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GENERALIZED DERIVATIONS AND COMMUTING ADDITIVE MAPS ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

УЗАГАЛЬНЕНІ ПОХІДНІ ТА КОМУТУЮЧІ АДИТИВНІ ВІДОБРАЖЕННЯ НА МУЛЬТИЛІНІЙНИХ ПОЛІНОМАХ У ПРОСТИХ КІЛЬЦЯХ

Let R be a prime ring with characteristic different from 2, U be its right Utumi quotient ring, C be its extended centroid, F and G be additive maps on R, $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, and I be a nonzero right ideal of R. We obtain information about the structure of R and describe the form of F and G in the following cases:

(1) $[(F^2+G)(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]=0$ for all $r_1,\ldots,r_n\in R$, where F and G are generalized derivations of R;

(2) $[(F^2+G)(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]=0$ for all $r_1,\ldots,r_n\in I$, where F and G are derivations of R.

Нехай R — просте кільце з характеристикою, що відмінна від 2, U — його праве фактор-кільце, C — його розширений центроїд, F та G — адитивні відображення на R, $f(x_1,\ldots,x_n)$ — мультилінійний поліном над C, а I — ненульовий правий ідеал для R. Отримано інформацію про структуру кільця R та описано форму F і G у таких випадках:

(1) $[(F^2+G)(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]=0$ для всіх $r_1,\ldots,r_n\in R$, де F та G — узагальнені похідні від R;

(2) $[(F^2+G)(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]=0$ для всіх $r_1,\ldots,r_n\in I$, де F та G — похідні від R.

1. Introduction. Throughout this paper, R always denotes a prime ring with center $Z(R)$ and extended centroid C, U its right Utumi quotient ring. By a derivation on R, we mean an additive map $G\colon R\longrightarrow R$ such that $G(xy)=G(x)y+xG(y)$ holds for all $x,y\in R$. A generalized derivation on R is an additive map $G\colon R\longrightarrow R$ such that $G(xy)=G(x)y+xd(y)$ holds for all $x,y\in R$, where d is a derivation of R. We denote $[a,b]=ab-ba$, the simple commutator of the elements $a,b\in R$ and $[a,b]_k=\big[[a,b]_{k-1},b\big]$, for $k>1$, the kth commutator of a,b. Let $T\subseteq R$. An additive map $F\colon R\longrightarrow R$ is said to be commuting in T (resp. centralizing in T) if $[F(x),x]=0$ for all $x\in T$ (resp. $[F(x),x]\in Z(R)$ for all $x\in T$).

Several authors have studied derivations and generalized derivations which are centralizing and commuting in some subsets of prime and semiprime rings (see [12, 17, 19, 21] for references). In this view, a well-known result proved by Posner [24] states that a prime ring R must be commutative, if it admits a non-zero centralizing derivation. In [16], Lee studied derivations with Engel conditions on polynomials $f(x_1,\ldots,x_n)$ in non-zero one-sided ideals of R. More precisely, he proved that if $\big[d\big(f(r_1,\ldots,r_n)\big),f(r_1,\ldots,r_n)\big]_k=0$ for all $r_1,\ldots,r_n\in L$, a non-zero left ideal of R, and $k\geq 1$ a fixed integer, then there exists an idempotent element e in the socle of RC such that $CL=RCe$ and one of the following holds: (i) $f(x_1,\ldots,x_n)$ is central valued in $eRCe$ unless C is finite or $0<\operatorname{char}(R)\leq k+1$; (ii) in case $\operatorname{char}(R)=p>0$, then $f(x_1,\ldots,x_n)^{p^s}$ is central valued in $eRCe$ for some $s\geq 0$, unless $\operatorname{char}(R)=2$ and $eRCe$ satisfies the identity s_4.

Recently in [6], the first author of the present paper studied the case when the Engel condition is satisfied by a generalized derivation on evaluations of multilinear polynomials. More precisely, he proved that if G is a non-zero generalized derivation of R, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C and I a non-zero right ideal of R such that G is commuting in $f(I)$, the set of all evaluations of $f(x_1, \ldots, x_n)$ over I, then either $G(x) = ax$ with $(a - \gamma)I = 0$ and a suitable $\gamma \in C$ or there exists an idempotent element $e \in \mathrm{soc}(RC)$ such that $IC = eRC$ and one of the following holds:

(1) $f(x_1, \ldots, x_n)$ is central valued in $eRCe$;

(2) $G(x) = cx + xb$, where $(c - b + \alpha)e = 0$ for $\alpha \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$;

(3) $\mathrm{char}(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.

Simultaneously in [1], Ali and Shah showed that if the generalized derivation G is centralizing in a one-sided ideal of the prime ring R, then R is commutative.

The main object of the present paper is to investigate the situation when the additive map $F^2 + G$ is commuting in $f(I)$, the set of all evaluations of $f(x_1, \ldots, x_n)$ over I, where $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C, I is a suitable subset of R and F, G two derivations or generalized derivations of R.

In [14] (Theorem 2.1), Lee et al. proved that if F and G are derivations of a $n!$-torsion free semiprime ring such that $[(F^2+G)(x), x^n] = 0$ for all $x \in R$, then F and G are both commuting in R.

Recently in [8], the first author of the present paper and Rehman extended the above result of [14] to generalized derivations. More precisely, in [8] (Theorem 3.1), it is proved that if R is a $n!$-torsion free semiprime ring, F and G two generalized derivations of R associated with non-zero derivations f and g respectively, such that $\big[(F^2 + G)(x), x^n\big] = 0$ for all $x \in R$, then either R contains a non-zero central ideal, or $f = 0$, $g(R) \subseteq Z(R)$ and there exist $a, b \in U$ such that $F(x) = ax$, $G(x) = bx + g(x)$ for all $x \in R$, with $a^2 + b \in C$.

Recently in [9], the second author and Sharma studied the case when F is a derivation of R, $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C and I is a right ideal of R.

They proved that if $\big[F^2\big(f(r_1, \ldots, r_n)\big), f(r_1, \ldots, r_n)\big] = 0$ for all $r_1, \ldots, r_n \in I$, then either $\big[f(x_1, \ldots, x_n), x_{n+1}\big]x_{n+2}$ is satisfied by I, or there exists $b \in U$ such that $F(x) = [b, x]$ for all $x \in R$, with $b^2 = 0$ and $bI = (0)$.

Being inspired by the above cited results, we shall prove the following theorem.

Theorem 1.1. *Let R be a prime ring of characteristic different from 2, U its right Utumi quotient ring, C its extended centroid, F and G two generalized derivations of R and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C. If $\big[(F^2+G)\big(f(r_1, \ldots, r_n)\big), f(r_1, \ldots, r_n)\big] = 0$ for all $r_1, \ldots, r_n \in$ $\in R$, then either $f(x_1, \ldots, x_n)$ is central valued on R or one of the following holds:*

(1) *there exist $c, p \in U$ such that $F(x) = xc$, $G(x) = xp$ for all $x \in R$, with $c^2 + p \in C$;*

(2) *there exist $c, p \in U$ and $\alpha \in C$ such that $F(x) = cx$, $G(x) = px$ for all $x \in R$, with $c^2 + p \in C$;*

(3) *$f(x_1, \ldots, x_n)^2$ is central valued on R and there exist $c, p, q \in U$ such that $F(x) = xc$, $G(x) = px + xq$ for all $x \in R$, with $c^2 + q - p \in C$;*

(4) *$f(x_1, \ldots, x_n)^2$ is central valued on R and there exist $c, p, q \in U$ such that $F(x) = cx$, $G(x) = px + xq$ for all $x \in R$, with $c^2 + p - q \in C$.*

Theorem 1.2. *Let R be a prime ring of characteristic different from 2, U its right Utumi quotient ring, C its extended centroid, F and G two derivations of R, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C and I a non-zero right ideal of R. If $\big[(F^2+G)\big(f(r_1,\ldots,r_n)\big), f(r_1,\ldots,r_n)\big]=0$ for all $r_1,\ldots,r_n\in I$, then one of the following holds*:

(1) *there exists an idempotent element $e\in\operatorname{soc}(RC)$ such that $IC=eRC$, moreover $f(x_1,\ldots,x_n)$ is central valued on $eRCe$*;

(2) *there exist $c,p\in U$ and $\alpha,\beta\in C$ such that $F(x)=[c,x]$, $G(x)=[p,x]$ for all $x\in R$, with $(c-\alpha)I=(p-\beta)I=(0)$ and $(c-\alpha)^2=(p-\beta)$.*

To prove our theorems, we shall use frequently the theory of generalized polynomial identities and differential identities (see [2, 4, 13, 20, 23]). In particular, we recall that if R is prime and I a non-zero right ideal of R, then I, IR and IU satisfy the same generalized polynomial identities [4].

In [17], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $g\colon I\longrightarrow U$ such that $g(xy)=g(x)y+xd(y)$ for all $x,y\in I$, where I is a dense right ideal of R and d is a derivation from I into U.

Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U.

More details about generalized derivations can be found in [11, 17, 21].

2. The case: pair of generalized derivations on multilinear polynomials in prime rings. In this section we will prove Theorem 1.1. We begin with the following lemma, which will be also used in the next section for the proof of Theorem 1.2.

Lemma 2.1. *Let R be a prime ring, $F(x)=ax+xb$ and $G(x)=px+xq$, for $a,b,p,q\in U$, be two inner generalized derivations of R. Let I be a right ideal of R such that $\big[(F^2+G)\big(f(r_1,\ldots,r_n)\big), f(r_1,\ldots,r_n)\big]=0$ for all $r_1,\ldots,r_n\in I$. Then R satisfies a nontrivial generalized polynomial identity, unless one of the following holds*:

(1) *there exist $\alpha,\beta\in C$ such that $(a^2+p-\alpha)I=(0)$, $(a-\beta)I=(0)$ and $b^2+q+2\beta b\in C$*;

(2) *$b,q\in C$ and there exists $\alpha\in C$ such that $(a^2+p+2ab-\alpha)I=(0)$.*

Proof. Let B be a basis of U over C. Then any element of $T=U*_C C\{x_1,\ldots,x_n\}$ can be written in the form $g=\sum_i \alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials, that is $m_i=q_0y_1\ldots y_hq_h$, with $q_i\in B$ and $y_i\in\{x_1,\ldots,x_n\}$. In [4], it is shown that a generalized polynomial $g=\sum_i\alpha_i m_i$ is the zero element of T if and only if all α_i are zeros. As a consequence, let $a_1,\ldots,a_k\in U$ be linearly independent over C and $a_1g_1(x_1,\ldots,x_n)+\ldots+a_kg_k(x_1,\ldots,x_n)=0\in T$, for some $g_1,\ldots,g_k\in T$.

If, for any i, $g_i(x_1,\ldots,x_n)=\sum_{j=1}^n x_jh_j(x_1,\ldots,x_n)$ and $h_j(x_1,\ldots,x_n)\in T$, then $g_1(x_1,\ldots,x_n)$, $\ldots$, $g_k(x_1,\ldots,x_n)$ are the zero elements of T. The same conclusion holds if $g_1(x_1,\ldots,x_n)a_1+\ldots+g_k(x_1,\ldots,x_n)a_k=0\in T$, and $g_i(x_1,\ldots,x_n)=\sum_{j=1}^n h_j(x_1,\ldots,x_n)x_j$ for some $h_j(x_1,\ldots,x_n)\in T$.

In all that follows we assume that R does not satisfy any nontrivial generalized polynomial identity with coefficients in U. Therefore by our hypothesis, for any $0\neq y\in I$,

$$\Phi(x_1,\ldots,x_n)=\Big[(a^2+p)f(yx_1,\ldots,yx_n)+2af(yx_1,\ldots,yx_n)b+$$

$$+f(yx_1,\ldots,yx_n)(b^2+q),f(yx_1,\ldots,yx_n)\Big]$$

is a trivial generalized polynomial identity for R. We rewrite it as

$$\Phi(x_1,\ldots,x_n)=(a^2+p)f(yx_1,\ldots,yx_n)^2+2af(yx_1,\ldots,yx_n)bf(yx_1,\ldots,yx_n)+$$

$$+f(yx_1,\ldots,yx_n)(b^2+q)f(yx_1,\ldots,yx_n)-f(yx_1,\ldots,yx_n)(a^2+p)f(yx_1,\ldots,yx_n)-$$

$$-2f(yx_1,\ldots,yx_n)af(yx_1,\ldots,yx_n)b-f(yx_1,\ldots,yx_n)^2(b^2+q). \tag{2.1}$$

If $b^2+q\in C$, then by applying previous argument to (2.1), we have $b\in C$. Analogously if $b\in C$, it follows $b^2+q\in C$. Hence $b\in C$ if and only if $b^2+q\in C$. On the other hand, by applying the same argument to (2.1), $\{(a^2+p)y,y\}$ is linearly C-dependent if and only if $\{ay,y\}$ is linearly C-dependent. Now we divide the proof into three cases:

Case 1. Suppose that $b^2+q,\ b\in C$. Then by (2.1), it follows that $\big\{(a^2+p+2ab)y,y\big\}$ is linearly C-dependent. Thus there exists $\alpha\in C$ such that $(a^2+p+2ab-\alpha)I=(0)$, which is our conclusion (2).

Case 2. Suppose that for any $y\in I$, $\big\{(a^2+p)y,y\big\}$ as well as $\{ay,y\}$ are two linearly C-dependent sets. In this case standard argument shows that there exist $\alpha,\lambda\in C$ such that $(a^2+p-\alpha)I=(0)$ and $(a-\lambda)I=(0)$. Then (2.1) reduces to

$$\big[f(yx_1,\ldots,yx_n)(2\lambda b+b^2+q),f(yx_1,\ldots,yx_n)\big]$$

which is a trivial generalized polynomial identity for R, implying $2\lambda b+b^2+q\in C$. Thus conclusion (1) is obtained.

Case 3. We denote $u=a^2+p$ and $v=b^2+q$. Finally, suppose that $b\notin C$, $b^2+q\notin C$ and there exists $y_0\in I$ such that $\{uy_0,y_0\}$ is linearly C-independent as well as $\{ay_0,y_0\}$ is linearly C-independent. Since R does not satisfy any nontrivial generalized polynomial identity, by (2.1) we have both the cases:

$\{b,v,1\}$ is linearly C dependent, so that there exist $\beta_1,\beta_2\in C$ such that $b=\beta_1v+\beta_2$. Moreover $\beta_1\neq 0$, since $b\notin C$;

$\{uy_0,ay_0,y_0\}$ is linearly C-dependent, so that there exist $\alpha_1,\alpha_2\in C$ such that $uy_0=\alpha_1ay_0+\alpha_2y_0$. Moreover $\alpha_1\neq 0$, since $uy_0\neq\alpha_2y_0$.

Hence by (2.1), R satisfies

$$(\alpha_1a+\alpha_2)f(y_0x_1,\ldots,y_0x_n)^2+2af(y_0x_1,\ldots,y_0x_n)(\beta_1v+\beta_2)f(y_0x_1,\ldots,y_0x_n)+$$

$$+f(y_0x_1,\ldots,y_0x_n)vf(y_0x_1,\ldots,y_0x_n)-$$

$$-f(y_0x_1,\ldots,y_0x_n)(\alpha_1a+\alpha_2)f(y_0x_1,\ldots,y_0x_n)-$$

$$-2f(y_0x_1,\ldots,y_0x_n)af(y_0x_1,\ldots,y_0x_n)(\beta_1v+\beta_2)-f(y_0x_1,\ldots,y_0x_n)^2v$$

which implies that $\{\beta_1v+\beta_2,v,1\}$ is linearly C-dependent. Since we assume $v\notin C$, it follows $\beta_1v+\beta_2=0$ and so $\beta_1=0$, a contradiction.

Lemma 2.1 is proved.

An easy consequence of the previous result is the following lemma.

Lemma 2.2. *Let R be a prime ring and $F(x) = ax + xb$, $G(x) = px + xq$, for $a, b, p, q \in U$ be two inner generalized derivations of R such that*

$$\big[(F^2 + G)\big(f(r_1, \ldots, r_n)\big), f(r_1, \ldots, r_n)\big] = 0$$

for all $r_1, \ldots, r_n \in R$. Then R satisfies a nontrivial generalized polynomial identity, unless one of the following holds:

(1) $a^2 + p \in C$, $a \in C$ *and* $b^2 + q + 2ab \in C$;

(2) $b, q \in C$ *and* $a^2 + p + 2ab \in C$.

Fact 2.1 (Theorem 1 in [6]). *Let R be a prime ring, $a, b \in R$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. If $\big[af(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)b, f(r_1, \ldots, r_n)\big] = 0$ for all $r_1, \ldots, r_n \in R$, then one of the following conclusions holds*:

(1) $a, b \in Z(R)$;

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on R and $a + b \in C$*;

(3) $\mathrm{char}(R) = 2$ *and R satisfies the standard identity s_4.*

Lemma 2.3. *Let R be a prime ring with $\mathrm{char}\,(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. Assume that $F(x) = xb$ and $G(x) = px + xq$, for $a, b, p, q \in U$. If $\big[(F^2 + G)\big(f(r_1, \ldots, r_n)\big), f(r_1, \ldots, r_n)\big] = 0$ for all $r_1, \ldots, r_n \in R$, then one of the following holds*:

(1) $p \in C$ *and* $b^2 + q \in C$;

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on R and $b^2 + q - p \in C$.*

Proof. In this case we have that R satisfies the generalized identity

$$\big[pf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)(b^2 + q), f(x_1, \ldots, x_n)\big].$$

Hence the required result follows from Fact 2.1.

Lemma 2.4. *Let R be a prime ring with $\mathrm{char}\,(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. Assume that $F(x) = ax$ and $G(x) = px + xq$, for $a, b, p, q \in U$. If $\big[(F^2 + G)\big(f(r_1, \ldots, r_n)\big), f(r_1, \ldots, r_n)\big] = 0$ for all $r_1, \ldots, r_n \in R$, then one of the following holds*:

(1) $q \in C$ *and* $a^2 + p \in C$;

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on R and $a^2 + p - q \in C$.*

Proof. In this case R satisfies the generalized identity

$$\big[(a^2 + p)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)q, f(x_1, \ldots, x_n)\big]$$

and as above we get the required conclusion by applying again Fact 2.1.

Lemma 2.5 (Lemma 1 in [7]). *Let C be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$, then there exists some invertible matrix $P \in M_m(C)$ such that any matrix $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all non-zero entries.*

Proposition 2.1. *Let $R = M_m(C)$ be the ring of all $(m \times m)$-matrices over the infinite field C and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. Assume that $F(x) = ax + xb$ and $G(x) = px + xq$, for $a, b, p, q \in R$. If $\big[(F^2 + G)\big(f(r_1, \ldots, r_n)\big), f(r_1, \ldots, r_n)\big] = 0$ for all $r_1, \ldots, r_n \in R$, then one of the following holds*:

(1) $a, p \in C \cdot I_m$ *and* $(a+b)^2 + q \in C \cdot I_m$;

(2) $b, q \in C \cdot I_m$ *and* $(a+b)^2 + p \in C \cdot I_m$;

(3) $f(x_1, \ldots, x_n)^2$ *is central valued on* R, $a \in C \cdot I_m$ *and* $(a+b)^2 + q - p \in C \cdot I_m$;

(4) $f(x_1, \ldots, x_n)^2$ *is central valued on* R, $b \in C \cdot I_m$ *and* $(a+b)^2 + p - q \in C \cdot I_m$.

Proof. By our assumption R satisfies the generalized identity

$$\big[(a^2+p)f(x_1,\ldots,x_n) + 2af(x_1,\ldots,x_n)b + f(x_1,\ldots,x_n)(b^2+q), f(x_1,\ldots,x_n)\big]. \tag{2.2}$$

If either $a \in Z(R)$ or $b \in Z(R)$, then the conclusions follow by Lemmas 2.3 and 2.4 respectively. Therefore we assume that $a \notin Z(R)$ and $b \notin Z(R)$. Now we shall show that this case leads a contradiction.

Since $a \notin Z(R)$ and $b \notin Z(R)$, by Lemma 2.5 there exists an C-automorphism φ of $M_m(C)$ such that $a' = \varphi(a)$, $b' = \varphi(b)$ have all non-zero entries. Clearly a', b', $p' = \varphi(p)$ and $q' = \varphi(q)$ must satisfy the condition (2.2). Without loss of generality we may replace a, b, p, q with a', b', p', q' respectively.

Here e_{kl} denotes the usual matrix unit with 1 in (k,l)-entry and zero elsewhere. Since $f(x_1,\ldots,x_n)$ is not central, by [20] (see also [22]), there exist $u_1,\ldots,u_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(u_1,\ldots,u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1,\ldots,r_n)\colon r_1,\ldots,r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1,\ldots,r_n \in M_m(C)$ such that $f(r_1,\ldots,r_n) = e_{ij}$. Hence by (2.2) we have

$$\big[(a^2+p)e_{ij} + 2ae_{ij}b + e_{ij}(b^2+q), e_{ij}\big] = 0$$

and then right multiplying by e_{ij}, it follows $2e_{ij}ae_{ij}be_{ij} = 0$, which is a contradiction, since a and b have all non-zero entries.

Proposition 2.1 is proved.

Proposition 2.2. *Let* $R = M_m(C)$ *be the ring of all matrices over the field* C *with* char $(R) \neq 2$ *and* $f(x_1,\ldots,x_n)$ *a noncentral multilinear polynomial over* C. *Assume that* $F(x) = ax + xb$ *and* $G(x) = px + xq$ *for* $a,b,p,q \in R$. *If* $\big[(F^2+G)\big(f(r_1,\ldots,r_n)\big), f(r_1,\ldots,r_n)\big] = 0$, *for all* $r_1,\ldots,r_n \in R$, *then one of the following holds*:

(1) $a, p \in C \cdot I_m$ *and* $(a+b)^2 + q \in C \cdot I_m$;

(2) $b, q \in C \cdot I_m$ *and* $(a+b)^2 + p \in C \cdot I_m$;

(3) $f(x_1, \ldots, x_n)^2$ *is central valued on* R, $a \in C \cdot I_m$ *and* $(a+b)^2 + q - p \in C \cdot I_m$;

(4) $f(x_1, \ldots, x_n)^2$ *is central valued on* R, $b \in C \cdot I_m$ *and* $(a+b)^2 + p - q \in C \cdot I_m$.

Proof. If one assumes that C is infinite, then the conclusions follow by Proposition 2.1.

Now let C be finite and K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1,\ldots,x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$P(x_1,\ldots,x_n) =$$

$$= \big[(a^2+p)f(x_1,\ldots,x_n) + 2af(x_1,\ldots,x_n)b + f(x_1,\ldots,x_n)(b^2+q), f(x_1,\ldots,x_n)\big]$$

which is a generalized polynomial identity for R.

Moreover, it is a multihomogeneous of multidegree $(2,\dots,2)$ in the indeterminates $x_1,\dots,x_n$.

Hence the complete linearization of $P(x_1,\dots,x_n)$ is a multilinear generalized polynomial $\Theta(x_1,\dots,x_n,y_1,\dots,y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1,\dots,x_n,x_1,\dots,x_n)=2^nP(x_1,\dots,x_n).$$

Clearly the multilinear polynomial $\Theta(x_1,\dots,x_n,y_1,\dots,y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\mathrm{char}(C)\neq 2$ we obtain $P(r_1,\dots,r_n)=0$ for all $r_1,\dots,r_n\in\overline{R}$ and then conclusion follows from Proposition 2.1.

Proposition 2.2 is proved.

Fact 2.2 (Reduced version of Theorem 3.1 in [8])**.** *Let R be a prime ring*, *F and G two generalized derivations of R such that $[(F^2+G)(x),x]=0$ for all $x\in R$. Then one of the following holds*:

(1) *R is commutative*;

(2) *there exist $a,b\in U$ such that $F(x)=ax$ and $G(x)=bx$ for all $x\in R$, with $a^2+b\in C$*;

(3) *there exist $a,b\in U$ such that $F(x)=xa$ and $G(x)=xb$ for all $x\in R$, with $a^2+b\in C$.*

Proposition 2.3. *Let R be a prime ring of characteristic different from 2 and $f(x_1,\dots,x_n)$ a noncentral multilinear polynomial over C. Assume that $F(x)=ax+xb$ and $G(x)=px+xq$ for $a,b,p,q\in U$. If $[(F^2+G)(f(r_1,\dots,r_n)),f(r_1,\dots,r_n)]=0$ for all $r_1,\dots,r_n\in R$, then one of the following holds*:

(1) *$a,p\in C$ and $(a+b)^2+q\in C$*;

(2) *$b,q\in C$ and $(a+b)^2+p\in C$*;

(3) *$f(x_1,\dots,x_n)^2$ is central valued on R, $a\in C$ and $(a+b)^2+q-p\in C$*;

(4) *$f(x_1,\dots,x_n)^2$ is central valued on R, $b\in C$ and $(a+b)^2+p-q\in C$.*

Proof. By Lemma 2.2, we may assume that R satisfies the nontrivial generalized polynomial identity

$$P(x_1,\dots,x_n)=$$

$$=\big[(a^2+p)f(x_1,\dots,x_n)+2af(x_1,\dots,x_n)b+f(x_1,\dots,x_n)(b^2+q),f(x_1,\dots,x_n)\big].$$

By a theorem due to Beidar (Theorem 2 in [3]) this generalized polynomial identity is also satisfied by U. In case C is infinite, we have $P(r_1,\dots,r_n)=0$ for all $r_1,\dots,r_n\in U\bigotimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U\bigotimes_C\overline{C}$ are centrally closed [10] (Theorems 2.5 and 3.5), we may replace R by U or $U\bigotimes_C\overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [23], R is a primitive ring having a non-zero socle H with C as the associated division ring and eHe is a simple central algebra finite dimensional over C, for any minimal idempotent element $e\in H$.

In light of Jacobson's theorem [12, p. 75], R is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies that $R\cong M_k(C)$, the ring of all $(k\times k)$-matrices over C. Since R is not commutative, we may assume $k\geq 2$. In this case the conclusion follows by Proposition 2.2.

Assume next that V is infinite-dimensional over C. As in Lemma 2 in [25], the set $f(R)$ is dense on R and so from $P(r_1,\ldots,r_n)=0$ for all $r_1,\ldots,r_n\in R$, we have that R satisfies the generalized identity

$$\big[(a^2+p)x+2axb+x(b^2+q),x\big]=0$$

that is

$$\big[(a^2+p)r+2arb+r(b^2+q),r\big]=0$$

for all $r\in R$. In this case, by Fact 2.2, it follows that either $b\in C$, $q\in C$ and $(a+b)^2+p\in C$; or $a\in C$, $p\in C$ and $(a+b)^2+q\in C$.

Proposition 2.3 is proved.

Now we extend the previous results to the general case: At first we need to recall the following notation: if $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C, then we write

$$f(x_1,\ldots,x_n)=x_1x_2\ldots x_n+\sum_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(n)}$$

for some $\alpha_\sigma\in C$. Moreover, if d is a derivation of R, we denote by $f^d(x_1,\ldots,x_n)$ the polynomial obtained from $f(x_1,\ldots,x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write $d\big(f(r_1,\ldots,r_n)\big)=f^d(r_1,\ldots,r_n)+\sum_i f\big(r_1,\ldots,d(r_i),\ldots,r_n\big)$, for all $r_1,r_2,\ldots,r_n$ in R. We also permit the following:

***Remark* 2.1** (Theorem 3 in [17]). Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x)=ax+d(x)$, for some $a\in U$ and a derivation d on U.

Fact 2.3. *Let R be a prime K-algebra of characteristic different from 2 and $f(x_1,\ldots,x_n)$ a multilinear polynomial over K. If for any $i=1,\ldots,n$,*

$$\big[f(r_1,\ldots,z_i,\ldots,r_n),f(r_1,\ldots,r_n)\big]\in Z(R)$$

for all $z_i,r_1,\ldots,r_n\in R$, then the polynomial $f(x_1,\ldots,x_n)$ is central-valued on R.

Proof. Let $s\in R$. Then by assumption

$$\big[s,f(r_1,\ldots,r_n)\big]_2=$$

$$=\left[\sum_i f(r_1,\ldots,[s,r_i],\ldots,r_n),f(r_1,\ldots,r_n)\right]\in Z(R).$$

Hence, $[s,f(r_1,\ldots,r_n)]_3=[[s,f(r_1,\ldots,r_n)]_2,f(r_1,\ldots,r_n)]=0$ and the result follows by [15] (Theorem).

Fact 2.3 is proved.

As a reduction of the result in [6] we get:

Fact 2.4. *Let R be a prime ring of characteristic different from 2, G a non-zero generalized derivation of R and $f(x_1,\ldots,x_n)$ a multilinear polynomial over C. If*

$$\big[G\big(f(x_1,\ldots,x_n)\big),f(x_1,\ldots,x_n)\big]=0$$

for all $x_1,\ldots,x_n\in R$, then either there exists $\alpha\in C$ such that $G(x)=\alpha x$ or one of the following holds:

(1) *$f(x_1,\ldots,x_n)$ is central valued in R;*

(2) *$G(x) = cx + xb$ with $c - b \in C$, and $f(x_1,\ldots,x_n)^2$ is central valued in R.*

In all that follows we denote by d and δ the derivations of U such that $F(x) = ax + d(x)$ and $G(x) = cx + \delta(x)$, for some $a, c \in U$ and for all $x \in R$. We would like to permit the following remark.

***Remark* 2.1.** If $F = 0$ in Theorem 1.1, then either $f(x_1,\ldots,x_n)$ is central valued in R or the particular cases of conclusions $1, 2, 4$ in Theorem 1.1 are obtained.

In this case $F^2 + G = G$ and then by Fact 2.4 either $f(x_1,\ldots,x_n)$ is central valued in R or one of the following holds:

(1) there exists $\alpha \in C$ such that $G(x) = \alpha x$ for all $x \in R$. Hence, by easy calculations $c = \alpha$ and $\delta = 0$ (particular case of conclusions 1 and 2);

(2) there exist $p, q \in U$ such that $G(x) = px + xq$ for all $x \in R$ with $p - q \in C$. Moreover, $f(x_1,\ldots,x_n)^2$ is central valued in R (particular case of conclusion 4).

***Remark* 2.2.** If $d = 0$ in Theorem 1.1, then either $f(x_1,\ldots,x_n)$ is central valued in R or we obtain particular cases of conclusions $2, 4$ in Theorem 1.1.

In this case $F(x) = ax$ and $(F^2 + G)(x) = (a^2 + c)x + \delta(x)$ for all $x \in R$. Therefore $F^2 + G$ is a generalized derivation of R and again by Fact 2.4 either $f(x_1,\ldots,x_n)$ is central valued in R or one of the following holds:

(1) there exists $\alpha \in C$ such that $(F^2 + G)(x) = \alpha x$ for all $x \in R$. By calculations, it follows $a^2 + c = \alpha$ and $\delta = 0$ (particular case of conclusion 2);

(2) there exist $p, q \in U$ such that $(F^2 + G)(x) = px + xq$ for all $x \in R$ with $p - q = \gamma \in C$. Moreover, $f(x_1,\ldots,x_n)^2$ is central valued in R. By calculations, it follows $G(x) = (q - a^2)x + x(q + \gamma)$ (particular case of conclusion 4).

***Proof of Theorem* 1.1.** We denote by d and δ the derivations of U such that $F(x) = ax + d(x)$ and $G(x) = cx + \delta(x)$, for some $a, c \in U$ and for all $x \in R$. In light of Remarks 2.1 and 2.2, we may assume in all follows that $F \neq 0$ and $d \neq 0$.

Let $f^d(x_1,\ldots,x_n), f^{d\delta}(x_1,\ldots,x_n)$ be the polynomials obtained from $f(x_1,\ldots,x_n)$ replacing each coefficient α_σ with $d(\alpha_\sigma)$ and $\delta(d(\alpha_\sigma))$ respectively. Thus we have

$$d\big(f(r_1,\ldots,r_n)\big) = f^d(r_1,\ldots,r_n) + \sum_i f\big(r_1,\ldots,d(r_i),\ldots,r_n\big)$$

and similarly for $\delta(f(r_1,\ldots,r_n))$. Moreover,

$$d^2\big(f(x_1,\ldots,x_n)\big) =$$

$$= f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,d(x_i),\ldots,x_n) +$$

$$+ \sum_i f(x_1,\ldots,d^2(x_i),\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,d(x_i),\ldots,d(x_j),\ldots,x_n).$$

By Remark 2.1, we have that R satisfies the following:

$$\Big[a^2 f(x_1,\ldots,x_n) + 2ad(f(x_1,\ldots,x_n)) + d(a)f(x_1,\ldots,x_n) +$$

$$+ d^2(f(x_1,\ldots,x_n)) + cf(x_1,\ldots,x_n) + \delta(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)\Big]$$

that is

$$\Bigg[a^2 f(x_1,\ldots,x_n) + 2af^d(x_1,\ldots,x_n) +$$

$$+2a\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n) + d(a)f(x_1,\ldots,x_n)+$$

$$+f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,d(x_i),\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,d^2(x_i),\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,d(x_i),\ldots,d(x_j),\ldots,x_n)+$$

$$+cf(x_1,\ldots,x_n) + f^{\delta}(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,\delta(x_i),\ldots,x_n), f(x_1,\ldots,x_n)\Bigg]. \qquad (2.3)$$

Suppose first that both d and δ are inner derivations of R, that is, there exist $b, q \in U$ such that $d(x) = [b, x]$ and $\delta(x) = [q, x]$ for all $x \in R$. In this case $F(x) = (a + b)x + x(-b)$ and $G(x) = (c + q)x + x(-q)$ for all $x \in R$. Then by Proposition 2.3, one of the following holds:

(1) $a + b, c + q \in C$, $a^2 + c \in C$ and $F(x) = xa$, $G(x) = xc$;

(2) $b, q \in C$, $a^2 + c \in C$ and $F(x) = ax$, $G(x) = cx$;

(3) $f(x_1,\ldots,x_n)^2$ is central valued on R, $a + b \in C$ and $F(x) = xa$ with $a^2 - 2q - c \in C$;

(4) $f(x_1,\ldots,x_n)^2$ is central valued on R, $b \in C$ and $F(x) = ax$ with $a^2 + 2q + c \in C$;

unless $f(x_1,\ldots,x_n)$ is central valued on R, as required.

To complete the proof, in all that follows we consider the case when at least one of either F or G is not an inner generalized derivation of R, that is, δ and d are not simultaneously inner derivations of R. We prove that if $f(x_1,\ldots,x_n)$ is not central valued on R, then this assumption leads to a number of contradictions.

Suppose first that δ and d are linearly C-independent modulo D_{int} (the set of inner derivations in U).

In case $\delta = 0$, by [13], (2.3) gives that R satisfies

$$\Big[a^2 f(x_1,\ldots,x_n) + 2af^d(x_1,\ldots,x_n)+$$

$$+2a\sum_i f(x_1,\ldots,y_i,\ldots,x_n) + d(a)f(x_1,\ldots,x_n)+$$

$$+f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,z_i,\ldots,x_n)+\sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n)+$$

$$+cf(x_1,\ldots,x_n),f(x_1,\ldots,x_n)\Big].$$

On the other hand, if $\delta\neq 0$, again by [13], (2.3) gives that R satisfies

$$\Big[a^2f(x_1,\ldots,x_n)+2af^d(x_1,\ldots,x_n)+$$

$$+2a\sum_i f(x_1,\ldots,y_i,\ldots,x_n)+d(a)f(x_1,\ldots,x_n)+$$

$$+f^{d^2}(x_1,\ldots,x_n)+2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,z_i,\ldots,x_n)+\sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n)+$$

$$+cf(x_1,\ldots,x_n)+f^\delta(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,t_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big].$$

Notice that in both cases R satisfies the blended component

$$\Big[f(x_1,\ldots,z_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]$$

for all $i=1,\ldots,n$. In light of Fact 2.3, this leads to the contradiction that $f(x_1,\ldots,x_n)$ is central valued on R.

Suppose next that δ and d are linearly C-dependent modulo D_{int}, that is, there exist $\alpha,\beta\in C$ and $q\in U$ such that $\alpha d+\beta\delta=ad(q)$, the inner derivation induced by q $\big($that is $ad(q)=[q,x]$ for all $x\in R\big)$. We divide this case into 3 subcases:

Case 1: $\alpha=0$. In this case $\delta(x)=[p,x]$ for all $x\in R$, with $p=\beta^{-1}q$. Moreover, d is not an inner derivation.

Since $d\neq 0$, by [13], (2.3) gives that R satisfies

$$\Big[a^2f(x_1,\ldots,x_n)+2af^d(x_1,\ldots,x_n)+$$

$$+2a\sum_i f(x_1,\ldots,y_i,\ldots,x_n)+d(a)f(x_1,\ldots,x_n)+$$

$$+f^{d^2}(x_1,\ldots,x_n)+2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,z_i,\ldots,x_n)+\sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n)+$$

$$+\, cf(x_1,\ldots,x_n)+[p,f(x_1,\ldots,x_n)],f(x_1,\ldots,x_n)\Big].$$

In particular R satisfies the component

$$\Big[f(x_1,\ldots,z_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]$$

for all $i=1,\ldots,n$ and then as above we get a contradiction.

Case 2: $\beta=0$. In this case $d=[p,x]$ for all $x\in R$, with $p=\alpha^{-1}q\notin C$. Moreover δ is not an inner derivation. Notice that in case $\delta=0$ then both F and G are inner generalized derivations of R, a contradiction. Thus $\delta\neq 0$. Then by [13], (2.3) gives that R satisfies

$$\Bigg[a^2f(x_1,\ldots,x_n)+2a[p,f(x_1,\ldots,x_n)]+d(a)f(x_1,\ldots,x_n)+$$

$$+\big[p,[p,f(x_1,\ldots,x_n)]\big]+$$

$$+cf(x_1,\ldots,x_n)+f^{\delta}(x_1,\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,y_i,\ldots,x_n),f(x_1,\ldots,x_n)\Bigg].$$

In particular R satisfies

$$\Big[f(x_1,\ldots,y_i,\ldots,x_n),f(x_1,\ldots,x_n)\Big]$$

for all $i=1,\ldots,n$, again leading a contradiction.

Case 3: $\alpha\neq 0$ and $\beta\neq 0$. In this case $\delta=\gamma d+ad(p)$, where $\gamma=-\alpha\beta^{-1}\neq 0$ and $ad(p)$ is the inner derivation induced by the element $p=\beta^{-1}q$, moreover d is not an inner derivation of R.

Also here we notice that, in case $\delta=0$ then both F and G are inner generalized derivations of R, a contradiction. Thus $\delta\neq 0$ and by equation (2.3), we have that R satisfies

$$\Bigg[a^2f(x_1,\ldots,x_n)+2af^d(x_1,\ldots,x_n)+$$

$$+2a\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)+d(a)f(x_1,\ldots,x_n)+$$

$$+f^{d^2}(x_1,\ldots,x_n)+2\sum_i f^d(x_1,\ldots,d(x_i),\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,d^2(x_i),\ldots,x_n)+\sum_{i\neq j} f(x_1,\ldots,d(x_i),\ldots,d(x_j),\ldots,x_n)+$$

$$+cf(x_1,\ldots,x_n)+\gamma f^d(x_1,\ldots,x_n)+$$

$$+\gamma\sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)+[p,f(x_1,\ldots,x_n)],f(x_1,\ldots,x_n)\Bigg].$$

Since d is not an inner derivation, by [13], from above relation, R satisfies

$$\Big[a^2 f(x_1,\ldots,x_n) + 2af^d(x_1,\ldots,x_n)+$$

$$+2a\sum_i f(x_1,\ldots,u_i,\ldots,x_n) + d(a)f(x_1,\ldots,x_n)+$$

$$+f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n)+$$

$$+\sum_i f(x_1,\ldots,z_i,\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n)+$$

$$+cf(x_1,\ldots,x_n) + \gamma f^d(x_1,\ldots,x_n)+$$

$$+\gamma\sum_i f(x_1,\ldots,y_i,\ldots,x_n) + \big[p, f(x_1,\ldots,x_n)\big], f(x_1,\ldots,x_n)\Big].$$

In particular, for all $i = 1,\ldots,n$, R satisfies

$$\Big[f(x_1,\ldots,z_i,\ldots,x_n), f(x_1,\ldots,x_n)\Big].$$

Then again by Fact 2.3, we have a contradiction.

3. The case: pair of derivations on multilinear polynomials in right ideals. We would like to point out the following reduced version of Theorem 1.1.

Theorem 3.1. *Let R be a prime ring of characteristic different from 2, U its right Utumi quotient ring, C its extended centroid, F and G two derivations of R, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C. If $\big[(F^2+G)\big(f(r_1,\ldots,r_n)\big), f(r_1,\ldots,r_n)\big] = 0$, for all $r_1,\ldots,r_n \in R$, then either $F = G = 0$ or $f(x_1,\ldots,x_n)$ is central valued on R.*

To prove Theorem 1.2, we begin with the following remark.

Remark **3.1.** Let R be a prime ring, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C and I a nonzero right ideal of R. By [18], following statements hold:

(1) if $f(x_1,\ldots,x_n)x_{n+1}$ is an identity for I, then there exists an idempotent element $e \in$ $\in \operatorname{soc}(RC)$ such that $IC = eRC$ and $f(x_1,\ldots,x_n)$ is an identity for $eRCe$, so that a fortiori $f(x_1,\ldots,x_n)$ is central valued in $eRCe$;

(2) if $\big[f(x_1,\ldots,x_n), x_{n+1}\big]x_{n+2}$ is an identity for I, then there exists $e^2 = e \in \operatorname{soc}(RC)$ such that $IC = eRC$ and $f(x_1,\ldots,x_n)$ is central valued in $eRCe$.

In light of Lemma 2.1, we have the following lemma.

Lemma 3.1. *Let R be a prime ring, $F(x) = cx - xc$ and $G(x) = px - xp$ for $c, p \in U$ be two inner derivations of R. Let I be a right ideal of R such that $\big[(F^2+G)\big(f(r_1,\ldots,r_n)\big), f(r_1,\ldots,r_n)\big] =$ $= 0$ for all $r_1,\ldots,r_n \in I$. Then either R satisfies a nontrivial generalized polynomial identity or there exist $\alpha, \beta \in C$ such that $(c-\alpha)I = (0)$, $(p-\beta)I = (0)$ and $(c-\alpha)^2 = (p-\beta)$.*

***Remark* 3.2.** We prefer to write the polynomial $f(x_1,\ldots,x_n)$ as follows:

$$f(x_1,\ldots,x_n)=\sum_i g_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i,$$

where g_i is a multilinear polynomial such that x_i never appears in any monomials of g_i. Note that if there exists an idempotent $e\in H=\mathrm{Soc}(RC)$ such that all g_is are the polynomial identities for eHe, then we get the conclusion that $f(x_1,\ldots,x_n)$ is a polynomial identity for eHe. Thus if $f(x_1,\ldots,x_n)$ is not a polynomial identity for eHe, there exists an index i and $r_1,\ldots,r_{n-1}\in eHe$ such that $g_i(r_1,\ldots,r_{n-1})\neq 0$. Without loss of generality we assume $i=n$, say $g_n(x_1,\ldots,x_{n-1})=$ $=t(x_1,\ldots,x_{n-1})$ and so $f(x_1,\ldots,x_n)=t(x_1,\ldots,x_{n-1})x_n+h(x_1,\ldots,x_n)$ where $t(eHe)\neq 0$ and $h(x_1,\ldots,x_n)$ is a multilinear polynomial such that x_n never appears as last variable in any monomials of h.

Lemma 3.2. *Let R be a prime ring of characteristic different from 2, U its right Utumi quotient ring, C its extended centroid, F and G two inner derivations of R induced by the elements $a,b\in U$ respectively, that is $F(x)=[a,x]$ and $G(x)=[b,x]$ for all $x\in R$. If $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C, I a non-zero right ideal of R such that $\big[(F^2+$ $+G)\big(f(r_1,\ldots,r_n)\big),f(r_1,\ldots,r_n)\big]=0$ for all $r_1,\ldots,r_n\in I$, then one of the following holds*:

(1) *there exists $e^2=e\in soc(RC)$ such that $IC=eRC$ and $f(x_1,\ldots,x_n)$ is central valued on $eRCe$*;

(2) *there exist $\alpha,\beta\in C$ such that $(a-\alpha)I=(b-\beta)I=(0)$ and $(a-\alpha)^2=(b-\beta)$.*

Proof. By Lemma 3.1, we may assume that R satisfies a nontrivial generalized polynomial identity, otherwise we get our conclusion (2). In this case by [23], RC is a primitive ring having a non-zero socle H with a non-zero right ideal $J=IH$. Note that H is simple, $J=HJ$ and J satisfies the same basic conditions as I. Thus without loss of generality we may replace R by H and I by J.

Since $R=H$ is a regular ring, then for any $a_1,\ldots,a_n\in I$ there exists $h=h^2\in R$ such that $\sum_{i=1}^n a_iR=hR$. Then $h\in IR=I$ and $a_i=ha_i$ for each $i=1,\ldots,n$.

By our assumption, I satisfies the following generalized identity with coefficients in U :

$$\big[(a^2+b)f(x_1,\ldots,x_n)-2af(x_1,\ldots,x_n)a+$$
$$+f(x_1,\ldots,x_n)(a^2-b),f(x_1,\ldots,x_n)\big]. \tag{3.1}$$

First we study the situation when there exists $\alpha\in C$ such that $(a-\alpha)I=(0)$. Notice that a and $c=a-\alpha$ induce the same derivation F. Thus we replace a by c and assume $cI=(0)$.

By calculations, (3.1) reduces to

$$\Big[bf(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)(c^2-b),f(x_1,\ldots,x_n)\Big]. \tag{3.2}$$

By Theorem 3 in [6], we have from (3.2) that either there exists $e=e^2\in\mathrm{Soc}(RC)$ such that $IC=eRC$ and $f(x_1,\ldots,x_n)$ is central valued in $eRCe$ or one of the following holds:

1. There exists $\beta\in C$ such that $(b-\beta)I=(0)$ and $c^2-b\in C$. Applying this to (3.1), it follows that I satisfies

$$f(x_1,\ldots,x_n)^2\big((a-\alpha)^2-(b-\beta)\big)=0$$

and by the main result in [5], we get the required conclusion $(a-\alpha)^2=(b-\beta)$, unless when there exists $e=e^2\in\mathrm{Soc}(RC)$ such that $IC=eRC$ and $f(x_1,\ldots,x_n)x_{n+1}$ is an identity for eRC.

2. There exists $\beta \in C$ such that $(c^2 - 2b - \beta)I = (0)$, that is $(b + \gamma)I = (0)$, with $\gamma = \dfrac{\beta}{2}$; moreover there exists $e = e^2 \in \operatorname{Soc}(RC)$ such that $IC = eRC$ and $f(x_1, \ldots, x_n)^2$ is central valued in $eRCe$. Since $eRCe$ satisfies (3.1), also in this case, by calculations we have that $eRCe$ satisfies

$$f(x_1, \ldots, x_n)^2 \big(c^2 - (b + \gamma)\big) = 0,$$

that is $(a - \alpha)^2 = (b + \gamma)$. In any case we obtain one of the required conclusions.

Therefore, in what follows we may assume that there exist $c, c_1, \ldots, c_{n+2} \in I$ such that $ac \neq \alpha c$ for all $\alpha \in C$ and $[f(c_1, \ldots, c_n), c_{n+1}]c_{n+2} \neq 0$.

By the above argument, there exists an idempotent element $e \in IH = IR$ such that $eR = \sum_{i=1}^{n+2} c_i R + cR + aR + bR$ and $c_i = ec_i$ (for any $i = 1, \ldots, n+2$), $c = ec$, $a = ea$, $b = eb$. Notice that

$$\Big[(a^2 + b) f(ex_1, \ldots, ex_n) - 2a f(ex_1, \ldots, ex_n) a +$$

$$+ f(ex_1, \ldots, ex_n)(a^2 - b), f(ex_1, \ldots, ex_n)\Big] \tag{3.3}$$

is satisfied by $R = H$. Now we write the polynomial $f(x_1, \ldots, x_n)$ as in Remark 3.2, and replace x_n by $x_n(1 - e)$. Hence

$$f\big(ex_1, \ldots, ex_{n-1}, ex_n(1 - e)\big) = t(ex_1, \ldots, ex_{n-1}) ex_n (1 - e). \tag{3.4}$$

By using (3.4) in (3.3) and right multiplying by e, we have that R satisfies

$$2t(ex_1, \ldots, ex_{n-1}) ex_n (1 - e) a t(ex_1, \ldots, ex_{n-1}) ex_n (1 - e) ae$$

and since $\operatorname{char}(R) \neq 2$ and $t(ex_1, \ldots, ex_n)e \neq 0$, it follows $(1 - e)ae = 0$, that is $ae = eae$ and $a^2 e = aeae = eaeae$.

In light of this, R satisfies

$$\Big[e(a^2 + b)e f(ex_1 e, \ldots, ex_n e) - 2eae f(ex_1 e, \ldots, ex_n e) eae +$$

$$+ f(ex_1 e, \ldots, ex_n e) e(a^2 - b)e, f(ex_1 e, \ldots, ex_n e)\Big]$$

that is $eRCe$ satisfies

$$\Big[\big(e(a^2 + b)e\big) f(x_1, \ldots, x_n) - 2(eae) f(x_1, \ldots, x_n)(eae) +$$

$$+ f(x_1, \ldots, x_n)\big(e(a^2 - b)e\big), f(x_1, \ldots, x_n)\Big].$$

Since $eRCe$ is a is a simple ring, by Theorem 3.1, we have that both $eae \in Z(eRCe)$ and $ebe \in Z(eRCe)$, since $f(x_1, \ldots, x_n)$ is not central valued on $eRCe$. In particular there exists $\alpha \in C$ such that $\alpha e = eae = ae$, that is $\alpha c = \alpha ec = aec = ac$, which is a contradiction.

Lemma 3.2 is proved.

We are finally ready for the proof of Theorem 1.2.

***Proof of Theorem* 1.2.** Of course we may assume that F and G are not simultaneously inner derivations of R, if not we end up by Lemma 3.2. Moreover in case either $F = 0$ or $G = 0$, the conclusion follows respectively from [16] (see also [6]) and [9]. Therefore we always assume that $F \neq 0$ and $G \neq 0$.

By our hypothesis, if $0 \neq c \in I$, R satisfies

$$\Big[F^2(f(cx_1,\ldots,cx_n)) + G(f(cx_1,\ldots,cx_n)), f(cx_1,\ldots,cx_n)\Big],$$

that is R satisfies

$$\Bigg[f^{F^2}(cx_1,\ldots,cx_n) + 2\sum_i f^F(cx_1,\ldots,F(c)x_i + cF(x_i),\ldots,cx_n) +$$

$$+\sum_i f(cx_1,\ldots,F^2(c)x_i + 2F(c)F(x_i) + cF^2(x_i),\ldots,cx_n) +$$

$$+\sum_{i\neq j} f(cx_1,\ldots,F(c)x_i + cF(x_i),\ldots,F(c)x_j + cF(x_j),\ldots,cx_n) +$$

$$+f^G(cx_1,\ldots,cx_n) + \sum_i f(cx_1,\ldots,G(c)x_i + cG(x_i),\ldots,cx_n), f(cx_1,\ldots,cx_n)\Bigg]. \tag{3.5}$$

In all that follows we consider the case when at least one of either F or G is not an inner derivation of R. Moreover, we assume that there exist $c_1,\ldots,c_{n+2} \in I$ such that

$$\big[f(c_1,\ldots,c_n), c_{n+1}\big]c_{n+2} \neq 0,$$

otherwise by Remark 3.1, we obtain conclusion (1).

Suppose first that F and G are linearly C-independent modulo D_{int}.

By [13], we have from (3.5) that R satisfies

$$\Bigg[f^{F^2}(cx_1,\ldots,cx_n) + 2\sum_i f^F(cx_1,\ldots,F(c)x_i + cy_i,\ldots,cx_n) +$$

$$+\sum_i f(cx_1,\ldots,F^2(c)x_i + 2F(c)y_i + cz_i,\ldots,cx_n) +$$

$$+\sum_{i\neq j} f(cx_1,\ldots,F(c)x_i + cy_i,\ldots,F(c)x_j + cy_j,\ldots,cx_n) +$$

$$+f^G(cx_1,\ldots,cx_n) + \sum_i f(cx_1,\ldots,G(c)x_i + ct_i,\ldots,cx_n), f(cx_1,\ldots,cx_n)\Bigg].$$

This implies that R satisfies the blended component

$$\left[\sum_i f(cx_1,\ldots,ct_i,\ldots,cx_n), f(cx_1,\ldots,cx_n)\right].$$

Suppose next that F and G are linearly C-dependent modulo D_{int}, that is, there exist $\alpha,\beta\in C$ and $q\in U$ such that $\alpha F+\beta G=ad(q)$, the inner derivation induced by q. We divide this case into 3 subcases:

Case 1: $\alpha=0$. In this case $G(x)=[p,x]$ for all $x\in R$, with $p=\beta^{-1}q$, moreover F is not an inner derivation. By (3.5), we have that R satisfies

$$\left[f^{F^2}(cx_1,\ldots,cx_n)+2\sum_i f^F(cx_1,\ldots,F(c)x_i+cy_i,\ldots,cx_n)+\right.$$

$$+\sum_i f(cx_1,\ldots,F^2(c)x_i+2F(c)y_i+cz_i,\ldots,cx_n)+$$

$$+\sum_{i\neq j} f(cx_1,\ldots,F(c)x_i+cy_i,\ldots,F(c)x_j+cy_j,\ldots,cx_n)+$$

$$\left.+\,pf(cx_1,\ldots,cx_n)-f(cx_1,\ldots,cx_n)p, f(cx_1,\ldots,cx_n)\right].$$

In particular, R satisfies the component

$$\left[f(cx_1,\ldots,cz_i,\ldots,cx_n), f(cx_1,\ldots,cx_n)\right]$$

for all $i=1,\ldots,n$.

Case 2: $\beta=0$. In this case $F=[p,x]$ for all $x\in R$, with $p=\alpha^{-1}q$, moreover G is not an inner derivation. By (3.5), we have that R satisfies

$$\left[p^2f(cx_1,\ldots,cx_n)-2pf(cx_1,\ldots,cx_n)p+f(cx_1,\ldots,cx_n)p^2+\right.$$

$$\left.+f^G(cx_1,\ldots,cx_n)+\sum_i f(cx_1,\ldots,G(c)x_i+cy_i,\ldots,cx_n), f(cx_1,\ldots,cx_n)\right].$$

In particular

$$\left[f(cx_1,\ldots,cy_i,\ldots,cx_n), f(cx_1,\ldots,cx_n)\right]$$

is satisfied by R, for all $i=1,\ldots,n$.

Case 3: $\alpha\neq 0$ and $\beta\neq 0$. In this case $G=\gamma F+ad(p)$, where $\gamma=-\alpha\beta^{-1}\neq 0$ and $ad(p)$ is the inner derivation induced by the element $p=\beta^{-1}q$, moreover F is not an inner derivation of R.

By equation (3.5), we have that R satisfies

$$\Bigg[f^{F^2}(cx_1,\ldots,cx_n)+2\sum_i f^F(cx_1,\ldots,F(c)x_i+cy_i,\ldots,cx_n)+$$

$$+\sum_i f(cx_1,\ldots,F^2(c)x_i+2F(c)y_i+cz_i,\ldots,cx_n)+$$

$$+\sum_{i\neq j} f(cx_1,\ldots,F(c)x_i+cy_i,\ldots,F(c)x_j+cy_j,\ldots,cx_n)+$$

$$+\gamma f^F(cx_1,\ldots,cx_n)+$$

$$+\gamma\sum_i f(cx_1,\ldots,F(c)x_i+cy_i,\ldots,cx_n)+[p,f(cx_1,\ldots,cx_n)],f(cx_1,\ldots,cx_n)\Bigg].$$

In particular, for all $i=1,\ldots,n$, R satisfies

$$\Big[f(cx_1,\ldots,cz_i,\ldots,cx_n),f(cx_1,\ldots,cx_n)\Big]. \tag{3.6}$$

All the previous argument says that in any case R satisfies (3.6). Thus R satisfies a nontrivial generalized polynomial identity. As remarked in the proof of Lemma 3.2, we may assume $H=R$ and $I=IR$. Moreover, for all $e^2=e\in I$ and by the above argument, R satisfies

$$\Bigg[\sum_i f(ex_1,\ldots,et_i,\ldots,ex_n),f(ex_1,\ldots,ex_n)\Bigg]. \tag{3.7}$$

Since $R=H$ is a regular ring, then there exists $h=h^2\in R$ such that $\sum_{i=1}^{n+2}c_iR=hR$. Then $h\in IR=I$ and $c_i=hc_i$ for each $i=1,\ldots,n+2$. In (3.7) and for all $i=1,\ldots,n$, we replace ht_i by $[hc_{n+1},hx_i]$, so that R satisfies

$$\big[c_{n+1},f(hx_1,\ldots,hx_n)\big]_2.$$

In particular the ring hRh satisfies $\big[c_{n+1},f(x_1,\ldots,x_n)\big]_2$. By [15], it follows $c_{n+1}\in Z(hRh)$, and a fortiori $\big[c_{n+1},f(c_1,\ldots,c_n)\big]c_{n+2}=0$, which is a contradiction.

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