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POLYNOMIAL APPROXIMATION IN BERGMAN SPACES*

ПОЛІНОМІАЛЬНІ НАБЛИЖЕННЯ У ПРОСТОРАХ БЕРГМАНА

The purpose of this work is to obtain Jackson and converse inequalities of the polynomial approximation in Bergman spaces. Some known results presented for the moduli of continuity are extended to the moduli of smoothness. We proved some simultaneous approximation theorems and obtained the Nikolskii – Stechkin inequality for polynomials in these spaces.

Метою даної роботи є встановлення нерівності Джексона та обернених нерівностей для поліноміальних наближень у просторі Бергмана. Деякі відомі результати для модулів неперервності узагальнено на модулі гладкості. Доведено деякі спільні теореми про наближення та встановлено нерівність Нікольського – Стечкіна для поліномів у цих просторах.

1. Statement of problem. Let $\Omega \subset \mathbb{C}$ be an arbitrary domain in the complex plane. The Bergman space $B^p(\Omega)$ consists of all functions f analytic in Ω for which

$$\|f\|_p := \left\{ \iint\limits_{\Omega} |f(z)|^p \, d\sigma(z) \right\}^{1/p} < \infty$$

for $0 < p < \infty$, where $d\sigma(z) := \dfrac{1}{\pi} dxdy$ is area measure on Ω. $B^\infty(\Omega)$ is the set of functions f bounded and analytic in Ω. In this case we set $\|f\|_\infty := \sup_{z\in\Omega} |f(z)|$. $B^0(\Omega)$ is the set of functions f analytic in Ω with

$$\|f\|_0 := \exp\left(\iint\limits_{\Omega} \log|f(z)| \, d\sigma(z) \right) < \infty,$$

where $\log|f|$ is summable on Ω. $\|f\|_p$ is called the norm of f and it is a true norm if $p \geq 1$. If Ω is the complex unit disc $\mathbb{D}$, then we will write B^p instead of $B^p(\mathbb{D})$.

For a function f, analytic in $\mathbb{D}$, the integral means are defined by

$$M_p(r,f) := \left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta \right\}^{1/p}, \qquad 0 < p < \infty, \qquad 0 \leq r < 1,$$

and $M_\infty(r,f) := \sup_{\theta\in[0,2\pi)} \left| f\left(re^{i\theta}\right)\right|$. If they are stay bounded as $r \to 1^-$, then f is said to belong to Hardy space H^p. Thus $H^\infty = B^\infty$ consists of all bounded analytic functions in $\mathbb{D}$. The norm $\|f\|_{H^p}$ of a function $f \in H^p$ is defined as the limit of $M_p(r,f)$ as $r \to 1^-$. It is a true norm if $p \geq 1$. H^0 is the set of functions f analytic in $\mathbb{D}$ with

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$$\|f\|_{H^0} := \exp\left(\frac{1}{2\pi}\int\limits_0^{2\pi}\log\left|f\left(e^{i\theta}\right)\right|d\theta\right) < \infty,$$

where $\log|f|$ is summable on $\mathbb{D}$. It is well known that $H^p \subset B^{2p}$ and $\|f\|_{2p} \leq \|f\|_{H^p}$ for $0 < p < \infty$.

The main initial problem in Approximation Theory is density of the set of polynomials in spaces investigated. The first results related with the density of the set of polynomials in Bergman spaces were obtained in 1934 by Farrel and Markushevich (see, e.g., [6, 17]). Also the same problems was considered by Al'per [1], Burbea [3] and Metzger [9] for Bergman spaces $B^p(\Omega)$, where $p \geq 1$ and Ω is a bounded simply connected region with a simply connected complement. In [1] Al'per defined a moduli of continuity in $B^p(\Omega)$ and proved the direct theorem in this space. Also in the same article Al'per also obtained the Quade type [10] converse inequalities in $B^p(\Omega)$. Later L. F. Zhong [25] construct a polynomial that near best approximant for functions of Bergman spaces $B^p(\Omega)$, where $1 < p < \infty$ and Ω is a bounded region with a sufficiently smooth boundary. For discs in $\mathbb{C}$ these problems in B^p was considered by many mathematicians: For $p = \infty$ Storozhenko [14] proved a Jackson-type direct theorem in B^∞ for boundary moduli of smoothness. She also defined [15] different moduli of smoothness on the unit circle and obtained a direct theorem of approximation in B^∞. Later Kryakin [7] extended the properties of Storozhenko's moduli of smoothness on whole $\mathbb{D}$ and he find Jackson-type direct theorem, simultaneous approximation theorem in B^∞. For $0 < p \leq \infty$ F. Ch. Xing and C. L. Su [23] ($p \geq 1$) and X. C. Shen and F. Ch. Xing [13] ($0 < p < 1$) was proved direct theorem for moduli of continuity in B^p. Also for $0 < p \leq \infty$ X. C. Shen and F. Ch. Xing [13] ($0 < p < 1$) and F. Ch. Xing and Z. Su [24] ($p \geq 1$) proved Quade-type converse inequalities in B^p. A different method has been applied by G. Ren and M. Wang [11] to obtain Jackson inequality with moduli of continuity in $B^p(\Omega_R)$, $0 < p \leq \infty$, where Ω_R is the arbitrary disc with radius R. For $1 \leq p \leq \infty$, M. Sh. Shabozov and O. Sh. Shabozov [12] find some exact constants of Jackson inequality with first and second degree moduli of smoothness in B^p. Also several questions related with the approximation by algebraic polynomials were considered by S. B. Vakarchuk [18] in B^p, $1 \leq p < \infty$. For $p > 0$, F. Ch. Xing [21] proved the Bernstein inequality and Quade-type converse inequalities [22] in B^p. For $0 < p \leq \infty$, M. Z. Wang and G. Ren [19] proved the direct theorem on polynomial approximation in $B^p(\Omega_R)$.

Above results are contain inequalities with moduli of continuity, except the results of Storozhenko [14] and Kryakin [7]. In this work we will generalize these results to the moduli of smoothness of arbitrary order. For $0 < p \leq \infty$, we will prove some direct and converse theorems of polynomial approximation in B^p for moduli of smoothness. In case of $1 < p < \infty$, we obtain simultaneous approximation theorem in B^p. The rests of the work organized as follows. Section 2 contains the main properties of the moduli of smoothness of functions in B^p, $0 < p \leq \infty$. In Section 3 we give a proof of first and second-type Jackson's direct theorems for B^p, $0 < p \leq \infty$. In Section 4 we prove simultaneous theorems of polynomial approximation in B^p, $1 < p < \infty$. In Section 5 using Szego composition we prove a Nikolskii–Stechkin inequality in B^p, $0 \leq p \leq \infty$. Then we obtain converse theorems of polynomial approximation in B^p, $0 < p \leq \infty$. Throughout the work we will denote by c and C positive constants which are different in different occurrences. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. By $j = \overline{0, m}$ we will mean $j = 0, 1, 2, 3, \ldots, m$.

2. Moduli of smoothness. Let $z, v \in \mathbb{C}$ and let f be a function defined on the points $z_j \in \mathbb{C}$, $j \in \mathbb{N}_0$. The divided difference of function f at the points z_j, $j \in \mathbb{N}_0$, is defined as

$$[z_0, z_1, \ldots, z_m]_f := \sum_{j=0}^{m} f(z_j) \prod_{i \neq j} (z_j - z_i)^{-1}. \tag{1}$$

Choosing $z_j = zv^j$, $j \in \mathbb{N}_0$, we define finite difference operator for $m \in \mathbb{N}$

$$[z, f]_v^0 := f(z), \quad [z, f]_v^m := [z, zv, \ldots, zv^m]_f \prod_{j=1}^{m} \left(z - zv^j\right).$$

From (1) we can see that

$$[z, f]_v^m = \sum_{j=0}^{m} (-1)^j P_{j,m}\left(v^{-1}\right) f\left(zv^j\right),$$

where

$$P_{j,m}(z) := \frac{z^{j(j-1)/2}\left(z^{j+1}-1\right)\ldots\left(z^m-1\right)}{(z-1)\ldots\left(z^{m-j}-1\right)}, \qquad j = \overline{0, m-1},$$

$$P_{m,m}(z) := z^{m(m-1)/2}.$$

We give the basic properties of finite difference operator: Let $z, v \in \mathbb{C}$ and $m, n \in \mathbb{N}$. Then

$$[z, f]_v^m = [z, f]_v^{m-1} - v^{-(m-1)} [zv, f]_v^{m-1}, \tag{2}$$

$$[z, f]_v^m = \sum_{j_1=0}^{1} \ldots \sum_{j_m=0}^{1} (-1)^{\sum_{l=1}^{m} j_l} v^{-\sum_{l=1}^{m}(l-1)j_l} f\left(zv^{\sum_{l=1}^{m} j_l}\right), \tag{3}$$

$$[z, f]_{v^n}^m = \sum_{j_1=0}^{n-1} \ldots \sum_{j_m=0}^{n-1} v^{-\sum_{l=1}^{m}(l-1)j_l} \left[zv^{\sum_{l=1}^{m} j_l}, f\right]_v^m. \tag{4}$$

If f is analytic in $\mathbb{D}$, and $|z| < 1$, $|v| \leq 1$, $i = \overline{0, m-1}$, we have

$$[z, f]_v^m = z^{m-i} \int_v^1 u_1^{m-i-1} du_1 \int_v^1 u_2^{m-i-2} du_2 \ldots \int_v^1 \left[zu_1 \ldots u_{m-i}, f^{(m-i)}\right]_v^i du_{m-i}. \tag{5}$$

Here $f^{(i)}(z) = \dfrac{d^i f(z)}{dz^i}$ is the ith derivative of $f(z)$.

We define generalized finite difference operator $[z, f]_v^{m,s}$ by

$$[z, f]_v^{0,s} := f(z), \quad [z, f]_v^{m,s} := [z, f]_v^{m-1,s} - v^{-(m+s-1)} [zv, f]_v^{m-1,s}.$$

By (2) we have $[z, f]_v^{m,0} = [z, f]_v^m$. On the other hand

$$[z, f]_v^{m,s} = \sum_{j_1=0}^{1} \ldots \sum_{j_m=0}^{1} (-1)^{\sum_{l=1}^{m} j_l} v^{-\sum_{l=1}^{m}(s+l-1)j_l} f\left(zv^{\sum_{l=1}^{m} j_l}\right).$$

If $k, m \in \mathbb{N}_0$ and $s \in \mathbb{Z}$, then

$$\frac{d}{dz}\left([z,f]_v^{m,s}\right)=\left[z,\frac{d}{dz}f\right]_v^{m,s-1} \qquad \text{and} \qquad [z,f]_v^{m+k}=\left[z,[\cdot,f]_v^{m,k}\right]_v^k. \tag{6}$$

All these properties can be proved by induction (see [7, 8, 15]). We define the moduli of smoothness as the following: Let $f\in B^p$, $0<p\le\infty$ and $\delta\ge 0$. The moduli of smoothness of order $m\in\mathbb{N}$ is defined as

$$\tilde{\omega}_m(\delta,f)_p:=\sup_{0\le t\le\delta}\left\|[\cdot,f]_{e^{it}}^m\right\|_p, \qquad 0<t\le 2\pi. \tag{7}$$

This moduli was first defined, on the unit circle $\mathbb{T}$, by Storozhenko [15] with $z=e^{i\varphi}$ and then Kryakin [7] proved its above properties on $\mathbb{D}$. Letting $m=1$ and $z=e^{i\varphi}$ in (7) this moduli coincides, on $\mathbb{T}$, with ordinary boundary moduli of continuity. If $m>1$, then $\tilde{\omega}_m(\cdot,f)_p$ and ordinary boundary moduli of smoothness are different.

It is easy to see that if $f,g\in B^p$, $0<p\le\infty$ and $0\le\delta\le\eta$, then, there exists a constant $c>0$, depending only on m and p, such that

$$\tilde{\omega}_m(\cdot,f)_p\le c\|f\|_p \text{ and } 0=\tilde{\omega}_m(0,f)_p\le\tilde{\omega}_m(\delta,f)_p\le\tilde{\omega}_m(\eta,f)_p,$$

$$\tilde{\omega}_m(\cdot,f+g)_p\le\tilde{\omega}_m(\cdot,f)_p+\tilde{\omega}_m(\cdot,f)_p.$$

Let $0<p\le\infty$ and $r\in\mathbb{N}$. We denote by B_r^p the class of functions $f\in B^p$ having the property $f^{(r)}\in B^p$.

Theorem 1. *Let $f\in B^p$, $0<p\le\infty$, $m,n\in\mathbb{N}$, $\delta\ge 0$ and $s:=\min\{1,p\}$. Then there exists a constant $c>0$, depending only on p, such that*

$$\tilde{\omega}_m(n\delta,f)_p\le cn^{m-1+(1/s)}\tilde{\omega}_m(\delta,f)_p.$$

Proof. Using (4) we have

$$\left\|[z,f]_{e^{int}}^m\right\|_p\le\left\{\iint\limits_{\mathbb{D}}\left[\sum_{j_1=0}^{n-1}\cdots\sum_{j_m=0}^{n-1}\left|\left[ze^{it\left(\sum_{l=1}^m j_l\right)},f\right]_{e^{it}}^m\right|\right]^p d\sigma(z)\right\}^{1/p}:=I^{1/p}.$$

For $p\ge 1$, $s=1$ we obtain

$$I^{1/p}\le 2^{(p-1)/p}\sum_{j_1=0}^{n-1}\cdots\sum_{j_m=0}^{n-1}\left\|\left[ze^{it\left(\sum_{l=1}^m j_l\right)},f\right]_{e^{it}}^m\right\|_p=$$

$$=2^{(p-1)/p}\sum_{j_1=0}^{n-1}\cdots\sum_{j_m=0}^{n-1}\left\|[z,f]_{e^{it}}^m\right\|_p\le 2^{(p-1)/p}n^m\left\|[z,f]_{e^{it}}^m\right\|_p=$$

$$=2^{(p-1)/p}n^{m-1+(1/s)}\left\|[z,f]_{e^{it}}^m\right\|_p$$

and hence $\tilde{\omega}_m(n\delta,f)_p\le cn^{m-1+(1/s)}\tilde{\omega}_m(\delta,f)_p$ for $p\ge 1$. Now we suppose $0<p<1$ and $s=p$. Then

$$\left\|[z,f]_{e^{int}}^m\right\|_p\le 2^{(1/p)-1}\sum_{j_1=0}^{n-1}\cdots\sum_{j_m=0}^{n-1}\left\|[z,f]_{e^{it}}^m\right\|_p.$$

Therefore

$$\tilde{\omega}_m\left(n\delta, f\right)_p \leq n^{m+(1/s)-1}\tilde{\omega}_m\left(\delta, f\right)_p.$$

Theorem 1 is proved.

Theorem 2. *Let $0 < p < 1$, $k \in \mathbb{N}$ and $f \in B_k^p$. Then there exist a constant $c > 0$ and a number $t_0 > 0$, depending only on p, k, such that*

$$\left\|[\cdot, f]_{e^{it}}^k\right\|_p \leq ct^k \left\|f^{(k)}\right\|_p$$

holds for all $0 < t \leq t_0$, where the constant c depends only on p.

Proof. We set $0 \leq r < R < 1$ and define $t_0 := (R-r)/Ak$, where $A \geq 1$ is some constant depending only on p. Then for all $0 < t \leq t_0$ we have [7] (Lemma 1)

$$M_p^p\left(r, [\cdot, f]_{e^{it}}^k\right) \leq c^p t^{kp} M_p^p\left(R, f^{(k)}\right).$$

Then

$$\int\limits_0^1 M_p^p\left(r, [\cdot, f]_{e^{it}}^k\right) rdr \leq c^p t^{kp} M_p^p\left(R, f^{(k)}\right) R,$$

$$\left\|[\cdot, f]_{e^{it}}^k\right\|_p^p \leq c^p t^{kp} \int\limits_0^1 M_p^p\left(R, f^{(k)}\right) RdR = c^p t^{kp} \left\|f^{(k)}\right\|_p^p.$$

Theorem 2 is proved.

Theorem 3. *Let $k \in \mathbb{N}$, $0 < p \leq \infty$, $f \in B_k^p$, $m \in \mathbb{N}_0$ and $\delta \geq 0$. Then there exist a constant $c > 0$ and a number $\delta_0 > 0$, depending only on p, k, such that*

$$\tilde{\omega}_{m+k}\left(\delta, f\right)_p \leq c\delta^k \tilde{\omega}_m\left(\delta, f^{(k)}\right)_p$$

holds for all $0 < \delta \leq \delta_0$, where the constant c depends only on p.

Proof. Let $p \geq 1$. Using (5) and the generalized Minkowski inequality for $|z| < 1$ we get

$$\left\|[z, f]_{e^{it}}^{k+m}\right\|_p = \left\|\int\limits_{e^{it}}^1 \ldots \int\limits_{e^{it}}^1 \left[zu_1 \ldots u_k, f^{(k)}\right]_{e^{it}}^m u_1^{k-1} \ldots z^k du_k \ldots du_1\right\|_p \leq$$

$$\leq \int\limits_{e^{it}}^1 \ldots \int\limits_{e^{it}}^1 \left\|\left[zu_1 \ldots u_k, f^{(k)}\right]_{e^{it}}^m\right\|_p du_k \ldots du_1 =$$

$$= \int\limits_{e^{it}}^1 \ldots \int\limits_{e^{it}}^1 \left\|\left[z, f^{(k)}\right]_{e^{it}}^m\right\|_p du_k \ldots du_1 \leq t^k \left\|\left[z, f^{(k)}\right]_{e^{it}}^m\right\|$$

and hence $\tilde{\omega}_{m+k}\left(\delta, f\right)_p \leq \delta^k \tilde{\omega}_m\left(\delta, f^{(k)}\right)_p$ for $p \geq 1$. Now let $0 < p < 1$, $\delta_0 := 1/(4Ak)$ and $0 < t \leq \delta_0$. Then using the second equality of (6), Theorem 2 and the first equality of (6), respectively, we have

$$\left\| [\cdot, f]_{e^{it}}^{k+m} \right\|_p = \left\| \left[\cdot, [\cdot, f]_{e^{it}}^{m,k} \right]_{e^{it}}^{k} \right\|_p \le ct^k \left\| \frac{d^k f(z)}{dz^k} \left([\cdot, f]_{e^{it}}^{m,k} \right) \right\|_p =$$

$$= ct^k \left\| \left(\left[\cdot, \frac{d^k f(z)}{dz^k} \right]_{e^{it}}^{m} \right) \right\|_p = ct^k \left\| \left(\left[\cdot, f^{(k)} \right]_{e^{it}}^{m} \right) \right\|_p$$

and $\tilde{\omega}_{m+k}(\delta, f)_p \le c\delta^k \tilde{\omega}_m \left(\delta, f^{(k)}\right)_p$ for $0 < p < 1$.

Theorem 3 is proved.

***Corollary* 1.** *Let $0 < p \le \infty$, $k \in \mathbb{N}$, $f \in B_k^p$ and $\delta \ge 0$. Then there exist a constant $c > 0$ and a number $\delta_0 > 0$, depending only on p, k, such that*

$$\tilde{\omega}_k(\delta, f)_p \le c\delta^k \left\| f^{(k)} \right\|_p$$

holds for all $0 < \delta \le \delta_0$, where the constant c depends only on p.

3. Direct theorems. In this section we will prove Jackson's fist- and second-type direct theorems in B^p, $0 < p \le \infty$. Let $E_n(f)_p := \inf \left\{ \|f - P_n\|_p : P_n \in \mathcal{P}_n \right\}$ for $n \in \mathbb{N}$, $0 < p \le \infty$ and $f \in B^p$. Here $\mathcal{P}_n$ is the set of the algebraic polynomials of degree at most n.

Lemma 1 [14]. *Let $0 < p < 1$, F be an analytic function in Ω_R. Then there exists a constant $c > 0$, depending only on p, such that*

$$\left(\int_{-\pi}^{\pi} \left| F\left(re^{i\varphi}\right) \right| d\varphi \right)^p \le C(\rho - r)^{p-1} \int_{-\pi}^{\pi} \left| F\left(\rho e^{i\varphi}\right) \right|^p d\varphi$$

hold for $0 \le r < \rho < R$. Furthermore, if $f \in H^p$, then the values $\rho = R = 1$ are assumed.

Jackson's fist-type direct theorem is the following theorem.

Theorem 4. *Let $0 < p \le \infty$, $f \in B^p$, $m, n \in \mathbb{N}$ and $n > m$. Then there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_p \le c\tilde{\omega}_m(1/n, f)_p . \tag{8}$$

Proof. Let $0 < r < 1$, $\alpha > 0$ and

$$K_n^\alpha\left(re^{it}\right) := \frac{1}{\begin{pmatrix} \alpha + n \\ n \end{pmatrix}} \frac{\left(re^{it}\right)^{-n}}{\left(1 - re^{it}\right)^{1+\alpha}}$$

be the kernels of $(C - \alpha)$-means of $f \in B^p$, $0 < p \le \infty$. In this case

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n^\alpha\left(re^{it}\right) dt = 1.$$

We set

$$\mathsf{P}_n(z) =: \sum_{j=1}^{m} (-1)^{j+1} \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{j,m}\left(\left(re^{it}\right)^{-1} \right) f\left(zr^j e^{ijt}\right) K_{n-m+1}^\alpha\left(re^{it}\right) dt.$$

The expression $\mathrm{P}_n(z)$ is a polynomial [8] of degree at most n. We will prove that

$$\|f - \mathrm{P}_n\|_p \leq c\tilde{\omega}_m(1/n, f)_p\,. \tag{9}$$

This implies (8). Let $0 < p < 1$, $\alpha > \dfrac{m+2}{p} - 1$ and $n_1 := n - m + 1$. Then

$$|f(z) - \mathrm{P}_n(z)| \leq \frac{1}{2\pi}\left|\int\limits_{-\pi}^{\pi}([z,f]^m_{re^{it}})\, K^{\alpha}_{n_1}\left(re^{it}\right)dt\right| =$$

$$= \frac{1}{\begin{pmatrix}\alpha+n_1\\ n_1\end{pmatrix}}\frac{1}{2\pi}\left|\int\limits_{-\pi}^{\pi}([z,f]^m_{re^{it}})\left(re^{it}\right)^{-n_1}\left(\frac{1-\left(re^{it}\right)^{n_1+1}}{1-re^{it}}\right)^{1+\alpha}dt\right| \leq$$

$$\leq \frac{r^{-n_1}}{\begin{pmatrix}\alpha+n_1\\ n_1\end{pmatrix}}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|[z,f]^m_{re^{it}}|\left|\left(\frac{1-\left(re^{it}\right)^{n_1+1}}{1-re^{it}}\right)\right|^{1+\alpha}dt.$$

Since the function $F(\varsigma) := [z,f]^m_{\varsigma}\left(\dfrac{1-\varsigma^{n+1}}{1-\varsigma}\right)^{1+\alpha}$ is analytic and belong to H^p, from Lemma 1 we obtain

$$|f(z) - \mathrm{P}_n(z)|^p \leq \frac{Cr^{-n_1p}}{\begin{pmatrix}\alpha+n_1\\ n_1\end{pmatrix}^p(1-r)^{1-p}}\int\limits_{-\pi}^{\pi}|[z,f]^m_{e^{it}}|^p\left|\left(\frac{1-\left(e^{it}\right)^{n_1+1}}{1-e^{it}}\right)\right|^{(1+\alpha)p}dt.$$

Integrating the last inequality with respect to z we get

$$\|f - \mathrm{P}_n\|_p^p \leq \frac{Cr^{-n_1p}}{\begin{pmatrix}\alpha+n_1\\ n_1\end{pmatrix}^p(1-r)^{1-p}}\int\limits_{0}^{\pi}\tilde{\omega}^p_m(t,f)_p\left|\frac{\sin n_1t/2}{\sin t/2}\right|^{(1+\alpha)p}dt.$$

Taking $r = 1 - (1/n_1)$ we have $(1-r)^{p-1} = (n_1)^{1-p}$ and $\begin{pmatrix}\alpha+n_1\\ n_1\end{pmatrix}^{-p} \leq c\,(n_1)^{-\alpha p}$. Hence

$$\|f - \mathrm{P}_n\|_p^p \leq C\,(n_1)^{1-(\alpha+1)p}\int\limits_{0}^{\pi}\tilde{\omega}^p_m(t,f)_p\left|\frac{\sin n_1t/2}{\sin t/2}\right|^{(1+\alpha)p}dt =$$

$$= C\,(n_1)^{1-(\alpha+1)p}\left\{\int\limits_{0}^{1/n_1}\tilde{\omega}^p_m(t,f)_p\left|\frac{\sin n_1t/2}{\sin t/2}\right|^{(1+\alpha)p}dt + \right.$$

$$\left. + \int\limits_{1/n_1}^{\pi}\tilde{\omega}^p_m(t,f)_p\left|\frac{\sin n_1t/2}{\sin t/2}\right|^{(1+\alpha)p}dt\right\}.$$

Therefore using the property $\tilde{\omega}_l^p(\lambda\delta, f)_p \le (\lambda+1)^l \tilde{\omega}_l^p(\delta, f)_p$, $\lambda, \delta > 0$, of moduli we obtain

$$\|f - \mathrm{P}_n\|_p^p \le C(n_1)^{1-(\alpha+1)p}\tilde{\omega}_m^p(1/n_1, f)_p \left\{ (n_1)^{(\alpha+1)p} \int\limits_0^{1/n_1} dt + \right.$$

$$\left. + \int\limits_{1/n_1}^{\pi} (n_1 t + 1)^m \left(\frac{t}{\pi}\right)^{-(1+\alpha)p} dt \right\} \le c\tilde{\omega}_m(1/n_1, f)_p \le c\tilde{\omega}_m(1/n, f)_p .$$

Let $p \ge 1$ and $\alpha > m$. Using generalized Minkowski's inequality we get

$$\|f - \mathrm{P}_n\|_p \le \frac{1}{\begin{pmatrix} \alpha + n_1 \\ n_1 \end{pmatrix}} \int\limits_{-\pi}^{\pi} \|[\cdot, f]_{e^{it}}^m\|_p \left|\frac{\sin n_1 t/2}{\sin t/2}\right|^{1+\alpha} dt \le$$

$$\le C(n_1)^{-\alpha}\tilde{\omega}_m(1/n_1, f)_p \left\{ (n_1)^{1+\alpha} \int\limits_0^{1/n_1} dt + (n_1)^m \int\limits_{1/n_1}^{\pi} \left(\frac{t}{\pi}\right)^{-(1+\alpha)} dt \right\} \le$$

$$\le c\tilde{\omega}_m(1/n_1, f)_p \le c\tilde{\omega}_m(1/n, f)_p$$

and (9) follows.

Theorem 4 is proved.

We note that the last theorem is a generalization of the results of Storozhenko [14, p. 207] (Theorem 1), Kryakin [7, p. 26] (Theorem 1), Xing and Su [23] ($1 \le p < \infty$) and X. C. Shen and F. Ch. Xing [13] ($0 < p < 1$), Kryakin and Trebels [8] (Theorem 2.3) ($0 < p < \infty$), Ren and Wang [11] (Theorem 3.5) ($0 < p \le \infty$), Ren and Wang [19] (Theorem 3.4) ($0 < p \le \infty$).

From Theorem 4 we have the following Jackson's second-type direct theorem.

Theorem 5. *Let $m, n, k \in \mathbb{N}$, $n > m + k$, $0 < p \le \infty$ and $f \in B_k^p$. Then there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_p \le cn^{-k}\tilde{\omega}_m\left(1/n, f^{(k)}\right)_p$$

holds.

This theorem generalizes Theorem 3 of [7].

Corollary 2. *Let $k, n \in \mathbb{N}$, $0 < p \le \infty$ and $f \in B_k^p$. Then there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_p \le cn^{-k}\left\|f^{(k)}\right\|_p$$

holds for $n > k$.

4. Simultaneous approximation. We suppose that $f \in B^p$, $0 < p \le \infty$, has Taylor expansion

$$\sum_{k=0}^{\infty} c_k(f) z^k. \tag{10}$$

Let $\mathrm{T}_n(f) := \mathrm{T}_n(f,z) := \sum_{k=0}^{n} c_k(f)\, z^k$ be the nth partial sum of (10). The operator $\mathrm{T}_n : B^p \to B^p,\ \ f \to \mathrm{T}_n(f,\cdot)$ will be called the nth partial sum operator for $f \in B^p$. We note that if $1 < p < \infty$, then the operator T_n is bounded [5] on B^p. Hence there exists a constant $c > 0$ such that

$$\|\mathrm{T}_n(f,\cdot)\|_p \le c\,\|f\|_p \tag{11}$$

holds for $f \in B^p$.

Let $\alpha_n^r := 1/\left(n(n-1)(n-2)(n-r+1)\right)$.

Theorem 6. *Let $n, r \in \mathbb{N}$, $1 \le p \le \infty$ and $f \in B_r^p$. Then*

$$\|f - \mathrm{T}_n(f)\|_p \le \alpha_n^r \left\| f^{(r)} - \mathrm{T}_{n-r-1}\left(f^{(r)}\right)\right\|_p$$

holds for $r < n$.

Proof. Let $\rho \in [0,1)$ and $f_\rho(z) := f(\rho z)$ be dilation operator. Since $f \in B^p$, $1 \le p \le \infty$, then $f_\rho \in H^p$. Hence [20, p. 158]

$$f_\rho\left(e^{i\theta}\right) - \mathrm{T}_n\left(f_\rho, e^{i\theta}\right) = \frac{e^{ir\theta}}{2\pi}\int_0^{2\pi}\left(f_\rho^{(r)}\left(e^{i(\theta+t)}\right) - \mathrm{T}_{n-r-1}\left(f_\rho^{(r)}, e^{i(\theta+t)}\right)\right)\frac{\varphi(t)}{e^{i(n-r)t}}dt,$$

where $\varphi(t) = \alpha_n^r + 2\sum_{k=1}^{\infty}\alpha_{k+n}^r \cos kt \ge 0$. Using generalized Minkowski inequality we get

$$M_p^p(\rho, f - \mathrm{T}_n(f)) = \frac{1}{2\pi}\int_0^{2\pi}\left|f_\rho\left(e^{i\theta}\right) - \mathrm{T}_n\left(f_\rho, e^{i\theta}\right)\right|^p d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{e^{ir\theta}}{2\pi}\int_0^{2\pi}\left(f_\rho^{(r)}\left(e^{i(\theta+t)}\right) - \mathrm{T}_{n-r-1}\left(f_\rho^{(r)}, e^{i(\theta+t)}\right)\right)e^{-i(n-r)t}\varphi(t)\,dt\right|^p d\theta \le$$

$$\le \frac{M_p^p\left(\rho, f^{(r)} - \mathrm{T}_{n-r-1}\left(f^{(r)}\right)\right)}{2\pi}\int_0^{2\pi}\left|e^{-i(n-r)t}\varphi(t)\right|dt \le$$

$$\le \alpha_n^r M_p^p\left(\rho, f^{(r)} - \mathrm{T}_{n-r-1}\left(f^{(r)}\right)\right)$$

and hence the required result

$$\|f - \mathrm{T}_n(f)\|_p \le \alpha_n^r \left\| f^{(r)} - \mathrm{T}_{n-r-1}\left(f^{(r)}\right)\right\|_p$$

follows.

Theorem 6 is proved.

From Theorem 6 and (11) we have the following corollary.

***Corollary* 3.** *Let $n, r \in \mathbb{N}$, $1 < p < \infty$ and $f \in B_r^p$. Then*

$$E_n(f)_p \le c n^{-r} E_{n-r}\left(f^{(r)}\right)_p$$

holds for $r < n$.

Bernstein inequality for B^p, $0<p\le\infty$.

Lemma 2. *Let* $n\in\mathbb{N}$ *and* $P_n\in\mathcal{P}_n$ *and* $0<p\le\infty$. *Then there exists a constant* $c>0$, *depending only on* p, *such that*

$$\left\|P_n'\right\|_p\le cn\left\|P_n\right\|_p.$$

When $p\ge1$ *the constant* c *can be chosen to be* 4.

This lemma was proved for $p\ge1$ in [23], for $p>0$ in [21, p. 432] (the main theorem).

Theorem 7. *Let* $n,r\in\mathbb{N}$, $1<p<\infty$, $f\in B_r^p$, $P_n^*\in\mathcal{P}_n$ *and* $E_n(f)_p=\|f-P_n^*\|_p$. *Then for all* $k=\overline{0,r}$

$$\left\|f^{(k)}-P_n^{*(k)}\right\|_p\le cn^{k-r}E_{n-r}\left(f^{(r)}\right)_p$$

holds for $r\le n$.

Proof. Let $q_{n-k}\in\mathcal{P}_{n-k}$, $E_{n-k}\left(f^{(k)}\right)_p=\left\|f^{(k)}-q_{n-k}\right\|_p$. Then using Corollary 3

$$\left\|f^{(k)}-P_n^{*(k)}\right\|_p\le\left\|f^{(k)}-\mathrm{T}_{n-k}\left(f^{(k)}\right)\right\|_p+\left\|\mathrm{T}_n^{(k)}(f)-P_n^{*(k)}\right\|_p\le$$

$$\le\left\|f^{(k)}-q_{n-k}\right\|_p+\left\|q_{n-k}-\mathrm{T}_{n-k}\left(f^{(k)}\right)\right\|_p+\left\|\left(\mathrm{T}_n(f)-P_n^*\right)^{(k)}\right\|_p\le$$

$$\le E_{n-k}\left(f^{(k)}\right)_p+\left\|\mathrm{T}_{n-k}(q_{n-k})-\mathrm{T}_{n-k}\left(f^{(k)}\right)\right\|_p+cn^k\left\|\mathrm{T}_n(f)-P_n^*\right\|_p\le$$

$$\le(1+c)E_{n-k}\left(f^{(k)}\right)_p+cn^k\left\|\mathrm{T}_n(f)-\mathrm{T}_n(P_n^*)\right\|_p\le$$

$$\le cn^{k-r}E_{n-r}\left(f^{(r)}\right)_p+cn^kE_n(f)_p\le cn^{k-r}E_{n-r}\left(f^{(r)}\right)_p$$

and the theorem is proved.

Theorem 8. *Let* $n,r\in\mathbb{N}$, $1<p<\infty$ *and* $f\in B_r^p$. *Then there exists a* $\Phi_n\in\mathcal{P}_n$ *such that for all* $k=\overline{0,r}$

$$\left\|f^{(k)}-\Phi_n{}^{(k)}\right\|_p\le cn^{k-r}\tilde{\omega}_{n-r}\left(1/n,f^{(r)}\right)_p$$

holds for $r\le n$.

Proof. Let $P_n^*\in\mathcal{P}_n$, $E_n(f)_p=\|f-P_n^*\|_p$ and $\Phi_n=\mathrm{P}_n$. From (9) and Theorem 5 we have

$$\|f-\mathrm{P}_n\|_p\le cn^{-r}\tilde{\omega}_{n-r}\left(1/n,f^{(r)}\right)_p\quad\text{and}\quad\|f-P_n^*\|_p\le cn^{-r}\tilde{\omega}_{n-r}\left(1/n,f^{(r)}\right)_p.$$

On the other hand we get

$$\|\mathrm{P}_n-P_n^*\|_p\le 2cn^{-r}\tilde{\omega}_{n-r}\left(1/n,f^{(r)}\right)_p.$$

Hence by Bernstein inequality in Lemma 2 we obtain

$$\left\|f^{(k)}-\mathrm{P}_n^{(k)}\right\|_p\le\left\|f^{(k)}-P_n^{*(k)}\right\|_p+\left\|\mathrm{P}_n^{(k)}-P_n^{*(k)}\right\|_p\le$$

$$\le cn^{k-r}E_{n-r}\left(f^{(r)}\right)_p+cn^{k-r}\tilde{\omega}_{n-r}\left(1/n,f^{(r)}\right)_p\le cn^{k-r}\tilde{\omega}_{n-r}\left(1/n,f^{(r)}\right)_p.$$

Theorem 8 is proved.

5. Converse theorems. Bernstein inequalities are play the central role in the proof of converse approximation theorems in Approximation Theory. For the spaces B^p these inequalities was established by [23] $(1 \le p)$ and [21] $(p > 0)$. But in some cases Bernstein inequalities are improved to Nikolskii – Stechkin inequalities. In the present section we prove a Nikolskii – Stechkin-type inequality in B^p, $0 \le p \le \infty$. For this purpose we use a Szego composition theorem for polynomials in H^p, $0 \le p \le \infty$. Let $P_n, \Lambda_n \in \mathcal{P}_n$ and we write them of the form $P_n(z) = \sum_{k=0}^{n} C_n^k c_k z^k$, $\Lambda_n(z) = \sum_{k=0}^{n} C_n^k \lambda_k z^k$. Here C_n^k are the Binom coefficients. The Szego composition of P_n and Λ_n will be defined as the polynomial $P_n \otimes \Lambda_n := \sum_{k=0}^{n} C_n^k c_k \lambda_k z^k$.

Theorem 9 [2]. *Let $n \in \mathbb{N}$, $P_n, \Lambda_n \in \mathcal{P}_n$ and $P_n \otimes \Lambda_n$ be the Szego composition of P_n and Λ_n. Then*

$$\|P_n \otimes \Lambda_n\|_{H^p} \le \|\Lambda_n\|_{H^0} \|P_n\|_{H^p}, \qquad 0 \le p \le \infty.$$

We prove a Nikolskii – Stechkin-type inequality.

Theorem 10. *Let $n, m \in \mathbb{N}$, $n > m$, $P_n \in \mathcal{P}_n$ and $0 < t < (2\pi/n)$. Then*

$$\left\|P_n^{(m)}(z)\right\|_p \le \prod_{j=1}^{m} \frac{(n-m+j)\, 2^{(n-m+j-1)/2}}{\sin(n-m+j)\, t/2} \|[z, P_n]_{e^{it}}^m\|_p, \quad 0 \le p \le \infty.$$

Proof. Let $m = 1$ and $q = e^{it}$. Since the polynomials $P_n'(z)$ and $[z, P_n]_q^1$ are independent of the constant term of $P_n(z)$, we may take $P_n(z)$ of the form

$$P_n(z) = z \sum_{k=0}^{n-1} C_{n-1}^k a_k z^k.$$

Then

$$P_n'(z) = \sum_{k=0}^{n-1} C_{n-1}^k a_k (k+1) z^k \text{ and } [z, P_n]_q^1 = z \sum_{k=0}^{n-1} C_{n-1}^k a_k \left(1 - e^{i(k+1)t}\right) z^k.$$

Now taking

$$Q_{n-1}(z) = \sum_{k=0}^{n-1} C_{n-1}^k \frac{k+1}{1 - e^{i(k+1)t}} z^k$$

we get

$$P_n'(z) = \frac{1}{z} [z, P_n]_q^1 \otimes Q_{n-1}(z).$$

Using Theorem 9

$$\left\|P_n'(z)\right\|_{H^p} \le \|Q_{n-1}(z)\|_{H^0} \left\|[z, P_n]_q^1\right\|_{H^p}.$$

From [16] (inequalities (8) and (10)) we have

$$\|Q_{n-1}(z)\|_{H^0} \le \frac{n 2^{(n-1)/2}}{\sin nt/2}$$

and hence

$$\left\|P_n'(z)\right\|_{H^p} \le \frac{n 2^{(n-1)/2}}{\sin nt/2} \left\|[z, P_n]_q^1\right\|_{H^p}.$$

By the dilation function method (see, for example, [26]) we conclude for $m=1$

$$\left\|P_n'\right\|_p \le \frac{n2^{(n-1)/2}}{\sin nt/2}\left\|[\cdot,P_n]_q^1\right\|_p, \qquad 0\le p\le\infty. \tag{12}$$

For $m>1$ using (12) and inequalities in (6) we find

$$\left\|P_n^{(m)}\right\|_p \le \frac{(n-m+1)\,2^{(n-m)/2}}{\sin(n-m+1)\,t/2}\left\|\left[\cdot,P_n^{(m-1)}\right]_q^1\right\|_p =$$

$$= \frac{(n-m+1)\,2^{(n-m)/2}}{\sin(n-m+1)\,t/2}\left\|\frac{d}{dz}\left(\left[\cdot,P_n^{(m-2)}\right]_q^{1,1}\right)\right\|_p \le$$

$$\le \prod_{j=1}^{2}\frac{(n-m+j)\,2^{(n-m+j-1)/2}}{\sin(n-m+j)\,t/2}\left\|\left[\cdot,\left[\cdot,P_n^{(m-2)}\right]_q^{1,1}\right]_q^1\right\|_p =$$

$$= \prod_{j=1}^{2}\frac{(n-m+j)\,2^{(n-m+j-1)/2}}{\sin(n-m+j)\,t/2}\left\|\left[\cdot,P_n^{(m-2)}\right]_q^2\right\|_p \le \ldots$$

$$\ldots \le \prod_{j=1}^{m}\frac{(n-m+j)\,2^{(n-m+j-1)/2}}{\sin(n-m+j)\,t/2}\left\|[\cdot,P_n]_q^m\right\|_p.$$

Theorem 10 is proved.

As a corollary we get the following Bernstein-type inequality:

***Corollary* 4.** *Let $n,m\in\mathbb{N}$, $P_n\in\mathcal{P}_n$, $0<t<(2\pi/n)$ and $n>m$. Then*

$$\left\|P_n^{(m)}(z)\right\|_p \le c\prod_{j=1}^{m}\frac{(n-m+j)\,2^{(n-m+j-1)/2}}{\sin(n-m+j)\,t/2}\left\|P_n\right\|_p$$

holds for $0\le p\le\infty$.

From Theorem 5.1 of [4, p. 216, 217] and Lemma 2 we have the following corollary.

***Corollary* 5.** *Let $r,n\in\mathbb{N}$, $0<p\le\infty$ and $f\in B^p$. Then there exists a constant $c>0$ independent of n such that*

$$K_r(f,1/n) \le \frac{c}{n^r}\left\{\sum_{k=1}^{n}\left[k^rE_k(f)_p\right]^\mu\frac{1}{k}\right\}^{1/\mu}$$

hold where $\mu:=\min\{1,p\}$ and $K_r(f,\delta):=\inf_{g\in B_r^p}\left\{\|f-g\|_p+\delta^r\left\|g^{(r)}\right\|_p\right\}$, $\delta\ge0$ is the Peetre K-functional.

Theorem 11. *Let $m\in\mathbb{N}$, $0<p\le\infty$ and $f\in B^p$. Then there exists a constant $c>0$ independent of n such that*

$$\tilde{\omega}_m(1/n,f)_p \le \frac{c}{n^r}\left\{\sum_{k=1}^{n}\left[k^rE_k(f)_p\right]^\mu\frac{1}{k}\right\}^{1/\mu}$$

hold where $\mu:=\min\{1,p\}$.

Proof. Using Corollary 1 we have

$$\tilde{\omega}_m^{\mu}\left(1/n, f\right)_p \leq c\left(\|f-g\|_p^{\mu} + (1/n)^{\mu m}\left\|g^{(m)}\right\|_p^{\mu}\right) \leq cK_m^{\mu}\left(1/n, f\right)_p$$

and the estimate $\tilde{\omega}_m\left(1/n, f\right)_p \leq cK_m\left(1/n, f\right)_p$ fulfilled. By Corollary 5 we conclude the required result.

Theorem 11 is proved.

This theorem is a generalization of converse theorem of [13] ($0 < p < 1$).

As an application of Theorems 11 and 4 we get the Marchaud inequality for $0 < p < 1$.

Corollary 6. *Let $r, m \in \mathbb{N}$, $r > m$, $0 < p < 1$ and $f \in B^p$. Then for all $0 < \delta < 1/r$*

$$\tilde{\omega}_m\left(\delta, f\right)_p \leq c\delta^m \left\{\int_{\delta}^{1/r} \frac{\tilde{\omega}_r\left(t, f\right)_p^p}{t^{mp+1}} dt\right\}^{1/p}$$

holds.

Definition 1. *Let $\varphi\left(t\right)$ be a positive function for $t > 0$ and $\lim_{t\to 0}\varphi(t) = 0$. We suppose that $0 < p \leq \infty$, $m \in \mathbb{N}$,*

$$H_{p,m}^{\varphi} := \left\{f \in B^p : \tilde{\omega}_m\left(t, f\right)_p = \mathcal{O}\left(\varphi\left(t\right)\right), t \to 0\right\} \qquad \textit{and} \qquad \mathrm{Lip}\left(\alpha, p\right) \equiv H_{p,m}^{t^{\alpha}},\ \alpha > 0.$$

Corollary 7. *Let $n \in \mathbb{N}$ and $f \in B^p$. In this case*

(A) *Suppose that* $0 < p < 1$. *Then*:

(i) *for $0 < \alpha < 1$ the conditions $E_n\left(f\right)_p = O\left(n^{-\alpha}\right)$ and $f \in \mathrm{Lip}\left(\alpha, p\right)$ are equivalent*;

(ii) *for $\alpha = 1$ the condition $E_n\left(f\right)_p = O\left(n^{-1}\right)$ implies $\tilde{\omega}_m\left(\delta, f\right)_p = O\left(\delta\left|\ln\delta\right|^{1/p}\right)$*;

(iii) *for $\alpha > 1$ the condition $E_n\left(f\right)_p = O\left(n^{-\alpha}\right)$ implies $f \in \mathrm{Lip}\left(1, p\right)$.*

(B) *Suppose that* $1 \leq p \leq \infty$. *Then*:

(i) *for $0 < \alpha < 1$ the conditions $E_n\left(f\right)_p = O\left(n^{-\alpha}\right)$ and $f \in \mathrm{Lip}\left(\alpha, p\right)$ are equivalent*;

(ii) *for $\alpha = 1$ the condition $E_n\left(f\right)_p = O\left(n^{-1}\right)$ implies $\tilde{\omega}_m\left(\delta, f\right)_p = O\left(\delta\left|\ln\delta\right|\right)$.*

Corollary 8. *Let $n, m \in \mathbb{N}$, $0 < p < 1$ and $f \in B^p$. If*

$$\int_{\delta}^{1} \frac{\varphi\left(t\right)^p}{t^{mp+1}} dt = O\left(\frac{\varphi\left(\delta\right)^p}{\delta^{mp}}\right), \qquad \delta \to 0,$$

then, the conditions $f \in H_{p,m}^{\varphi}$ and $E_{2^n}\left(f\right)_p = O\left(\varphi\left(2^{-n}\right)\right)$, $n \to \infty$, are equivalent.

6. Concluding remarks. Theorems 7 and 8 for $0 < p < 1$ and for $p = \infty$, remain open. It may be interesting to investigate above problems in B^0 or $B^p\left(\Omega\right)$ for general domains Ω.

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