+q_{n-2^{A-1}}2^{A-1}K_{2^{A-1}}^w(\tau_{A-1}(x^2))\right)-$$

$$-D_{2^A}(x^2)w_{2^A-1}(x^1)\left(\sum_{j=1}^{2^{A-1}-1}(q_{n-2^A+j}-q_{n-2^A+j+1})jK_j^w(\tau_{A-1}(x^1))+\right.$$

$$\left.+q_{n-2^{A-1}}2^{A-1}K_{2^{A-1}}^w(\tau_{A-1}(x^1))\right)+$$

$$+w_{2^A-1}(x^1)w_{2^A-1}(x^2)\left(\sum_{j=1}^{2^{A-1}-1}(q_{n-2^A+j}-q_{n-2^A+j+1})j\mathcal{K}_j^w(\tau_{A-1}(x^1),\tau_{A-1}(x^2))+\right.$$

$$\left.+q_{n-2^{A-1}}2^{A-1}\mathcal{K}_{2^{A-1}}^w(\tau_{A-1}(x^1),\tau_{A-1}(x^2))\right).$$

To discuss the expression I_2, we set $s=2^A-j$ and use Abel's transformation:

$$I_2=\sum_{s=1}^{2^{A-1}-1}q_{n-s}D_s^\kappa(x^1)D_s^\kappa(x^2)=$$

$$= \sum_{s=1}^{2^{A-1}-2} (q_{n-s} - q_{n-s-1}) s \mathcal{K}_s^{\kappa}(x^1, x^2) + q_{n-2^{A-1}+1}(2^{A-1} - 1)\mathcal{K}_{2^{A-1}-1}^{\kappa}(x^1, x^2).$$

Lemma 1 is proved.

By the help of this lemma we have our main theorem, which states that the approximation behavior of the two-dimensional Walsh – Kaczmarz – Nörlund means of Marcinkiewicz type is as good as the approximation behavior of the one-dimensional Walsh – Nörlund means. The last one was investigated by Móricz and Siddiqi [17] and recently by Fridli, Manchanda and Siddiqi [3]. Moreover, the rate of the approximation of Nörlund means of Marcinkiewicz type are close to each other for both rearrengement of the Walsh system (see also [19]).

Theorem 1. *Let* $f \in L^p$, $1 \le p \le \infty$ (*with the notation* $L^\infty = C$), $|n| = A \ge 1$ *and* $\{q_k : k \ge 1\}$ *be a sequence of nonnegative numbers.*

If $\{q_k\}$ *is nondecreasing, in sign* $\uparrow$, *then*

$$\|\mathbf{t}_n^{\kappa}(f) - f\|_p \le \frac{c}{Q_n} \sum_{l=0}^{A-1} q_{n-2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

If $\{q_k\}$ *is nonincreasing, in sign* $\downarrow$, *such that*

$$\frac{n}{Q_n^2} \sum_{k=1}^{n-1} q_k^2 = O(1), \tag{4}$$

then

$$\|\mathbf{t}_n^{\kappa}(f) - f\|_p \le \frac{c}{Q_n} \sum_{l=0}^{A-1} q_{n-2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

To prove our theorem we need the following lemmas proved by Schipp, Móricz [16], Yano [31], Simon [25], Glukhov [8] and Gát, Goginava, Nagy [7].

Lemma 2 [16]. *If the condition* (4) *is satisfied, then there exists a constant* C *such that* $\|L_n^w\|_1 \le$ $\le C$, $n \ge 1$.

Lemma 3 [31]. *Let* $n \ge 1$, *then* $\|K_n^w\|_1 \le 2$.

Lemma 4 [25]. *There is a constant* C *such that* $\|K_n^{\kappa}\|_1 \le C$, $n \ge 1$.

Lemma 5 [8]. *Let* $\alpha_1, \ldots, \alpha_n$ *be real numbers. Then*

$$\frac{1}{n} \left\| \sum_{k=1}^{n} \alpha_k D_k^w \otimes D_k^w \right\|_1 \le \frac{c}{\sqrt{n}} \left(\sum_{k=1}^{n} \alpha_k^2 \right)^{1/2},$$

where c *is an absolute constant.*

As corollary of the lemma of Glukhov, we get that there exists a constant C such that $\|\mathcal{K}_n^w\|_1 \le C$, $n \ge 1$, and the fact that condition (4) implies $\|\mathcal{L}_n^w\|_1 \le C$, $n \ge 1$), where C is an absolute constant.

Lemma 6 [7]. *There exists a constant* C *such that*

$$\|\mathcal{K}_n^{\kappa}\|_1 \le C, \quad n \ge 1.$$

***Proof of Theorem* 1.** Clearly, condition (4) implies the regularity of the summability method. We make the proof for $1 \leq p < \infty$, for $p = \infty$ the proof is analogous (where $L^\infty = C$), thus we omit to write it.

Let $n \in \mathbb{N}$ be fixed and set $|n| = A$. By Lemma 1 and Minkowski inequality we may write that

$$Q_n \|\mathbf{t}_n^\kappa(f) - f\|_p \leq$$

$$\leq Q_{n-2^{A-1}+1} \left\| \int\limits_{G^2} (f(. + x) - f(.)) D_{2^A}(x^1) D_{2^A}(x^2) d\mu(x) \right\|_p +$$

$$+ \sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j} - q_{n-2^A+j+1}| j \times$$

$$\times \left\| \int\limits_{G^2} (f(. + x) - f(.)) D_{2^A}(x^1) w_{2^A-1}(x^2) K_j^w(\tau_{A-1}(x^2)) d\mu(x) \right\|_p +$$

$$+ \sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j} - q_{n-2^A+j+1}| j \times$$

$$\times \left\| \int\limits_{G^2} (f(. + x) - f(.)) D_{2^A}(x^2) w_{2^A-1}(x^1) K_j^w(\tau_{A-1}(x^1)) d\mu(x) \right\|_p +$$

$$+ q_{n-2^{A-1}} 2^{A-1} \left\| \int\limits_{G^2} (f(. + x) - f(.)) D_{2^A}(x^1) w_{2^A-1}(x^2) K_{2^{A-1}}^w(\tau_{A-1}(x^2)) d\mu(x) \right\|_p +$$

$$+ q_{n-2^{A-1}} 2^{A-1} \left\| \int\limits_{G^2} (f(. + x) - f(.)) D_{2^A}(x^2) w_{2^A-1}(x^1) K_{2^{A-1}}^w(\tau_{A-1}(x^1)) d\mu(x) \right\|_p +$$

$$+ \sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j} - q_{n-2^A+j+1}| j \times$$

$$\times \left\| \int\limits_{G^2} (f(. + x) - f(.)) w_{2^A-1}(x^1) w_{2^A-1}(x^2) \mathcal{K}_j^w(\tau_{A-1}(x^1), \tau_{A-1}(x^2)) d\mu(x) \right\|_p +$$

$$+ q_{n-2^{A-1}} 2^{A-1} \left\| \int\limits_{G^2} (f(. + x) - f(.)) w_{2^A-1}(x^1) w_{2^A-1}(x^2) \mathcal{K}_{2^{A-1}}^w(\tau_{A-1}(x^1), \tau_{A-1}(x^2)) d\mu(x) \right\|_p +$$

$$+\sum_{j=1}^{2^{A-1}-2}|q_{n-j}-q_{n-j-1}|j\left\|\int_{G^2}(f(.+x)-f(.))\mathcal{K}_j^{\kappa}(x)d\mu(x)\right\|_p+$$

$$+\,q_{n-2^{A-1}+1}(2^{A-1}-1)\left\|\int_{G^2}(f(.+x)-f(.))\mathcal{K}_{2^{A-1}-1}^{\kappa}(x)d\mu(x)\right\|_p+$$

$$+\,Q_{n-2^A}\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^1)r_A(x^2)L_{n-2^A}^w(\tau_A(x^2))d\mu(x)\right\|_p+$$

$$+\,Q_{n-2^A}\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^2)r_A(x^1)L_{n-2^A}^w(\tau_A(x^1))d\mu(x)\right\|_p+$$

$$+Q_{n-2^A}\left\|\int_{G^2}(f(.+x)-f(.))r_A(x^1)r_A(x^2)\mathcal{L}_{n-2^A}^w(\tau_A(x^1),\tau_A(x^2))d\mu(x)\right\|_p=:$$

$$=:\sum_{i=1}^{12}A_{n,i}.$$

Now, we discuss the expression $A_{n,1}$. By (1) and generalized Minkowski inequality we find

$$\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^1)D_{2^A}(x^2)d\mu(x)\right\|_p\le$$

$$\le\int_{I_A^2}D_{2^A}(x^1)D_{2^A}(x^2)\left(\int_{G^2}|f(y+x)-f(y)|^pd\mu(y)\right)^{1/p}d\mu(x)\le c\omega_p(2^{-A},f).$$

Thus, we immediately have

$$A_{n,1}\le cQ_{n-2^{A-1}+1}\omega_p(2^{-A},f).$$

To discuss $A_{n,2}$, $A_{n,3}$, $A_{n,4}$, $A_{n,5}$, for any $\varepsilon\in G$, $y\in G^2$ and $A\in\mathbb{P}$ we write the following:

$$\left|\int_{I_A(\varepsilon)\times I_A}(f(y+x)-f(y))r_A(x^1)d\mu(x)\right|=\left|\int_{I_A(\varepsilon)\times I_A}f(y+x)r_A(x^1)d\mu(x)\right|=$$

$$=\left|\int_{I_{A+1}(\varepsilon)\times I_A}f(y+x)r_A(x^1)d\mu(x)+\int_{I_{A+1}(\varepsilon+e_A)\times I_A}f(y+x)r_A(x^1)d\mu(x)\right|=$$

$$=\left|\int\limits_{I_{A+1}(\varepsilon)\times I_A} f(y+x)-f(y+x+e_A^1)d\mu(x)\right|\le$$

$$\le\int\limits_{I_{A+1}(\varepsilon)\times I_A}|f(y+x)-f(y+x+e_A^1)|d\mu(x), \tag{5}$$

where $e_A^1:=(e_A,0)$ (and $e_A^2:=(0,e_A)$ we will use it later). Now, for any $j\le 2^{A-1}$ we write that

$$B_j^{A-1}:=\left\|\int\limits_{G^2}(f(.+x)-f(.))D_{2^A}(x^2)w_{2^A-1}(x^1)K_j^w(\tau_{A-1}(x^1))d\mu(x)\right\|_p=$$

$$=\left\|\sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\ldots,A-2\}}}^{1}2^A\int\limits_{I_{A-1}(\varepsilon)\times I_A}(f(.+x)-f(.))r_{A-1}(x^1)w_{2^{A-1}-1}(x^1)K_j^w(\tau_{A-1}(x^1))d\mu(x)\right\|_p.$$

The function $w_{2^{A-1}-1}(x^1)K_j^w(\tau_{A-1}(x^1))$ is constant on the sets $I_{A-1}(\varepsilon)$ $(\varepsilon\in G, |j|\le A-1)$. Thus, the method of (5) and Lemma 3 imply

$$B_j^{A-1}=$$

$$=\left\|\sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\ldots,A-2\}}}^{1}2^A w_{2^{A-1}-1}(\varepsilon)K_j^w(\tau_{A-1}(\varepsilon))\int\limits_{I_{A-1}(\varepsilon)\times I_A}(f(.+x)-f(.))r_{A-1}(x^1)d\mu(x)\right\|_p\le$$

$$\le\sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\ldots,A-2\}}}^{1}2^A|K_j^w(\tau_{A-1}(\varepsilon))|\times$$

$$\times\left(\int\limits_{G^2}\left|\int\limits_{I_{A-1}(\varepsilon)\times I_A}(f(y+x)-f(y))r_{A-1}(x^1)d\mu(x)\right|^p d\mu(y)\right)^{1/p}\le$$

$$\le\sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\ldots,A-2\}}}^{1}2^A|K_j^w(\tau_{A-1}(\varepsilon))|\times$$

$$\times\left(\int\limits_{G^2}\left(\int\limits_{I_A(\varepsilon)\times I_A}|f(y+x)-f(y+x+e_{A-1}^1)|d\mu(x)\right)^p d\mu(y)\right)^{1/p}\le$$

$$\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\ldots,A-2\}}}^{1} 2^A|K_j^w(\tau_{A-1}(\varepsilon))|\times$$

$$\times \int\limits_{I_A(\varepsilon)\times I_A} \left(\int\limits_{G^2} |f(y+x)-f(y+x+e^1_{A-1})|^p d\mu(y)\right)^{1/p} d\mu(x) \leq$$

$$\leq c \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\ldots,A-2\}}}^{1} 2^A|K_j^w(\tau_{A-1}(\varepsilon))|\omega_p(2^{-A+1},f) \int\limits_{I_A(\varepsilon)\times I_A} d\mu(x) \leq$$

$$\leq c\omega_p(2^{-A+1},f)\|K_j^w\circ\tau_{A-1}\|_1 \leq$$

$$\leq c\omega_p(2^{-A+1},f)\|K_j^w\|_1 \leq c\omega_p(2^{-A+1},f). \tag{6}$$

This yields that

$$A_{n,4}, A_{n,5} \leq q_{n-2^{A-1}}2^{A-1}B^{A-1}_{2^{A-1}} \leq cq_{n-2^{A-1}}2^{A-1}\omega_p(2^{-A+1},f),$$

$$A_{n,2}, A_{n,3} = \sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j}-q_{n-2^A+j+1}|jB_j^{A-1} \leq$$

$$\leq c\sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j}-q_{n-2^A+j+1}|j\omega_p(2^{-A},f).$$

If $q_k\uparrow$, we get that

$$\sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j}-q_{n-2^A+j+1}|j \leq 2^{A-1}q_{n-2^{A-1}} - \sum_{j=1}^{2^{A-1}-1} q_{n-2^A+j} \leq 2^{A-1}q_{n-2^{A-1}} \tag{7}$$

and

$$A_{n,3} \leq c2^{A-1}q_{n-2^{A-1}}\omega_p(2^{-A+1},f),$$

while in the case when $q_k\downarrow$

$$\sum_{j=1}^{2^{A-1}-1} |q_{n-2^A+j}-q_{n-2^A+j+1}|j = \sum_{j=1}^{2^{A-1}-1} q_{n-2^A+j} - (2^{A-1}-1)q_{n-2^{A-1}} \leq$$

$$\leq Q_{n-2^{A-1}} - Q_{n-2^A+1} \tag{8}$$

and

$$A_{n,3} \leq c(Q_{n-2^{A-1}} - Q_{n-2^A+1})\omega_p(f,2^{-A}).$$

Now, we introduce the notation $\tilde{B}_j^A$ (see inequality (6)) to discuss $A_{n,10}, A_{n,11}$. First, let $q_k \downarrow$

$$\tilde{B}_j^A := \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^A}(x^2) r_A(x^1) L_j^w(\tau_A(x^1)) d\mu(x) \right\|_p ,$$

for $|j| \le A$. The method presented for inequality (6) (we note that $|n - 2^A| \le A - 1$) and Lemma 2 imply

$$\tilde{B}_{n-2^A}^A \le c\omega_p(2^{-A}, f) \|L_{n-2^A}^w \circ \tau_A\|_1 \le c\omega_p(2^{-A}, f) \|L_{n-2^A}^w\|_1 \le c\omega_p(2^{-A}, f)$$

and

$$A_{n,10}, A_{n,11} \le cQ_{n-2^A} \omega_p(2^{-A}, f).$$

Now, let $q_k \uparrow$. We use Abel's transformation for the expression $Q_{n-2^A} L_{n-2^A}^w$:

$$Q_{n-2^A} L_{n-2^A}^w = \sum_{j=1}^{n-2^A-2} (q_{n-2^A-j} - q_{n-2^A-j-1}) j K_j^w + q_1(n - 2^A - 1) K_{n-2^A-1}^w,$$

and define $\tilde{\tilde{B}}_j^A$ by

$$\tilde{\tilde{B}}_j^A := \left\| \int_{G^2} (f(.+x) - f(.)) D_{2^A}(x^2) r_A(x^1) K_j^w(\tau_A(x^1)) d\mu(x) \right\|_p , \quad |j| \le A.$$

The method of the discussion B_j^A (see inequality (6)), Lemma 3 and the fact that the transformation τ_A is measure-preserving [26] immediately give that

$$\tilde{\tilde{B}}_j^A \le c\omega_p(2^{-A}, f) \|K_j^w \circ \tau_A\|_1 \le c\omega_p(2^{-A}, f) \|K_j^w\|_1 \le c\omega_p(2^{-A}, f)$$

and

$$A_{n,10}, A_{n,11} \le c\omega_p(2^{-A}, f) \left(\sum_{j=1}^{n-2^A-2} |q_{n-2^A-j} - q_{n-2^A-j-1}| j + q_1(n - 2^A - 1) \right) \le$$

$$\le c\omega_p(2^{-A}, f) \left(\sum_{j=1}^{n-2^A-2} q_{n-2^A-j} + q_1(n - 2^A - 1) \right) \le$$

$$\le c\omega_p(2^{-A}, f)(Q_{n-2^A} + q_1(n - 2^A - 1)).$$

We note that $Q_n \ge (n-1)q_1$ for increasing sequence $\{q_k\}$.

Now, we discuss the terms $A_{n,8}$ and $A_{n,9}$. Let us set $j < 2^{A-1}$. We use Lemma 6 of Gát, Goginava and the author and the fact that the functions $\mathcal{K}_j^\kappa$ are constant on the sets $I_{|j|}(\varepsilon) \times I_{|j|}(\rho)$ for any $\varepsilon, \rho \in G$:

$$E_j := \left\| \int\limits_{G^2} (f(.+x) - f(.))\mathcal{K}_j^{\kappa}(x)d\mu(x) \right\|_p =$$

$$= \left\| \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\ldots,j-1\}}}^{1} \sum_{\substack{\rho_l=0 \\ l\in\{0,\ldots,j-1\}}}^{1} \int\limits_{I_{|j|}(\varepsilon)\times I_{|j|}(\rho)} (f(.+x) - f(.))\mathcal{K}_j^{\kappa}(x)d\mu(x) \right\|_p \le$$

$$\le \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\ldots,j-1\}}}^{1} \sum_{\substack{\rho_l=0 \\ l\in\{0,\ldots,j-1\}}}^{1} |\mathcal{K}_j^{\kappa}(\varepsilon,\rho)| \times$$

$$\times \left(\int\limits_{G^2} \left(\int\limits_{I_{|j|}(\varepsilon)\times I_{|j|}(\rho)} |f(y+x) - f(y)| d\mu(x) \right)^p d\mu(y) \right)^{1/p} \le$$

$$\le \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\ldots,j-1\}}}^{1} \sum_{\substack{\rho_l=0 \\ l\in\{0,\ldots,j-1\}}}^{1} |\mathcal{K}_j^{\kappa}(\varepsilon,\rho)| \int\limits_{I_{|j|}(\varepsilon)\times I_{|j|}(\rho)} \left(\int\limits_{G^2} |f(y+x) - f(y)|^p d\mu(y) \right)^{1/p} d\mu(x) \le$$

$$\le c \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\ldots,j-1\}}}^{1} \sum_{\substack{\rho_l=0 \\ l\in\{0,\ldots,j-1\}}}^{1} |\mathcal{K}_j^{\kappa}(\varepsilon,\rho)| \omega_p(2^{-|j|}, f) \int\limits_{I_{|j|}(\varepsilon)\times I_{|j|}(\rho)} d\mu(x) \le$$

$$\le c\|\mathcal{K}_j^{\kappa}\|_1 \omega_p(2^{-|j|}, f) \le c\omega_p(2^{-|j|}, f).$$

This yields that

$$A_{n,9} \le q_{n-2^{A-1}+1}(2^{A-1}-1)E_{2^{A-1}-1} \le cq_{n-2^{A-1}+1}2^{A-1}\omega_p(2^{-(A-2)}, f).$$

If $q_k \uparrow$, then

$$A_{n,9} \le cq_{n-2^{A-2}}2^{A-2}\omega_p(2^{-(A-2)}, f).$$

If $q_k \downarrow$, then

$$A_{n,9} \le cq_{n-2^{A-1}}2^{A-1}\omega_p(2^{-A+1}, f).$$

Moreover, we get

$$A_{n,8} \le c \sum_{j=1}^{2^{A-1}-2} |q_{n-j} - q_{n-j-1}| jE_j \le \sum_{j=0}^{A-2} \sum_{l=2^j}^{2^{j+1}-1} |q_{n-l} - q_{n-l-1}| lE_l \le$$

$$\le c \sum_{j=0}^{A-2} \omega_p(2^{-j}, f) \sum_{l=2^j}^{2^{j+1}-1} |q_{n-l} - q_{n-l-1}| l.$$

If $q_k \uparrow$, then

$$\sum_{l=2^j}^{2^{j+1}-1} |q_{n-l} - q_{n-l-1}| l \leq \sum_{l=0}^{2^j-1} q_{n-2^j-l} \leq 2^j q_{n-2^j},$$

$$A_{n,8} \leq c \sum_{j=0}^{A-2} q_{n-2^j} 2^j \omega_p(2^{-j}, f).$$

If $q_k \downarrow$, then

$$\sum_{l=2^j}^{2^{j+1}-1} |q_{n-l} - q_{n-l-1}| l \leq 2^{j+1} q_{n-2^{j+1}},$$

$$A_{n,8} \leq c \sum_{j=0}^{A-2} q_{n-2^{j+1}} 2^{j+1} \omega_p(2^{-j-1}, f).$$

At last, we discuss the expressions $A_{n,6}, A_{n,7}, A_{n,12}$. Now, we investigate $A_{n,12}$ and the other two term can be treated analogously. But, we will write some words about it later.

First, let $q_k \downarrow$. We note that $\mathcal{L}^w_{n-2^A}(\tau_A(x^1), \tau_A(x^2))$ is constant on the sets $I_A(\varepsilon) \times I_A(\rho)$ for any $\varepsilon, \rho \in G$. This and the generalized Minkowski inequality give

$$F^A_{n-2^A} := \left\| \int_{G^2} (f(. + x) - f(.)) r_A(x^1) r_A(x^2) \mathcal{L}^w_{n-2^A}(\tau_A(x^1), \tau_A(x^2)) d\mu(x) \right\|_p =$$

$$= \left\| \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\ldots,A-1\}}}^{1} \sum_{\substack{\rho_j=0 \\ j\in\{0,\ldots,A-1\}}}^{1} \int_{I_A(\varepsilon)\times I_A(\rho)} (f(. + x) - f(.)) \times \right.$$

$$\left. \times\, r_A(x^1) r_A(x^2) \mathcal{L}^w_{n-2^A}(\tau_A(x^1), \tau_A(x^2)) d\mu(x) \right\|_p \leq$$

$$\leq \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\ldots,A-1\}}}^{1} \sum_{\substack{\rho_j=0 \\ j\in\{0,\ldots,A-1\}}}^{1} |\mathcal{L}^w_{n-2^A}(\tau_A(\varepsilon), \tau_A(\rho))| \times$$

$$\times \left\| \int_{I_A(\varepsilon)\times I_A(\rho)} (f(. + x) - f(.)) r_A(x^1) r_A(x^2) d\mu(x) \right\|_p \leq$$

$$\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}^w_{n-2^A}(\tau_A(\varepsilon),\tau_A(\rho))|\times$$

$$\times\left(\int\limits_{G^2}\left|\int\limits_{I_A(\varepsilon)\times I_A(\rho)}(f(y+x)-f(y))r_A(x^1)r_A(x^2)d\mu(x)\right|^p d\mu(y)\right)^{1/p}.$$

In the way of estimation (5) we easily get

$$\left|\int\limits_{I_A(\varepsilon)\times I_A(\rho)}(f(y+x)-f(y))r_A(x^1)r_A(x^2)d\mu(x)\right|\leq \int\limits_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)}\Delta_A f(x,y)d\mu(x), \tag{9}$$

where

$$\Delta_A f(x,y):=|f(x+y)-f(x+y+e_A^2)-f(x+y+e_A^1)+f(x+y+e_A^1+e_A^2)|.$$

Inequality (9), condition (4) and Lemma 5 imply that

$$F^A_{n-2^A}\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}^w_{n-2^A}(\tau_A(\varepsilon),\tau_A(\rho))|\times$$

$$\times\left(\int\limits_{G^2}\left(\int\limits_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)}\Delta_A f(x,y)d\mu(x)\right)^p d\mu(y)\right)^{1/p}\leq$$

$$\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}^w_{n-2^A}(\tau_A(\varepsilon),\tau_A(\rho))|\times$$

$$\times\int\limits_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)}\left(\int\limits_{G^2}(\Delta_A f(x,y))^p d\mu(y)\right)^{1/p} d\mu(x)\leq$$

$$\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} \int\limits_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)}|\mathcal{L}^w_{n-2^A}(\tau_A(\varepsilon),\tau_A(\rho))|d\mu(x)\omega^p_{1,2}(2^{-A},2^{-A},f)\leq$$

$$\leq \|\mathcal{L}^w_{n-2^A}\circ(\tau_A\times\tau_A)\|_1\omega^p_{1,2}(2^{-A},2^{-A},f)\leq$$

$$\leq \|\mathcal{L}^w_{n-2^A}\|_1\omega^p_{1,2}(2^{-A},2^{-A},f)\leq c\omega^p_{1,2}(2^{-A},2^{-A},f). \tag{10}$$

We note that we used that the transformation τ_A is measure-preserving [26]. From the above written

$$A_{n,12} \le cQ_{n-2^A}\omega^p_{1,2}(2^{-A},2^{-A},f) \le cQ_{n-2^A}\omega_p(2^{-A},f).$$

Now, we discuss the expression $A_{n,12}$ for sequence $q_k \uparrow$. By Abel's transformation we write

$$Q_{n-2^A}\mathcal{L}^w_{n-2^A} = \sum_{j=1}^{n-2^A-2}(q_{n-2^A-j}-q_{n-2^A-j-1})j\mathcal{K}^w_j + q_1(n-2^A-1)\mathcal{K}^w_{n-2^A-1}.$$

Let us set for $|j| \le A$

$$\tilde{F}^A_j := \left\| \int_{G^2}(f(.+x)-f(.))r_A(x^1)r_A(x^2)\mathcal{K}^w_j(\tau_A(x^1),\tau_A(x^2))d\mu(x)\right\|_p .$$

The method of the discussion of F^A_j (see inequality (10)) and Lemma 5 give immediately

$$\tilde{F}^A_j \le c\omega^p_{1,2}(2^{-A},2^{-A},f)\|\mathcal{K}^w_j\circ(\tau_A\times\tau_A)\|_1 \le c\omega^p_{1,2}(2^{-A},2^{-A},f)\|\mathcal{K}^w_j\|_1 \le c\omega_p(2^{-A},f)$$

and

$$A_{n,12} \le c\omega_p(2^{-A},f)\left(Q_{n-2^A}+q_1(n-2^A-1)\right).$$

(For more details see $A_{n,10}, A_{n,11}$.)

Let us define $\tilde{\tilde{F}}^{A-1}_j$ (for any $|j| \le A-1$) by

$$\tilde{\tilde{F}}^{A-1}_j :=$$

$$:= \left\| \int_{G^2}(f(.+x)-f(.))r_{A-1}(x^1+x^2)\omega_{2^{A-1}-1}(x^1+x^2)\mathcal{K}^w_j(\tau_{A-1}(x^1),\tau_{A-1}(x^2))d\mu(x)\right\|_p .$$

The method presented for discussion F^A_j (see inequality (10)) and Lemma 5 give

$$\tilde{\tilde{F}}^{A-1}_j \le c\|\mathcal{K}^w_j\circ(\tau_{A-1}\times\tau_{A-1})\|_1\omega^p_{1,2}(2^{-A+1},2^{-A+1},f) \le$$

$$\le c\|\mathcal{K}^w_j\|_1\omega^p_{1,2}(2^{-A+1},2^{-A+1},f) \le c\omega^p_{1,2}(2^{-A+1},2^{-A+1},f).$$

Thus,

$$A_{n,7} \le cq_{n-2^{A-1}}2^{A-1}\omega^p_{1,2}(2^{-A+1},2^{-A+1},f) \le cq_{n-2^{A-1}}2^{A-1}\omega_p(2^{-A+1},f)$$

and

$$A_{n,6} \le c\sum_{j=1}^{2^{A-1}-1}|q_{n-2^A+j}-q_{n-2^A+j+1}|j\omega^p_{1,2}(2^{-A+1},2^{-A+1},f).$$

If $q_k \uparrow$, then by (7)

$$A_{n,6} \le c2^{A-1}q_{n-2^{A-1}}\omega_p(2^{-A+1},f).$$

If $q_k \downarrow$, then by (8)

$$A_{n,6} \leq c(Q_{n-2^{A-1}} - Q_{n-2^A+1})\omega_p(2^{-A+1}, f) \leq c(Q_{n-2^{A-1}} - Q_{n-2^A+1})\omega_p(2^{-A}, f).$$

Summarising our results on $A_{n,i}$, $i = 1, \ldots, 12$, we complete the proof of our main theorem.

Now, we discuss the following cases:

(A) The nondecreasing $\{q_k\}$, in sign $q_k \uparrow$, satisfies the condition

$$\frac{nq_{n-1}}{Q_n} = O(1). \tag{11}$$

In particular (11) is true if

$$q_k \asymp k^\beta \text{ or } (\log k)^\beta \text{ for some } \beta > 0.$$

(B) The nonincreasing $\{q_k\}$, in sign $q_k \downarrow$, satisfies

(B_i) $q_k \asymp k^{-\beta}$ for some $0 < \beta < 1$, or

(B_{ii}) $q_k \asymp (\log k)^{-\beta}$ for some $0 < \beta$.

(We note that the condition (4) is satisfied in these cases.) For more details see [17, 19].

The one-dimensional analogue of the following theorem was proven for Walsh – Paley system by Móricz and Siddiqi in [17] for Walsh – Kaczmarz system by the author [18]. We mention that as special case (set $q_k := 1$ for all k) we get Marcinkiewicz means of Walsh – Kaczmarz – Fourier series. More generally, when $q_k := A_k^\beta := \binom{\beta+k}{k}$ for $k \geq 1$, $\beta \neq -1, -2, \ldots$, we have the (C, β) mean of Marcinkiewicz type discussed by Goginava [13] with respect to the double Walsh – Paley system and by the author [22] with respect to double Walsh – Kaczmarz system.

Theorem 2. *Let $f \in \mathrm{Lip}\,(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$.*

Let $\{q_k : k \geq 1\}$ be a sequence of nonnegative numbers such that in case $q_k \uparrow$ the condition (11) *is satisfied, while in case $q_k \downarrow$ the condition* (B_i) *or* (B_{ii}) *is satisfied, then*

$$\|\mathbf{t}_n^\kappa(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \textit{if} \quad 0 < \alpha < 1, \\ O(n^{-1}\log n), & \textit{if} \quad \alpha = 1, \\ O(n^{-1}), & \textit{if} \quad \alpha > 1. \end{cases}$$

Proof. Let $f \in \mathrm{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$.

First, let $q_k \uparrow$, which satisfies the condition (11). From Theorem 1 by the method of Móricz and Siddiqi [17] Theorem 2 can be proven.

Second, let $q_k \downarrow$, which satisfies the condition (B_i), that is,

$$q_k \asymp k^{-\beta} \quad \text{for some} \quad 0 < \beta < 1, \quad \text{then} \quad Q_n \asymp n^{1-\beta}.$$

From Theorem 1 it follows that

$$\|\mathbf{t}_n^\kappa(f) - f\|_p \leq \frac{c}{Q_n} \sum_{l=0}^{|n|-1} q_{n-2^l} 2^l 2^{-l\alpha} + O(2^{-|n|\alpha}).$$

For $0 \leq l \leq |n| - 1$ we have $2^{|n|-1} \leq n - 2^l$ and $q_{n-2^l} \leq c2^{-\beta(|n|-1)}$. Thus,

$$\|\mathbf{t}_n^\kappa(f) - f\|_p \leq \frac{c}{n^{1-\beta}} \sum_{l=0}^{|n|-1} 2^{-\beta|n|} 2^{l(1-\alpha)} + O(2^{-|n|\alpha}) \leq \frac{c}{n} \sum_{l=0}^{|n|-1} 2^{l(1-\alpha)} + O(2^{-|n|\alpha}) =$$

$$= \begin{cases} O\left(\dfrac{2^{|n|(1-\alpha)}}{n}\right), & \text{if} \quad 0 < \alpha < 1, \\ O\left(\dfrac{|n|}{n}\right), & \text{if} \quad \alpha = 1, \\ O\left(\dfrac{1}{n}\right), & \text{if}\, \alpha > 1. \end{cases}$$

Let the condition (B_{ii}) be satisfied, that is,

$$q_k \asymp (\log k)^{-\beta} \text{ for some } 0 < \beta, \text{ then } Q_n \asymp n(\log n)^{-\beta}.$$

From now the proof goes along the same lines as that of case (B_i).

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