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EXISTENCE AND NONEXISTENCE OF SOLUTIONS OF REACTION-DIFFUSION EQUATION WITH ROBIN BOUNDARY CONDITION

ІСНУВАННЯ ТА НЕІСНУВАННЯ РОЗВ'ЯЗКІВ РЕАКЦІЙНО-ДИФУЗІЙНОГО РІВНЯННЯ З ГРАНИЧНИМИ УМОВАМИ РОБЕНА

We investigate the long-time behavior of the reaction-diffusion equation, which has a nonlinearity of polynomial growth of any order, with Robin boundary condition. Sufficient conditions are obtained for the solutions of the problem to be bounded or approaching infinity at a finite time.

Досліджено довготривалу поведінку реакційно-дифузійного рівняння з граничними умовами Робена, яке містить нелінійність поліноміального росту будь-якого порядку. Отримано достатні умови для того, щоб розв'язки граничної задачі були обмеженими, або прямували до нескінченності на скінченному проміжку часу.

1. Introduction. We consider the following reaction-diffusion equation with Robin boundary condition:

$$u_t - \Delta u + a(x,t)|u|^{\rho}u - b(x,t)|u|^{\nu}u = h(x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

$$\left(\frac{\partial u}{\partial \eta} + k(x',t)u\right)\bigg|_{\partial\Omega} = \varphi(x',t), \quad (x',t) \in \Sigma_T = \partial\Omega \times [0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$; $\rho, \nu > 0$ are given some constants; T is a positive number; Δ is the n-dimensional Laplace operator; $a\colon Q_T \to \mathbb{R}^1_+$, $b\colon Q_T \to \mathbb{R}^1_+$ and $k\colon \Sigma_T \to \mathbb{R}^1$ are given functions; h and φ are given generalized functions. $\dfrac{\partial u}{\partial \eta}$ denotes the normal derivative of the function u in direction of the outer normal vector η. Here $u(x,t)$ is an unknown function which can represent temperature, population density, or in general the quantity of a substance.

Existence and nonexistence of solutions of nonlinear parabolic problems extensively investigated during the past few decades. We refer the reader to the survey paper of Galaktionov and Vazquez [5], Levine [7] and books of Quittner, Souplet [17] and Samarskii et al. [19]. There are many studies on blow-up of solutions of semilinear parabolic equation without time-dependent coefficients under homogeneous Dirichlet or Neumann boundary condition (see [1, 2, 4, 11, 12]). Also, some nonlinear initial value parabolic problems with time-dependent coefficients under homogeneous Dirichlet boundary condition and homogeneous Neumann boundary condition were investigated in [16, 14] respectively.

One of the first papers on this subject is due to Hale, Rocha in [6], with homogeneous Robin boundary condition in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \leq 3$, and where it is shown the existence of attractor.

In [3], blow-up phenomena was investigated for the nonlinear parabolic equation under homogeneous Robin boundary condition without time dependent coefficients.

In [8], Marras and Piro considered the semilinear parabolic equation $u_t = \Delta u + f(u)$ under the Robin boundary conditions by taking constant coefficient on the bounded domain of $\mathbb{R}^2$. They determined sufficient conditions on the geometry and data to preclude the blow-up of the solution and to obtain an exponential decay bound for the solution and its gradient.

In [18], Rault investigated the Fujita phenomenon for nonlinear parabolic problems $\partial_t u = \Delta u + u^p$ in an exterior domain of $\mathbb{R}^n$ under the homogeneous Robin boundary conditions with time dependent coefficient in the superlinear case for positive solutions.

In [13], Payne and Schaefer studied the semilinear parabolic equation $u_t = \Delta u + f(u)$ under a Robin boundary condition without time dependent coefficient where f satisfies the constraint $uf(u) \geq 2(1+\alpha)F(u)$, $F(u) = \int_0^u f(s)ds$, $\alpha > 0$. They determined sufficient conditions which ensure that blow-up does occur or does not occur.

After that in [15], Payne and Philippin dealt with the time dependent semilinear parabolic equation $u_t = \Delta u + b(t)f(u)$ under a Robin boundary condition. They determined upper and lower bounds for the blow-up time on a region $\Omega \subset \mathbb{R}^3$.

In [9], a lower bound for the blow-up time was derived for a nonlinear parabolic problem with a gradient term; $u_t = \Delta u + k_1(t)u^p - k_2(t)|\nabla u|^q$, $p, q > 1$ under the Robin boundary conditions by taking constant coefficient.

In a recent paper [10], we considered problem (1.1)–(1.3) and showed the existence of generalized solution by using a general result in [20] and the existence of global attractor for the autonomous case.

The purpose of this paper is to investigate the asymptotic behavior of solutions and to give some condition for blow-up of solutions of the problem (1.1)–(1.3) in finite time. In this study, we investigate Robin type boundary-value problem for reaction-diffusion equation by taking Yamabe type polynom as the nonlinear part of the equation. Also, differently from articles above, we consider the problem in nonstationary case, i.e., the coefficients in the equation and boundary condition depend on time. Moreover, here the asymptotic behavior is studied in the space where the solution exists.

The plan of this paper is as follows: In the next section we give some results on the existence and uniqueness of the solution of the problem (1.1)–(1.3). In Section 3, we give some conditions under which the solutions of the problem (1.1)–(1.3) is bounded in $L_2(\Omega)$ for all $t \geq 0$. We also obtain additional condition under which the solutions tend to zero as $t \to \infty$. In Section 4, we investigated the effect of exponents ρ, ν and the data on the behavior of the solutions, some sufficient conditions are obtained for solutions of the problem tending to infinity at a finite time.

2. Preliminaries. First, we give the definition of the generalized solution and then recall the existence and uniqueness theorems since we will investigate the behavior of the solution which exists. For more details, we refer to [10].

We shall assume $h \in L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\rho+2}{\rho+1}}(Q_T)$, $\varphi \in L_2(0,T;W_2^{-1/2}(\partial\Omega))$ and define the following class of functions $u\colon Q_T \to \mathbb{R}^1$:

$$P_0 \equiv L_2(0,T;W_2^1(\Omega)) \cap L_{\rho+2}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u\colon u(x,0) = u_0(x)\}.$$

We will understand the solution of the considered problem in the following sense:

Definition 2.1. *A function $u \in P_0$ is called the generalized solution of problem* (1.1)–(1.3) *if it satisfies the equality*;

$$-\int_0^T\int_\Omega u\frac{\partial v}{\partial t}dxdt+\int_\Omega u(x,T)v(x,T)dx+\int_0^T\int_\Omega Du\,Dv\,dx\,dt+$$

$$+\int_0^T\int_\Omega \left(a(x,t)\left|u\right|^\rho u-b(x,t)\left|u\right|^\nu u\right)vdxdt-$$

$$-\int_\Omega u(x,0)v(x,0)dx+\int_0^T\int_{\partial\Omega} k(x',t)uvdx'dt=$$

$$=\int_0^T\int_\Omega hvdxdt+\int_0^T\int_{\partial\Omega}\varphi vdx'dt,$$

for all $v \in W_2^1(0,T;(W_2^1(\Omega))^*)\cap L_2(0,T;W_2^1(\Omega))\cap L_{\rho+2}(Q_T)$.

Theorem 2.1. *Assume that the following conditions are satisfied with $0<\nu\le\rho$:*

(i) *a and b are positive functions,*

$$a\in L_\infty(\mathbb{R}^+;L_\infty(\Omega)),\qquad b\in\begin{cases}L_{\frac{\rho+2}{\rho-\nu}}(\mathbb{R}^+;L_{\frac{\rho+2}{\rho-\nu}}(\Omega)), & \text{if}\quad \nu<\rho,\\ L_\infty(\mathbb{R}^+;L_\infty(\Omega)), & \text{if}\quad \nu=\rho.\end{cases}$$

If $\nu<\rho$, then there exists a number $a_0>0$ such that $a(x,t)\ge a_0$ for almost every $(x,t)\in$ $\in\Omega\times\mathbb{R}^+$.

If $\nu=\rho$, then there exists a number $b_0>0$ such that $a(x,t)-b(x,t)\ge b_0$ for almost every $(x,t)\in\Omega\times\mathbb{R}^+$.

(ii) *$k\in L_\infty(\mathbb{R}^+;L_{n-1}(\partial\Omega))$ and there exists a number $k_0\ge 0$ such that $k(x',t)\ge -k_0$ for almost every $(x',t)\in\partial\Omega\times\mathbb{R}^+$,*

$$k_0<\begin{cases}\dfrac{\min\{a',\theta_1\}}{c_3^2}, & \text{if}\quad 0<\nu<\rho,\\[2mm] \dfrac{\min\{b',\theta_1\}}{c_3^2}, & \text{if}\quad \nu=\rho.\end{cases}$$

Then problem (1.1)–(1.3) *is solvable in P_0 for any $(h,\varphi)\in[L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\rho+2}{\rho+1}}(Q_T)]\times$ $\times L_2(0,T;W_2^{-1/2}(\partial\Omega))$ and $u_0\in W_2^1(\Omega)\cap L_{\rho+2}(\Omega)$ (here θ_1, a' and b' are positive numbers such that $a'<a_0$, $b'<b_0$, $\theta_1<1$ and c_3 comes from Sobolev's embedding inequality $\|u\|_{L_2(\partial\Omega)}\le$ $\le c_3\|u\|_{W_2^1(\Omega)}$).*

Theorem 2.2. *Assume that the conditions of Theorem* 2.1 *are satisfied. If there exists a positive number* b_1 *such that* $b(x,t) \le b_1 < a_0$ *for almost every* $(x,t) \in Q_T$ *when* $0 < \nu < \rho$, *then the solution is unique. Moreover, if u and v are solutions of problem* (1.1)–(1.3), *with initial data* u_0 *and* v_0, *respectively, then*

$$|u(x,t) - v(x,t)|^2_{L_2(\Omega)} \le \|u_0 - v_0\|^2_{L_2(\Omega)} e^{2(b_1(\rho+1)+1)t} \quad as \quad \nu < \rho,$$

$$|u(x,t) - v(x,t)|^2_{L_2(\Omega)} \le \|u_0 - v_0\|^2_{L_2(\Omega)} e^{2t} \quad as \quad \nu = \rho.$$

3. Asymptotic behavior of solutions in homogeneous case. In this section, we show that under some conditions the solutions of problem (1.1)–(1.3) is bounded in $L_2(\Omega)$ for all $t \ge 0$ in the case of $h(x,t) = 0$, $\varphi(x',t) = 0$. We also describe the asymptotic behavior of these solutions.

The main result of this section is the following theorem.

Theorem 3.1. *Let condition* (i) *of Theorem* 2.2 *be fulfilled. Assume that function k belongs to* $L_\infty(\mathbb{R}^+; L_{n-1}(\partial\Omega))$ *and there exist positive numbers* b_1 *and* k_0 *such that* $b(x,t) \le b_1$, $k(x',t) \ge k_0$ *hold for almost every* $(x,t) \in \Omega \times \mathbb{R}^+$ *and* $(x',t) \in \partial\Omega \times \mathbb{R}^+$. *Then we have the following inequalities for the solution of problem* (1.1)–(1.3) *for all* $t \ge 0$:

$$|u|^2_{L_2(\Omega)}(t) \le \frac{2}{\left[\dfrac{K_2}{K_1} + e^{-\frac{\rho}{2}K_1 t}\left(\left(\dfrac{1}{2}\displaystyle\int_\Omega u_0^2 dx\right)^{-\rho/2} - \dfrac{K_2}{K_1}\right)\right]^{2/\rho}}, \tag{3.1}$$

where $K_1 = K_1(\tilde{c}, c_2, k_0, b_1, \rho, \nu) > 0$, $K_2 = K_2(a_0, c_5, \rho) > 0$ *as* $0 < \nu < \rho$ *and* $K_1 = K_1(\tilde{c}, c_2, k_0) < 0$, $K_2 = K_2(b_0, c_5, \rho) > 0$ *as* $\nu = \rho$. (*Here* $\tilde{c}$, c_2, c_5 *come from inequalities* $\|u\|_{L_2(\Omega)} \le c_2\|u\|_{W_2^1(\Omega)}$, $\|u\|^2_{W_2^1(\Omega)} \le \tilde{c}(\|Du\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\partial\Omega)})$, $\|u\|_{L_2(\Omega)} \le c_5\|u\|_{L_{\rho+2}(\Omega)}$).

Proof. Conditions of Theorem 2.1 provide that problem (1.1)–(1.3) has a solution in P_0. Let define the auxiliary function:

$$E(t) = \frac{1}{2}\int_\Omega u^2 dx,$$

where $u(x,t)$ is solution of problem (1.1)–(1.3) and compute $E'(t) = \int_\Omega u u_t dx$:

$$E'(t) = \int_\Omega u\Delta u dx - \int_\Omega a(x,t)|u|^{\rho+2} dx + \int_\Omega b(x,t)|u|^{\nu+2} dx,$$

after applying the integrating by parts, and using the conditions of theorem 3.1, we get

$$E'(t) \le -\frac{1}{\tilde{c}c_2^2}\min\{1, k_0\}\|u\|^2_{L_2(\Omega)} - \int_\Omega a(x,t)|u|^{\rho+2} dx + \int_\Omega b(x,t)|u|^{\nu+2} dx.$$

Here first consider the case $0 < \nu < \rho$. By using the assumptions of Theorem 2.1 we have

$$E'(t) \le -\frac{1}{\tilde{c}c_2^2}\min\{1, k_0\}\|u\|^2_{L_2(\Omega)} - a_0|u|^{\rho+2}_{L_{\rho+2}(\Omega)} + b_1\int_\Omega |u|^{\nu+2} dx.$$

We separate the end term of the right-hand side such that the following:

$$E'(t) \le -\frac{1}{\tilde{c}c_2^2}\min\{1,k_0\}\|u\|^2_{L_2(\Omega)} - a_0|u|^{\rho+2}_{L_{\rho+2}(\Omega)} + b_1\int\limits_\Omega |u|^{\frac{2}{\rho}(\rho-\nu)}|u|^{\frac{\nu}{\rho}(\rho+2)}dx.$$

Applying Hцlder and Young inequalities for the last term, we deduce that

$$E'(t) \le \left(-\frac{1}{\tilde{c}c_2^2}\min\{1,k_0\} + c(\varepsilon)b_1^{\rho/(\rho-\nu)}\right)\|u\|^2_{L_2(\Omega)} - (a_0-\varepsilon)\frac{1}{c_5^{\rho+2}}|u|^{\rho+2}_{L_2(\Omega)}, \qquad (3.2)$$

where $\varepsilon < \min\left\{a_0, \dfrac{\nu}{\rho}\left(\dfrac{b_1^{\rho/(\rho-\nu)}(\rho-\nu)\tilde{c}c_2^2}{\rho\min\{1,k_0\}}\right)^{(\rho-\nu)/\nu}\right\}$, thus we obtain

$$E'(t) \le 2\left(-\frac{1}{\tilde{c}c_2^2}\min\{1,k_0\} + c(\varepsilon)b_1^{\rho/(\rho-\nu)}\right)E(t) - (a_0-\varepsilon)\left(\frac{2}{c_5^2}\right)^{(\rho+2)/2}(E(t))^{(\rho+2)/2},$$

for convenience we denote coefficients by K_1, K_2:

$$K_1 = 2\left(-\frac{1}{\tilde{c}c_2^2}\min\{1,k_0\} + c(\varepsilon)b_1^{\rho/(\rho-\nu)}\right),$$

$$K_2 = (a_0-\varepsilon)\left(\frac{2}{c_5^2}\right)^{(\rho+2)/2},$$

then we have

$$E'(t) \le K_1E(t) - K_2(E(t))^{(\rho+2)/2},$$

where $K_1 > 0$, $K_2 > 0$ by depending on choosing of ε. Now we solve the following inequality:

$$E'(t) \le K_1E(t) - K_2(E(t))^{(\rho+2)/2}$$

with

$$E(0) = \frac{1}{2}\int\limits_\Omega u_0^2dx,$$

making use of the substitution $v = (E(t))^{-\rho/2}$, then we obtain

$$v' + \frac{\rho}{2}K_1v \ge K_2\frac{\rho}{2},$$

that is

$$\frac{d}{dt}(ve^{\frac{\rho}{2}K_1t}) \ge K_2e^{\frac{\rho}{2}K_1t}\frac{\rho}{2},$$

by integrating we have

$$ve^{\frac{\rho}{2}K_1t} - v(0) \ge \frac{K_2}{K_1}e^{\frac{\rho}{2}K_1t} - \frac{K_2}{K_1},$$

that is

$$v \geq \left[\frac{K_2}{K_1}e^{\frac{\rho}{2}K_1 t} - \frac{K_2}{K_1} + v(0)\right] e^{-\frac{\rho}{2}K_1 t},$$

it follows that

$$|u|^2_{L_2(\Omega)}(t) \leq \frac{2}{\left[\frac{K_2}{K_1} + e^{-\frac{\rho}{2}K_1 t}\left(\left(\frac{1}{2}\int_\Omega u_0^2 dx\right)^{-\rho/2} - \frac{K_2}{K_1}\right)\right]^{2/\rho}}$$

which completes the first part of the proof.

Now consider the case $\nu = \rho$. By making use of same arguments as in the case $0 < \nu < \rho$, we obtain a differential inequality of the following form:

$$E'(t) \leq K_1 E(t) - K_2 (E(t))^{(\rho+2)/2}$$

with

$$K_1 = -2\frac{1}{\tilde{c}c_2^2}\min\{1, k_0\}, \quad K_2 = b_0\left(\frac{2}{c_5^2}\right)^{(\rho+2)/2},$$

if we solve this inequality then we arrive at the desired result for the case $\nu = \rho$.

Theorem 3.1 is proved.

Now, we investigate the decay to zero of the solutions under some extra conditions.

***Corollary* 3.1.** *Assume that the following inequality is satisfied with* $0 < \nu < \rho$:

$$\frac{\nu}{\rho}\left(\frac{b_1^{\rho/(\rho-\nu)}(\rho-\nu)\tilde{c}c_2^2}{\rho\min\{1, k_0\}}\right)^{(\rho-\nu)/\nu} < a_0. \tag{3.3}$$

Then

$$\|u\|_{L_2(\Omega)}(t) \leq \|u_0\|_{L_2(\Omega)} \quad \forall t \geq 0, \tag{3.4}$$

and

$$u(x,t) \to 0 \quad as \quad t \to \infty, \tag{3.5}$$

under the assumptions of Theorem 3.1.

By using the assumption (3.3) we can apply the Young inequality to (3.2) with

$$\frac{\nu}{\rho}\left(\frac{b_1^{\rho/(\rho-\nu)}(\rho-\nu)\tilde{c}c_2^2}{\rho\min\{1, k_0\}}\right)^{(\rho-\nu)/\nu} < \varepsilon < a_0,$$

then we have negative constant K_1. Considering this in (3.1), we obtain (3.4), (3.5) immediately.

***Corollary* 3.2.** *If* $u_0 = 0$, *then the solution is zero regardless of the sign of* K_1 *under the assumptions of Theorem* 3.1.

In light of inequality (3.1) we have the following:

$$|u|^2_{L_2(\Omega)} \leq \frac{2|u_0|^2_{L_2(\Omega)}}{\left[\dfrac{K_2}{K_1}|u_0|^{\rho}_{L_2(\Omega)}(1-e^{-\frac{\rho}{2}K_1 t})+2^{\rho/2}e^{-\frac{\rho}{2}K_1 t}\right]^{2/\rho}}.$$

***Corollary* 3.3.** *If $\nu=\rho$, then $\|u\|_{L_2(\Omega)}(t)\leq\|u_0\|_{L_2(\Omega)}$ for all $t\geq 0$ and $u(x,t)\to 0$ as $t\to\infty$ under the assumptions of Theorem* 3.1.

By using the inequality (3.1), we give the following results on the existence of invariant set and the asymptotic behavior of the solution.

***Corollary* 3.4.** *Under the assumptions of Theorem* 3.1,

$$\|u\|^2_{L_2(\Omega)}(t)\leq 2\left(\frac{K_1}{K_2}\right)^{2/\rho} \quad \text{when} \quad \|u_0\|^2_{L_2(\Omega)}\leq 2\left(\frac{K_1}{K_2}\right)^{2/\rho}$$

for all $t\geq 0$ and

$$\|u\|^2_{L_2(\Omega)}(t)\leq 2\left(\frac{K_1}{K_2}\right)^{2/\rho} \quad \text{as} \quad t\to\infty.$$

4. On blow-up. In this section we will give sufficient conditions which ensure that the solution of problem (1.1)–(1.3) blows-up at some finite time t^* in the case of $h(x,t)=0$, $\varphi(x',t)=0$, for the sufficiently smooth solution;

$$\begin{cases} u_t-\Delta u+a(x,t)|u|^{\rho}u-b(x,t)|u|^{\nu}u=0, & (x,t)\in\Omega\times(0,t^*),\\ \left.\left(\dfrac{\partial u}{\partial\eta}+k(x',t)u\right)\right|_{\partial\Omega}=0, & (x',t)\in\partial\Omega\times(0,t^*),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a bounded domain with smooth boundary $\partial\Omega$.

Theorem 4.1. *Assume that the following conditions are satisfied for every $(x',t)\in\partial\Omega\times(0,t^*]$, $(x,t)\in\Omega\times(0,t^*]$:*

(i) $a(x,t)\geq 0$, $b(x,t)>0$, $k(x',t)>0$, $\dfrac{\partial}{\partial t}a(x,t)\leq 0$, $\dfrac{\partial}{\partial t}b(x,t)\geq 0$, $\dfrac{\partial}{\partial t}k(x',t)\leq 0$;

(ii) $\nu\geq\rho>0$ *and let inequality* $a(x,t)<b(x,t)$ *hold when* $\nu=\rho$;

(iii) $u_0>0$ *satisfies given inequality*

$$2\int\limits_{\Omega} b(x,0)u_0^{\nu+2}dx-\frac{2(\nu+2)}{\rho+2}\int\limits_{\Omega} a(x,0)u_0^{\rho+2}dx>$$

$$>(\nu+2)\left(\int\limits_{\Omega}(\nabla u_0)^2dx+\int\limits_{\partial\Omega}k(x',0)u_0^2dx'\right).$$

If u is a positive solution of problem (1.1)–(1.3), *then u blows-up in $L_2(\Omega)$ in finite time t^*.*

Proof. Let define the auxiliary function $G(t) = \int_\Omega u^2 dx$ where $u(x,t)$ is solution of the problem and compute $G'(t)$:

$$G'(t) = -2\int\limits_\Omega (\nabla u)^2 dx - 2\int\limits_{\partial\Omega} k(x',t)u^2 dx' +$$

$$+2\int\limits_\Omega b(x,t)u^{\nu+2}dx - 2\int\limits_\Omega a(x,t)u^{\rho+2}dx,$$

$$G'(t) \geq -2\left(1+\frac{\nu}{2}\right)\left(\int\limits_\Omega (\nabla u)^2 dx + \int\limits_{\partial\Omega} k(x',t)u^2 dx'\right) +$$

$$+2\int\limits_\Omega b(x,t)u^{\nu+2}dx - 2\left(\frac{\nu+2}{\rho+2}\right)\int\limits_\Omega a(x,t)u^{\rho+2}dx.$$

We denote right side of this inequality by $H(t)$ and compute $H'(t)$, then we get

$$H'(t) = 2(\nu+2)\int\limits_\Omega (u_t)^2 dx - 2\left(1+\frac{\nu}{2}\right)\int\limits_{\partial\Omega} \frac{\partial k}{\partial t}(x',t)u^2 dx' -$$

$$-2\left(\frac{\nu+2}{\rho+2}\right)\int\limits_\Omega \frac{\partial a}{\partial t}(x,t)u^{\rho+2}dx +$$

$$+2\int\limits_\Omega \frac{\partial b}{\partial t}(x,t)u^{\nu+2}dx.$$

In the last inequality, by using condition (i), we obtain

$$H'(t) \geq 2(\nu+2)\int\limits_\Omega (u_t)^2 dx.$$

Since $H'(t) \geq 0$ and $H(0) > 0$ (condition (iii)), it follows that $H(t) > 0$ for $t \geq 0$.

By using $G(t)$, (3.1) and Schwarz inequality, we have

$$(G'(t))^2 = \left(2\int\limits_\Omega u u_t dx\right)^2 \leq 4\int\limits_\Omega u^2 dx \int\limits_\Omega u_t^2 dx \leq \frac{2}{\nu+2}G(t)H'(t)$$

and since $G'(t) \geq H(t)$, we get

$$\frac{2}{\nu+2}G(t)H'(t) \geq G'(t)H(t).$$

It follows that

$$\frac{H'(t)}{H(t)} \geq \left(\frac{2+\nu}{2}\right) \frac{G'(t)}{G(t)}.$$

Integrating the above inequality from 0 to t, we obtain that

$$\frac{H(t)}{H(0)} \geq \left[\frac{G(t)}{G(0)}\right]^{\left(\frac{2+\nu}{2}\right)}. \tag{4.1}$$

By using $G'(t) \geq H(t)$ in (4.1), we have,

$$\frac{G'(t)}{[G(t)]^{\left(\frac{2+\nu}{2}\right)}} \geq M, \qquad M = \frac{H(0)}{[G(0)]^{\left(\frac{2+\nu}{2}\right)}}.$$

We integrate above inequality, we get

$$\frac{1}{[G(0)]^{\nu/2}} - \frac{1}{[G(t)]^{\nu/2}} \geq \frac{\nu}{2} Mt$$

or

$$\frac{1}{[G(t)]^{\nu/2}} \leq \frac{1}{[G(0)]^{\nu/2}} - \frac{\nu}{2} Mt. \tag{4.2}$$

Since inequality (4.2) does not exist for all time $t \geq 0$, we say that u blows-up at some finite time t^* and t^* is bounded above by

$$t^* \leq \frac{2G(0)}{\nu H(0)}.$$

Theorem 4.1 is proved.

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