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EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR THIRD-ORDER m-POINT IMPULSIVE BOUNDARY VALUE PROBLEMS ON TIME SCALES

ІСНУВАННЯ ДОДАТНИХ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ m-ТОЧКОВИХ ІМПУЛЬСИВНИХ ГРАНИЧНИХ ЗАДАЧ ТРЕТЬОГО ПОРЯДКУ НА ЧАСОВИХ МАСШТАБАХ

In the paper, the four functionals fixed-point theorem is used to study the existence of positive solutions for nonlinear third-order m-point impulsive boundary-value problems on time scales. As an application, we give an example demonstrating our results.

Теорема про нерухому точку для чотирьох функціоналів застосовано для дослідження задачі існування додатних розв'язків нелінійних m-точкових імпульсивних граничних задач третього порядку на часових масштабах. Як застосування, наведено приклад, який ілюструє результати, що отримані в роботі.

1. Introduction. Impulsive differential equations, which arise in physics, chemical technology, population dynamics, biotechnology, economics and so on (see [3] and references therein), have become more important in recent years in some mathematical models of real processes. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [12], Samoilenko and Perestyuk [19] and the references therein.

The theory of time scales was introduced by Stefan Hilger [10] in his PhD thesis in 1988 in order to unify continuous and discrete analysis. We refer to the books by Bohner and Peterson [5, 6] and Lakshmikantham et al. [13].

Recently, the existence and multiplicity of positive solutions for linear and nonlinear second-order impulsive differential equations have been studied extensively. To identify a few, we refer to the reader to see [8, 9, 11, 16, 20]. However, there is not work on third-order with m-point impulsive boundary-value problems except that in [17] by Liang and Zhang. On the other hand, there is not much reported concerning the boundary-value problems for impulsive dynamic equations on time scales, see [4, 7, 14, 15]. Especially the existence of positive solutions for third-order with m-point impulsive boundary-value problems on time scales still remains unknown.

In [9], Guo studied the following two-point boundary-value problem:

$$-x'' = f(t,x,x'), \quad t \neq t_k,$$

$$\Delta x|_{t=t_k} = I_k\big(x(t_k)\big),$$

$$\Delta x'|_{t=t_k} = \bar{I}_k\big(x(t_k)x'(t_k)\big), \quad k = 1,2,\ldots,m,$$

$$ax(0) - bx'(0) = x_0, \quad cx(1) + dx'(1) = x_0^*.$$

By using the Darbo fixed point theorem, Guo obtained the existence criteria of at least one solution.

In [11], Hu, Liu and Wu studied second-order two-point impulsive boundary-value problem

$$-u'' = h(t)f(t,u), \quad t \in J',$$

$$-\Delta u'|_{t=t_k} = I_k\big(u(t_k)\big),$$
$$\Delta u|_{t=t_k} = \bar{I}_k\big(u(t_k)\big), \quad k = 1, 2, \ldots, m,$$
$$\alpha u(0) - \beta u'(0) = 0$$
$$\gamma u(1) + \delta u'(1) = 0.$$

By using the fixed point theorem in cone, they obtained the existence criteria of one or two positive solutions.

In [18], Ma considered the existence and multiplicity of positive solutions for the m-boundary-value problems

$$\big(p(t)u'\big)' - q(t)u + f(t, u) = 0, \quad 0 < t < 1,$$
$$au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$
$$cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i).$$

The main tool is Guo – Krasnoselskii fixed point theorem.

In [17], Liang and Zhang studied the following third-order impulsive boundary-value problem

$$\big(\varphi(-u''(t))\big)' + a(t)f\big(u(t)\big) = 0, \quad t \neq t_k, \quad 0 < t < 1,$$
$$\Delta u|_{t=t_k} = I_k\big(u(t_k)\big), \quad k = 1, 2, \ldots, N,$$
$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$
$$u'(1) = 0, \quad u''(0) = 0,$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is the increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$. By using the five functionals fixed point theorem, they provided sufficient conditions for the existence of three positive solutions.

In [15], Li and Li studied the following boundary-value problem for the nonlinear third-order impulsive dynamic system on time scales

$$-u^{\triangle^3}(t) = f\big(t, u(t), u^{\triangle}(t), u^{\triangle^2}(t)\big), \quad t \in [0, T]_{\mathbb{T}} \setminus \Omega,$$
$$\Delta u(t_k) = I_k, \quad \Delta u^{\triangle}(t_k) = J_k, \quad \Delta u^{\triangle^2}(t_k) = L_k, \quad k = 1, 2, \ldots, m,$$
$$u(0) = \lambda u(\sigma(T)), \quad u^{\triangle}(0) = \lambda u^{\triangle}(\sigma(T)), \quad u^{\triangle^2}(0) = \lambda u^{\triangle^2}(\sigma(T)).$$

They obtained some sufficient conditions for the existence of solutions by using Schauder's fixed point theorem.

Motivated by the above results, in this study, we consider the following third-order impulsive boundary-value problem (BVP) on time scales:

$$\begin{aligned}
&\Big(\phi_p\Big(u^{\triangle\triangle}(t)\Big)\Big)^{\triangle} + q(t)f(t,u(t),u^{\triangle}(t)) = 0, \qquad t \in J\colon = [0,1]_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1,2,\ldots,n,\\
&\Delta u\,(t_k) = I_k\big(u(t_k)\big),\\
&\Delta u^{\triangle}(t_k) = -J_k\big(u(t_k),u^{\triangle}(t_k)\big),\\
&au(0) - bu^{\triangle}(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i),\\
&cu(1) + du^{\triangle}(1) = \sum_{i=1}^{m-2}\beta_i u(\xi_i),\\
&u^{\triangle\triangle}(0) = 0,
\end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0,1 \in \mathbb{T}$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p>1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $t_k \in (0,1)_{\mathbb{T}}$, $k = 1,2,\ldots,n$ with $0 < t_1 < t_2 < \ldots < t_n < 1$, $\Delta u(t_k)$ and $\Delta u^{\triangle}(t_k)$ denote the jump of $u(t)$ and $u^{\triangle}(t)$ at $t = t_k$, i.e.,

$$\Delta u(t_k) = u(t_k^+) - u(t_k^-), \qquad \Delta u^{\triangle}(t_k) = u^{\triangle}(t_k^+) - u^{\triangle}(t_k^-),$$

where $u(t_k^+)$, $u^{\triangle}(t_k^+)$ and $u(t_k^-)$, $u^{\triangle}(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ and $u^{\triangle}(t)$ at $t = t_k$, $k = 1,2,\ldots,n$, respectively.

Throughout this paper we assume that following conditions hold:

(C_1) $a,b,c,d \in [0,\infty)$ with $ac+ad+bc > 0$; $\alpha_i, \beta_i \in [0,\infty)$, $\xi_i \in (0,1)_{\mathbb{T}}$ for $i \in \{1,2,\ldots,m-2\}$

(C_2) $f \in \mathcal{C}([0,1]_{\mathbb{T}} \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$,

(C_3) $q \in \mathcal{C}([0,1]_{\mathbb{T}}, \mathbb{R}^+)$,

(C_4) $I_k \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ is a bounded function, $J_k \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ such that $(c(1-t_k) + d)J_k\big(u(t_k),u^{\triangle}(t_k)\big) > cI_k\big(u(t_k)\big)$, $k = 1,2,\ldots,n$.

By using the four functional fixed point theorem [1], we get the existence of at least one positive solution for the impulsive BVP (1.1). In fact, our result is also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the result can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. We give and prove our main result in Section 3. Finally, in Section 4, we give an example to demonstrate our result.

2. Preliminaries. In this section, we present auxiliary lemmas which will be used later.

Throughout the rest of this paper, we assume that the points of impulse t_k are right dense for each $k = 1,2,\ldots,n$. Let $J = [0,1]_{\mathbb{T}}$, $J' = J \setminus \{t_1,t_2,\ldots,t_n\}$.

Set

$$PC(J) = \big\{u\colon [0,1]_{\mathbb{T}} \to \mathbb{R}; u \in C(J'), u(t_k^+)\,\text{and}\,u(t_k^-)\,\text{exist, and}\,u(t_k^-) = u(t_k), 1 \le k \le n\big\},$$

$$PC^1(J) = \big\{u \in PC(J)\colon u^{\Delta} \in C(J'), u^{\Delta}(t_k^+)\,\text{and}\,u^{\Delta}(t_k^-)\,\text{exist, and}\,u^{\Delta}(t_k^-) = u^{\Delta}(t_k), 1 \le k \le n\big\}.$$

Obviously, $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_{PC} = \max_{t\in[0,1]_{\mathbb{T}}} |u(t)|, \qquad \|u\|_{PC^1} = \max\left\{\|u\|_{PC}, \|u^{\Delta}\|_{PC}\right\},$$

respectively. A function $u \in PC^1(J) \cap C^2(J')$ is called a solution to (1.1) if it satisfies all equations of (1.1).

Define the cone $\mathcal{P} \subset PC^1(J)$ by

$$\mathcal{P} = \Bigg\{ u \in PC^1(J) \colon u(t) \;\text{ is nonnegative,} \\ \text{nondecreasing on } [0,1]_{\mathbb{T}} \text{ and } u^{\triangle}(t) \text{ is nonincreasing on } [0,1]_{\mathbb{T}}, \\ au(0) - bu^{\triangle}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) \Bigg\}.$$

Denote by θ and φ, the solutions of the corresponding homogeneous equation

$$\left(\phi_p\left(u^{\triangle\triangle}(t)\right)\right)^{\triangle} = 0, \qquad t \in J\colon = [0,1]_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1, 2, \ldots, n, \tag{2.1}$$

under the initial conditions

$$\begin{aligned} &\theta(0) = b, \qquad \theta^{\Delta}(0) = a, \\ &\varphi(1) = d, \qquad \varphi^{\Delta}(1) = -c. \end{aligned} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for θ and φ the following equations:

$$\theta(t) = b + at, \qquad \varphi(t) = d + c(1-t). \tag{2.3}$$

Set

$$\Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1-\xi_i)) \\ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) & -\sum_{i=1}^{m-2} \beta_i (d + c(1-\xi_i)) \end{vmatrix}, \tag{2.4}$$

and

$$\rho := ad + ac + bc. \tag{2.5}$$

Lemma 2.1. *Let* (C_1) – (C_4) *hold. Assume that*

(C_5) $\Delta \neq 0$.

If $u \in PC^1(J) \cap C^2(J')$ is a solution of the equation

$$u(t) = \int_0^1 G(t,s)\,\phi_q\left(\int_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s +$$

$$+\sum_{k=1}^{n} W_k(t,t_k) + A(f)(b+at) + B(f)(d+c(1-t)), \tag{2.6}$$

where

$$W_k(t,t_k) = \frac{1}{\rho}\begin{cases}(b+at)\left(-cI_k(u(t_k)) + (d+c(1-t_k))J_k\left(u(t_k),u^{\triangle}(t_k)\right)\right), & t<t_k,\\ (d+c(1-t))\left(aI_k(u(t_k)) + (b+at_k)J_k\left(u(t_k),u^{\triangle}(t_k)\right)\right), & t_k \le t,\end{cases} \tag{2.7}$$

$$G(t,s) = \frac{1}{\rho}\begin{cases}(b+a\sigma(s))(d+c(1-t)), & \sigma(s)\le t,\\ (b+at)(d+c(1-\sigma(s))), & t\le s,\end{cases} \tag{2.8}$$

$$A(f) := \frac{1}{\Delta}\begin{vmatrix}\sum\limits_{i=1}^{m-2}\alpha_i\mathcal{K}_i & \rho-\sum\limits_{i=1}^{m-2}\alpha_i(d+c(1-\xi_i))\\ \sum\limits_{i=1}^{m-2}\beta_i\mathcal{K}_i & -\sum\limits_{i=1}^{m-2}\beta_i(d+c(1-\xi_i))\end{vmatrix}, \tag{2.9}$$

$$B(f) := \frac{1}{\Delta}\begin{vmatrix}-\sum\limits_{i=1}^{m-2}\alpha_i(b+a\xi_i) & \sum\limits_{i=1}^{m-2}\alpha_i\mathcal{K}_i\\ \rho-\sum\limits_{i=1}^{m-2}\beta_i(b+a\xi_i) & \sum\limits_{i=1}^{m-2}\beta_i\mathcal{K}_i\end{vmatrix}, \tag{2.10}$$

and

$$\mathcal{K}_i := \int_0^1 G(\xi_i,s)\phi_q\left(\int_0^s q(\tau) f\left(\tau,u(\tau),u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s + \sum_{k=1}^{n} W_k(\xi_i,t_k), \tag{2.11}$$

then u is a solution of the impulsive BVP (1.1).

Proof. Let u satisfies the integral equation (2.6), then u is a solution of the impulsive BVP (1.1). Then we have

$$u(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s q(\tau) f\left(\tau,u(\tau),u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s +$$

$$+\sum_{k=1}^{n} W_k(t,t_k) + A(f)(b+at) + B(f)(d+c(1-t)),$$

i.e.,

$$u(t) = \int\limits_0^t \frac{1}{\rho}(b + a\sigma(s))(d + c(1-t))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \int\limits_t^1 \frac{1}{\rho}(b + at)(d + c(1 - \sigma(s)))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \sum_{0<t_k<t} (d + c(1-t)) \left(aI_k(u(t_k)) + (b + at_k) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) +$$

$$+ \sum_{t<t_k<1} (b + at) \left(-cI_k(u(t_k)) + (d + c(1 - t_k)) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) +$$

$$+ A(f)(b + at) + B(f)(d + c(1-t)),$$

$$u^{\triangle}(t) = -\int\limits_0^t \frac{c}{\rho}(b + a\sigma(s))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau, u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \int\limits_t^1 \frac{a}{\rho}(d + c(1 - \sigma(s)))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s -$$

$$- \sum_{0<t_k<t} c \left(aI_k(u(t_k)) + (b + at_k) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) +$$

$$+ \sum_{t<t_k<1} a \left(-cI_k(u(t_k)) + (d + c(1 - t_k)) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) + A(f)a - B(f)c,$$

$$u^{\triangle\triangle}(t) = \frac{1}{\rho} \left(-c\left(b + a\sigma(t)\right) - a(d + c(1 - \sigma(t))) \right) \phi_q \left(\int\limits_0^t q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) -$$

$$-\phi_q \left(\int\limits_0^t q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right),$$

$$u^{\triangle\triangle}(0) = 0.$$

So that

$$\left(\phi_p \left(u^{\triangle\triangle}(t) \right) \right)^{\triangle} = \left(-\int\limits_0^t q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right)^{\triangle} = -q(t) f(t, u(t), u^{\triangle}(t)),$$

$$\left(\phi_p \left(u^{\triangle\triangle}(t) \right) \right)^{\triangle} + q(t) f(t, u(t), u^{\triangle}(t)) = 0.$$

Since

$$u(0) = \int\limits_0^1 \frac{b}{\rho}(d + c(1 - \sigma(s)))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \sum_{k=1}^{n} b \left(-cI_k(u(t_k)) + (d + c(1 - t_k)) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) + A(f)b + B(f)(d + c),$$

$$u^{\triangle}(0) = \int\limits_0^1 \frac{a}{\rho}(d + c(1 - \sigma(s)))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \sum_{k=1}^{n} a \left(-cI_k(u(t_k)) + (d + c(1 - t_k)) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) + A(f)a - B(f)c,$$

we have that

$$au(0) - bu^{\triangle}(0) =$$

$$= B(f)(ad + ac + bc) = \sum_{i=1}^{m-2} \alpha_i \left[\int\limits_0^1 G(\xi_i, s)\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s + \right.$$

$$\left. + \sum_{k=1}^{n} W_k(\xi_i, t_k) + A(f)(b + a\xi_i) + B(f)(d + c(1 - \xi_i)) \right]. \qquad (2.12)$$

Since

$$u(1) = \int\limits_0^1 \frac{d}{\rho}(b + a\sigma(s))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau, u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \sum_{k=1}^{n} d \left(aI_k(u(t_k)) + (b + at_k) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) + A(f)(b + a) + B(f)d,$$

$$u^{\triangle}(1) = -\int\limits_0^1 \frac{c}{\rho}(b + a\sigma(s))\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau, u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s +$$

$$+ \sum_{k=1}^{n} -c \left(aI_k(u(t_k)) + (b + at_k) J_k \left(u(t_k), u^{\triangle}(t_k) \right) \right) + A(f)a - B(f)c,$$

we have that

$$cu(1) + du^{\triangle}(1) = A(f)(ad + ac + bc) =$$

$$= \sum_{i=1}^{m-2} \beta_i \left[\int\limits_0^1 G(\xi_i, s)\phi_q \left(\int\limits_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right) \triangle\tau \right) \triangle s + \right.$$

$$+\sum_{k=1}^{n} W_k(\xi_i, t_k) + A(f)(b + a\xi_i) + B(f)(d + c(1 - \xi_i))\Bigg]. \tag{2.13}$$

From (2.5), (2.12) and (2.13), we get that

$$\left[-\sum_{i=1}^{m-2}\alpha_i(b + a\xi_i)\right] A(f) + \left[\rho - \sum_{i=1}^{m-2}\alpha_i(d + c(1 - \xi_i))\right] B(f) = \sum_{i=1}^{m-2}\alpha_i\mathcal{K}_i,$$

$$\left[\rho - \sum_{i=1}^{m-2}\beta_i(b + a\xi_i)\right] A(f) + \left[-\sum_{i=1}^{m-2}\beta_i(d + c(1 - \xi_i))\right] B(f) = \sum_{i=1}^{m-2}\beta_i\mathcal{K}_i,$$

which implies that $A(f)$ and $B(f)$ satisfy (2.9) and (2.10), respectively.

Lemma 2.1 is proved.

Lemma 2.2. *Let* (C_1)–(C_4) *hold. Assume*

(C_6) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2}\beta_i(b + a\xi_i) > 0$, $a - \sum_{i=1}^{m-2}\alpha_i > 0$.

Then for $u \in PC^1(J) \cap C^2(J')$ *with* f, $q \geq 0$, *the solution* u *of the problem* (1.1) *satisfies*

$$u(t) \geq 0 \quad \textit{for} \quad t \in [0, 1]_{\mathbb{T}}.$$

Proof. It is an immediate subsequence of the facts that $G \geq 0$ on $[0, 1]_{\mathbb{T}} \times [0, 1]_{\mathbb{T}}$ and $A(f) \geq 0$, $B(f) \geq 0$.

Lemma 2.3. *Let* (C_1)–(C_4) *and* (C_6) *hold. Assume*

(C_7) $c - \sum_{i=1}^{m-2}\beta_i < 0$.

Then the solution $u \in PC^1(J) \cap C^2(J')$*of the problem* (1.1) *satisfies* $u^{\triangle}(t) \geq 0$ *for* $t \in [0, 1]_{\mathbb{T}}$.

Proof. Assume that the inequality $u^{\triangle}(t) < 0$ holds. Since $u^{\triangle}(t)$ is nonincreasing on $[0, 1]_{\mathbb{T}}$, one can verify that

$$u^{\triangle}(1) \leq u^{\triangle}(t), \quad t \in [0, 1]_{\mathbb{T}}.$$

From the boundary conditions of the problem (1.1), we have

$$-\frac{c}{d}u(1) + \frac{1}{d}\sum_{i=1}^{m-2}\beta_i u(\xi_i) \leq u^{\triangle}(t) < 0.$$

The last inequality yields

$$-cu(1) + \sum_{i=1}^{m-2}\beta_i u(\xi_i) < 0.$$

Therefore, we obtain that

$$\sum_{i=1}^{m-2}\beta_i u(1) < \sum_{i=1}^{m-2}\beta_i u(\xi_i) < cu(1),$$

i.e.,

$$\left(c - \sum_{i=1}^{m-2} \beta_i\right) u(1) > 0.$$

According to Lemma (2.2), we have that $u(1) \geq 0$. So, $c - \sum_{i=1}^{m-2} \beta_i > 0$. However, this contradicts to condition (C_7). Consequently, $u^{\triangle}(t) \geq 0$ for $t \in [0,1]_{\mathbb{T}}$.

Lemma 2.3 is proved.

Lemma 2.4. *If* $(\mathrm{C}_1)-(\mathrm{C}_7)$ *hold, then* $\max_{t\in[0,1]_{\mathbb{T}}} u(t) \leq M \max_{t\in[0,1]_{\mathbb{T}}} u^{\Delta}(t)$ *for* $u \in \mathcal{P}$, *where*

$$M = 1 + \frac{b + \sum_{i=1}^{m-2} \alpha_i \xi_i}{a - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.14}$$

Proof. For $u \in \mathcal{P}$, since $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$ one arrives at

$$\frac{u(\xi_i) - u(0)}{\xi_i} \leq u^{\triangle}(0),$$

i.e., $u(\xi_i) - u(0) \leq \xi_i u^{\triangle}(0)$. Hence,

$$\sum_{i=1}^{m-2} \alpha_i u(\xi_i) - \sum_{i=1}^{m-2} \alpha_i u(0) \leq \sum_{i=1}^{m-2} \alpha_i \xi_i u^{\triangle}(0).$$

By $au(0) - bu^{\triangle}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$, we get

$$u(0) \leq \frac{b + \sum_{i=1}^{m-2} \alpha_i \xi_i}{a - \sum_{i=1}^{m-2} \alpha_i} u^{\triangle}(0).$$

Hence

$$u(t) = \int_0^t u^{\triangle}(s)\triangle s + u(0) \leq tu^{\triangle}(0) + u(0) \leq$$

$$\leq tu^{\triangle}(0) + \frac{b + \sum_{i=1}^{m-2} \alpha_i \xi_i}{a - \sum_{i=1}^{m-2} \alpha_i} u^{\triangle}(0) \leq \left(1 + \frac{b + \sum_{i=1}^{m-2} \alpha_i \xi_i}{a - \sum_{i=1}^{m-2} \alpha_i}\right) u^{\triangle}(0) = Mu^{\triangle}(0),$$

i.e.,

$$\|u\|_{PC} = \max_{t\in[0,1]_{\mathbb{T}}} u(t) \leq Mu^{\triangle}(0) \leq M \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t).$$

Lemma 2.4 is proved.

From Lemma (2.4), we obtain

$$\|u\|_{PC^1} = \max\left\{\|u\|_{PC}, \|u^{\Delta}\|_{PC}\right\} = \max\left\{\max_{t\in[0,1]_{\mathbb{T}}} |u(t)|,\ \max_{t\in[0,1]_{\mathbb{T}}} |u^{\Delta}(t)|\right\} \leq$$

$$\le \max\left\{ M \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t),\ \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t) \right\} = M \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t).$$

Now define an operator $T\colon \mathcal{P} \longrightarrow PC^1(J)$ by

$$Tu(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s + \sum_{k=1}^{n} W_k\left(t, t_k\right) +$$

$$+A(f)(b+at) + B(f)(d+c(1-t)), \tag{2.15}$$

where W_k, G, $A(f)$, $B(f)$ and θ, φ are defined as in (2.7), (2.8), (2.9), (2.10) and (2.3) respectively.

Lemma 2.5. *Let* (C_1)–(C_7) *hold. Then* $T\colon \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof. By Arzela–Ascoli theorem, we can easily prove that operator T is completely continuous.

3. Main Results. We are now ready to apply the four functionals fixed point theorem [1] to the operator T in order to get sufficient conditions for the existence of at least one positive solution to the problem (1.1).

Let α and Ψ be nonnegative continuous concave functionals on $\mathcal{P}$, and let β and Φ be nonnegative continuous convex functionals on $\mathcal{P}$, then for positive numbers r, j, l and R, we define the sets:

$$\begin{aligned} Q(\alpha,\beta,r,R) &= \{u \in \mathcal{P} : r \le \alpha(u),\ \beta(u) \le R\}, \\ U(\Psi, j) &= \{u \in Q(\alpha,\beta,r,R) : j \le \Psi(u)\}, \\ V(\Phi, l) &= \{u \in Q(\alpha,\beta,r,R) : \Phi(u) \le l\}. \end{aligned} \tag{3.1}$$

Lemma 3.1 [1]. *If $\mathcal{P}$ is a cone in a real Banach space $\mathbb{B}$, α and Ψ are nonnegative continuous concave functionals on $\mathcal{P}$, β and Φ are nonnegative continuous convex functionals on $\mathcal{P}$ and there exist positive numbers r, j, l and R, such that*

$$T\colon Q(\alpha,\beta,r,R) \to \mathcal{P}$$

is a completely continuous operator, and $Q(\alpha,\beta,r,R)$ is a bounded set. If

(i) $\{u \in U(\Psi, j) : \beta(u) < R\} \cap \{u \in V(\Phi, l) : r < \alpha(u)\} \neq \varnothing$;

(ii) $\alpha(Tu) \ge r$, *for all* $u \in Q(\alpha,\beta,r,R)$, *with* $\alpha(u) = r$ *and* $l < \Phi(Tu)$;

(iii) $\alpha(Tu) \ge r$, *for all* $u \in V(\Phi, l)$, *with* $\alpha(u) = r$;

(iv) $\beta(Tu) \le R$, *for all* $u \in Q(\alpha,\beta,r,R)$, *with* $\beta(u) = R$ *and* $\Psi(Tu) < j$;

(v) $\beta(Tu) \le R$, *for all* $u \in U(\Psi, j)$, *with* $\beta(u) = R$.

Then T has a fixed point u in $Q(\alpha,\beta,r,R)$.

Suppose ω, $z \in \mathbb{T}$ with $0 < \omega < z < 1$. For the convenience, we take the notations

$$A := \frac{1}{\Delta}\left|\begin{matrix} \sum_{i=1}^{m-2}\alpha_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\times \right. \\ \left. \times\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s + \frac{n}{\rho}(c+d)(2a+b)\right) & \rho - \sum_{i=1}^{m-2}\alpha_i(d+c(1-\xi_i)) \\ \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 G(\xi_i,s)\,\phi_q\times \right. \\ \left. \times\left(\int_0^s q(\tau)\triangle\tau\right)\triangle s + \frac{n}{\rho}(c+d)(2a+b)\right) & -\sum_{i=1}^{m-2}\beta_i(d+c(1-\xi_i)) \end{matrix}\right|$$

$$\Omega = \int_w^z G(\omega,s)\phi_q\left(\int_w^z q(\tau)\triangle\tau\right)\triangle s,$$

$$\Lambda = \int_0^1 \frac{1}{\rho}\left(c(1-\sigma(s))+d\right)\phi_q\left(\int_0^1 q(\tau)\triangle\tau\right)\triangle s + na(c+d) + Aa,$$

$$L = \frac{a-\sum_{i=1}^{m-2}\alpha_i}{b+\sum_{i=1}^{m-2}\alpha_i\xi_i},$$

and define the maps

$$\alpha(u) = \min_{t\in[\omega,z]_{\mathbb{T}}} u(t), \qquad \Phi(u) = \max_{t\in[0,1]_{\mathbb{T}}} u(t), \qquad \beta(u) = \Psi(u) = \max_{t\in[0,1]_{\mathbb{T}}} u^{\Delta}(t). \tag{3.2}$$

and let $Q(\alpha,\beta,r,R),\ U(\Psi,j)$ and $V(\Phi,l)$ be defined by (3.1).

Theorem 3.1. *Assume* (C_1)–(C_7) *hold. If there exist constants* $r,\ j,\ l,\ R$ *with* $\max\left\{\frac{r}{\omega},R\right\}\le l$, $\max\left\{\frac{L+1}{L}j,\frac{L+1}{L\omega+1}r\right\} < R$ *and suppose that* f *satisfies the following conditions*:

(C_8) $f(t,u,u^{\triangle}) \ge \phi_p\left(\frac{r}{\Omega}\right)$ *for* $(t,u,u^{\triangle})\in[\omega,z]_{\mathbb{T}}\times[r,l]\times[0,R]$;

(C_9) $f(t,u,u^{\triangle}) \le \phi_p\left(\frac{R}{\Lambda}\right)$, $I_k\big(u(t_k)\big)\le\frac{R}{\Lambda}$, $J_k\big(u(t_k),u^{\triangle}(t_k)\big)\le\frac{R}{\Lambda}$ *for* $(t,u,u^{\triangle})\in[0,1]_{\mathbb{T}}\times$ $\times[0,MR]\times[0,R]$.

Then the BVP (1.1) *has at least one positive solution* $u\in\mathcal{P}$ *such that*

$$\min_{t\in[\omega,z]_{\mathbb{T}}} u(t)\ge r, \qquad \max_{t\in[0,1]_{\mathbb{T}}} u(t)\le R.$$

Proof. The impulsive BVP (1.1) has a solution $u=u(t)$ if and only if u solves the operator equation $u=Tu$. Thus we set out to verify that the operator T satisfies four functionals fixed point theorem which will prove the existence of a fixed point of T.

We first show that $Q(\alpha,\beta,r,R)$ is bounded and $T\colon Q(\alpha,\beta,r,R)\to\mathcal{P}$ is completely continuous. For all $u\in Q(\alpha,\beta,r,R)$ with Lemma 2.4, we have

$$\|u\|_{PC^1}\leq M\max_{t\in[0,1]_{\mathbb{T}}}u^{\Delta}(t)=M\beta(u)\leq MR,$$

which means that $Q(\alpha,\beta,r,R)$ is a bounded set. According to Lemma (2.5), it is clear that $T\colon Q(\alpha,\beta,r,R)\to\mathcal{P}$ is completely continuous.

Let

$$u_0=\frac{R}{L+1}(Lt+1).$$

Clearly, $u_0\in\mathcal{P}$. By direct calculation,

$$\alpha(u_0)=u_0(\omega)=\frac{R}{L+1}(L\omega+1)>r,$$

$$\beta(u_0)=\frac{R}{L+1}L<R,$$

$$\Psi(u_0)=\beta(u_0)=\frac{R}{L+1}L\geq j,$$

$$\Phi(u_0)=u_0(1)=\frac{R}{L+1}(L+1)=R\leq l.$$

So, $u_0\in\big\{u\in U(\Psi,j)\colon\beta(u)<R\big\}\cap\big\{u\in V(\Phi,l)\colon r<\alpha(u)\big\}$, which means that (i) in Lemma (3.1) is satisfied.

For all $u\in Q(\alpha,\beta,r,R)$, with $\alpha(u)=r$ and $l<\Phi(Tu)$, since $u^{\triangle}$ is nonincreasing on $[0,1]_{\mathbb{T}}$ we have

$$\alpha(Tu)=Tu(\omega)\geq\omega Tu(1)=\omega\Phi(Tu)>\omega l\geq r.$$

So, $\alpha(Tu)>r$. Hence (ii) in Lemma 3.1 is fulfilled.

For all $u\in V(\Phi,l)$, with $\alpha(u)=r$,

$$\alpha(Tu)=\min_{t\in[\omega,z]_{\mathbb{T}}}Tu(t)=(Tu)(\omega)=$$

$$=\int\limits_0^1G(\omega,s)\,\phi_q\left(\int\limits_0^s q(\tau)f\left(\tau,u(\tau),u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s+$$

$$+\sum_{k=1}^{n}W_k(\omega,t_k)+A(f)\theta(\omega)+B(f)\varphi(\omega)\geq$$

$$\geq\int\limits_0^1G(\omega,s)\,\phi_q\left(\int\limits_0^s q(\tau)f\left(\tau,u(\tau),u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s\geq$$

$$\geq\int\limits_\omega^zG(\omega,s)\,\phi_q\left(\int\limits_\omega^s q(\tau)f\left(\tau,u(\tau),u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s\geq$$

$$\geq \frac{r}{\Omega}\int\limits_{\omega}^{z} G(\omega, s)\,\phi_q\left(\int\limits_{\omega}^{s} q(\tau)\triangle\tau\right)\triangle s = r,$$

and for all $u \in U(\Psi, j)$, with $\beta(u) = R$,

$$\beta(Tu) = \max_{t\in[0,1]_{\mathbb{T}}} (Tu)^{\Delta}(t) = (Tu)^{\Delta}(0) \leq$$

$$\leq \int\limits_{0}^{1} \frac{1}{\rho} a\left(c(1-\sigma(s)) + d\right)\phi_q\left(\int\limits_{0}^{1} q(\tau) f\left(\tau, u(\tau), u^{\triangle}(\tau)\right)\triangle\tau\right)\triangle s +$$

$$+\sum_{k=1}^{n} a\left(-cI_k(u(t_k)) + \left(c(1-t_k)+d\right) J_k(u(t_k), u^{\triangle}(t_k))\right) + A(f)a \leq$$

$$\leq \frac{R}{\Lambda}\int\limits_{0}^{1}\frac{1}{\rho} a\left(c(1-\sigma(s))+d\right)\phi_q\left(\int\limits_{0}^{1} q(\tau)\triangle\tau\right)\triangle s + na(c+d)\frac{R}{\Lambda} + a\frac{R}{\Lambda}A =$$

$$= \frac{R}{\Lambda}\left(\int\limits_{0}^{1}\frac{1}{\rho} a\left(c(1-\sigma(s))+d\right)\phi_q\left(\int\limits_{0}^{1} q(\tau)\triangle\tau\right)\triangle s + na(c+d) + aA\right) = R.$$

Thus (iii) and (v) in Lemma 3.1 hold. We finally prove that (iv) in Lemma 3.1 holds.

For all $u \in Q(\alpha, \beta, r, R)$, with $\beta(u) = R$ and $\Psi(Tu) < j$, we have

$$\beta(Tu) = \Psi(Tu) < j < \frac{L}{L+1}R < R.$$

Thus, all conditions of Lemma 3.1 are satisfied. T has a fixed point u in $Q(\alpha, \beta, r, R)$. Therefore, the BVP (1.1) has at least one positive solution $u \in \mathcal{P}$ such that

$$\min_{t\in[\omega,z]_{\mathbb{T}}} u(t) \geq r, \qquad \max_{t\in[0,1]_{\mathbb{T}}} u(t) \leq R.$$

Theorem 3.1 is proved.

4. An example.

***Example* 4.1.** In BVP (1.1), suppose that $\mathbb{T} = [0,1]$, $p = 2$, $m = 3$, $n = 1$, $q(t) = 1$, $a = b = c = d = 1$, $\xi_1 = \frac{1}{4}$, $\alpha_1 = \frac{1}{10}$, $\beta_1 = 2$, $t_1 = \frac{1}{2}$, i.e.,

$$\begin{gathered}\left(u^{\triangle\triangle}(t)\right)^{\triangle} + f(t, u(t), u^{\triangle}(t)) = 0, \qquad t \in [0,1], \quad t \neq \frac{1}{2},\\ \Delta u\left(\frac{1}{2}\right) = I_1\left(u\left(\frac{1}{2}\right)\right),\\ \Delta u^{\triangle}\left(\frac{1}{2}\right) = -J_1\left(u\left(\frac{1}{2}\right), u^{\triangle}\left(\frac{1}{2}\right)\right),\end{gathered} \tag{4.1}$$

$$u(0) - u^{\triangle}(0) = \frac{1}{10} u\left(\frac{1}{4}\right),$$

$$u(1) + u^{\triangle}(1) = 2u\left(\frac{1}{4}\right),$$

$$u^{\triangle\triangle}(0) = 0,$$

where

$$f(t,u,u^{\triangle}) = \begin{cases} 0{,}09, & u \in \left[0, \dfrac{1}{100}\right], \\ \dfrac{11}{1799}u + \dfrac{809}{8995}, & u \geq \dfrac{1}{100}, \end{cases}$$

$$I_1(u) = \frac{1}{45}u, \quad u \geq 0,$$

$$J_1(u,u^{\triangle}) = \frac{3}{180}u, \qquad (u,u^{\triangle}) \in [0,\infty) \times [0,\infty).$$

Set $\omega = \frac{1}{5}$, $z = \frac{1}{3}$, by simple calculation we get

$$A = 14{,}64957265, \qquad \Omega = 0{,}03375593836, \qquad \Lambda = 17{,}14957265, \qquad L = \frac{36}{41}, \qquad M = \frac{77}{36},$$

and

$$G(t,s) = \frac{1}{3} \begin{cases} (1+\sigma(s))(2-t), & \sigma(s) \leq t, \\ (1+t)(2-\sigma(s)), & t \leq s. \end{cases}$$

Choose $r = \frac{1}{100}$, $l = 10$, $j = 2$ and $R = 8$, it is easy to check that $\max\left\{\frac{1}{20}, 8\right\} \leq 10$, $\max\left\{\frac{154}{36}, \frac{77}{4820}\right\} < 8$,

$$f(t,u(t),u^{\triangle}(t)) = 0{,}09 \geq \phi_p\left(\frac{r}{\Omega}\right) = 0{,}08776060928$$

$$\text{for} \quad (t,u(t),u^{\triangle}(t)) \in \left[\frac{1}{5}, \frac{1}{3}\right]\left[\frac{1}{100}, 10\right] \times [0,8]\,;$$

$$f(t,u(t),u^{\triangle}(t)) \leq 0{,}2 \leq \phi_p\left(\frac{R}{\Lambda}\right) = 0{,}2176072544,$$

$$I_1\left(u\left(\frac{1}{2}\right)\right) = 0{,}3802469136 \leq 0{,}4532687651 = \frac{R}{\Lambda},$$

$$J_1\left(u\left(\frac{1}{2}\right), u^{\triangle}\left(\frac{1}{2}\right)\right) =$$

$$= 0{,}2851851852 \leq 0{,}4532687651 = \frac{R}{\Lambda} \text{ for}(t,u(t),\, u^{\triangle}(t)) \in [0,1] \times \left[0, \frac{154}{9}\right] \times [2,8].$$

So, all conditions of Theorem 3.1 hold. Thus by Theorem 3.1, the BVP (4.1) has at least one positive solution u such that

$$\min_{t\in\left[\frac{1}{5},\frac{1}{3}\right]} u(t) \geq \frac{1}{100}, \qquad \max_{t\in[0,1]} u(t) \leq 8.$$

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