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A NOTE ON DEGREE OF APPROXIMATION BY MATRIX MEANS IN GENERALIZED HÖLDER METRIC

ПРО СТУПІНЬ АПРОКСИМАЦІЇ МАТРИЧНИМИ СЕРЕДНІМИ В УЗАГАЛЬНЕНІЙ МЕТРИЦІ ГЕЛЬДЕРА

The aim of the paper is to determine the degree of approximation of functions by matrix means of their Fourier series in a new space of functions introduced by Das, Nath, and Ray. In particular, we extend some results of Leindler and some other results by weakening the monotonicity conditions in results obtained by Singh and Sonker for some classes of numerical sequences introduced by Mohapatra and Szal and present new results by using matrix means.

Визначено ступінь апроксимації функцій матричними середніми їх рядів Фур'є в новому просторі функцій, уведених Дасом, Насом та Реєм. Зокрема, розширено деякі результати Лейндлера, а також деякі інші результати шляхом послаблення умов монотонності в результатах, отриманих Сінгхом та Сонкером для деяких класів числових послідовностей, що були введені Могапатра та Салом, та наведено нові результати, отримані за допомогою матричних середніх.

1. Notations and background. Let f be a 2π-periodic function and $f \in L_p := L_p\,(0, 2\pi)$ for $p \geq 1$. Then we write

$$s_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} U_k(f;x)$$

partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L_p$, $p \geq 1$, at a point x. There are numerous papers devoted to the approximations by partial sums of Fourier series by many mathematicians, as Quade [8], Chandra [1], Das, Nath and Ray [2], Leindler [4, 5].

Throughout this work $\|.\|_p$ will denote L_p-norm with respect to x and will be defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}.$$

Moreover, modulus of continuity of $f \in C_{2\pi}$ is defined by

$$\omega(f, \delta) := \sup_{|h| \leq \delta} |f(x+h) - f(x)|,$$

where $C_{2\pi}$ is space of all 2π-periodic and continuous functions defined on $\mathbb{R}$ with the supremum norm. The class of functions H^ω is defined as the following:

$$H^\omega := \{f \in C_{2\pi} : \omega(f, \delta) = O(\omega(\delta))\},$$

where $\omega(\delta)$ is a modulus of continuity. The degree of approximation of functions from classes H^ω in various spaces has been studied by Leindler [5], Mazhar and Totik [6], Das, Nath and Ray [2]. A further generalization of H^ω space has been given by Das, Nath and Ray in [2]. They defined the following notations: If $f \in L_p(0, 2\pi)$, $p \geq 1$, then denote

$$H_p^{(\omega)} := \{f \in L_p(0, 2\pi), p \geq 1 \colon A(f, \omega) < \infty\},$$

where ω is a modulus of continuity and

$$A(f, \omega) := \sup_{t \neq 0} \frac{||f(\cdot + t) - f(\cdot)||_p}{\omega(|t|)}.$$

The norm in the space $H_p^{(\omega)}$ is defined by

$$||f||_p^{(\omega)} := ||f||_p + A(f, \omega).$$

In [2] they proved the following theorem.

Theorem 1.1. *Let* v *and* ω *be moduli of continuity such that* $\dfrac{\omega(t)}{v(t)}$ *is nondecreasing. Moreover, if* $f \in H_p^{(\omega)}$, $p \geq 1$, *then*

$$\|s_n - f\|_p^{(v)} = O\left(\frac{\omega(\pi/n)}{v(\pi/n)} \log n + \frac{1}{n} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t^2 v(t)} dt\right).$$

In [5], Leindler has established the following result improving Theorem 1.1.

Theorem 1.2. *Let* v *and* ω *be moduli of continuity such that* $\dfrac{\omega(t)}{v(t)}$ *is nondecreasing. Moreover, let the function*

$$\eta(t) := \eta(v, \omega, \varepsilon; t) := t^{-\varepsilon} \frac{\omega(t)}{v(t)}$$

be nonincreasing for some $0 < \varepsilon \leq 1$. *If* $f \in H_p^{(\omega)}$, $p \geq 1$, *then*

$$\|s_n - f\|_p^{(v)} \leq \frac{\omega(\pi/n)}{v(\pi/n)} \log n \tag{1.1}$$

for all $n \geq 2$.

Due to this theorem, Leindler showed that if there exists $\varepsilon > 0$ such that $t^{-\varepsilon} \dfrac{\omega(t)}{v(t)}$ is also nonincreasing, then in Theorem 1.1 the second term can be omitted. In this paper of Leindler, he also considered the degree of approximation of $f \in H_p^{(\omega)}$ by Nörlund[1], Riesz and generalized de la Vallée Poussin means defined as follows:

$$N_n(f; x) := \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} s_m(f; x),$$

$$R_n(f; x) := \frac{1}{P_n} \sum_{m=0}^{n} p_m s_m(f; x),$$

where $P_n = p_0 + p_1 + p_2 + ... + p_n \neq 0$, $n \geq 0$, and by convention $p_{-1} = P_{-1} = 0$; and

[1]Called the "Woronoi's transformations" instead of "Nörlund's transformations (or Nörlund means)" by Russian mathematicians. See [10] for more detailed information.

$$V_n(x) := V_n(\lambda, f; x) := \frac{1}{\lambda_{n+1}} \sum_{m=n-\lambda_n}^{n} s_m(f; x),$$

where (λ_n) is a nondecreasing sequence of positive integers with $\lambda_0 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. In this studying, we shall consider the degree of approximation of $f \in H_p^{(\omega)}$ with the norm in the space $H_p^{(\omega)}$ by trigonometrical polynomials $\tau_n(f; x)$, where

$$\tau_n(f; x) = \tau_n(f, T; x) := \sum_{m=0}^{n} a_{n,m} s_m(f; x) \quad \forall n \geq 0.$$

Throughout $T \equiv (a_{n,m})$ is a lower triangular infinite matrix of nonnegative real numbers such that:

$$a_{n,m} = \begin{cases} \geq 0, & m \leq n, \\ 0, & m > n, \end{cases} \qquad n, m = 0, 1, 2, \ldots, \tag{1.2}$$

$$\sum_{m=0}^{n} a_{n,m} = 1, \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

The Fourier series of signal f is called to be T-summable to s, if $\tau_n(f; x) \to s$ as $n \to \infty$. On the other hand, we know that a summability method is regular, if for every convergent sequence (s_n), $\lim_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} \tau_n = s$. If we take

$$a_{n,m} = \begin{cases} \dfrac{1}{n+1}, & m \leq n, \\ 0, & m > n, \end{cases} \qquad n, m = 0, 1, 2, \ldots,$$

$\tau_n(f; x)$ gives us

$$\sigma_n(f; x) = \frac{1}{n+1} \sum_{m=0}^{n} s_m(f; x) \quad \forall n \geq 0.$$

Throughout this studying, we shall use notations $D \ll R$ $(R \ll D)$ in inequalities if there exists a positive constant K such that $D \leq KR$ $(R \leq KD)$ where D and R are depend on n. However, K may be different in different occurrences of " $\ll$ ".

Now, let's recall the definitions of some classes of numerical sequences discussed in detail in [5] and [7]. Let $u := (u_n)$ be a nonnegative infinite sequence and $C := (C_n) = \dfrac{1}{n+1} \sum_{m=0}^{n} u_m$.

A sequence u is called almost monotone decreasing (briefly $u \in AMDS$) (increasing (briefly $u \in AMIS$)), if there exists a constant $K := K(u)$ which only depends on u such that

$$u_n \leq K u_m \qquad (u_n \geq K u_m)$$

for all $n \geq m$.

If $C \in AMDS$ $(C \in AMIS)$, then we say that the sequence u is almost monotone decreasing (increasing) mean sequence and denoted by $C \in AMDMS$ $(C \in AMIMS)$.

A sequence u tending to zero is called a rest bounded variation sequence $(RBVS)$ (rest bounded variation mean sequence $(RBVMS)$), if it has the property

$$\sum_{m=k}^{\infty} |\Delta u_m| \le K(u) u_k \quad \left(\sum_{m=k}^{\infty} |\Delta C_m| \le K(u) C_k \right)$$

for all natural numbers k, where $\Delta u_n = u_n - u_{n+1}$.

A sequence u is called a head bounded variation sequence ($HBVS$) (head bounded variation mean sequence ($HBVMS$)), if it has the property

$$\sum_{m=0}^{k-1} |\Delta u_m| \le K(u) u_k \quad \left(\sum_{m=0}^{k-1} |\Delta C_m| \le K(u) C_k \right)$$

for all natural numbers k, or only for all $k \le N$ if the sequence u has only finite nonzero terms and the last nonzero term u_N.

When a matrix $T = (a_{n,m})$ belongs to one of the above classes, it means that it satisfies the required conditions from the above definitions with respect to $m = 0, 1, 2, \ldots, n$ for all n. Accordingly if $(a_{n,m})_{m=0}^{\infty}$ belongs to $RBVS$ $(RBVMS)$ or $HBVS$ $(HBVMS)$, respectively then

$$\sum_{m=k}^{\infty} |\Delta_m a_{n,m}| \le K a_{n,k} \quad \left(\sum_{m=k}^{\infty} |\Delta_m A_{n,m}| \le K A_{n,k} \right),$$

$$\sum_{m=0}^{k-1} |\Delta_m a_{n,m}| \le K a_{n,k} \quad \left(\sum_{m=0}^{k-1} |\Delta_m A_{n,m}| \le K A_{n,k} \right),$$

where

$$A_{n,m} = \frac{1}{m+1} \sum_{k=0}^{m} a_{n,k}$$

for all n $(0 \le k \le n)$ and $\Delta_m a_{n,m} = a_{n,m} - a_{n,m+1}$.

It is clear that the following inclusions are true for the above classes of numerical sequences:

$$RBVS \subset AMDS\,, \quad RBVMS \subset AMDMS$$

and

$$HBVS \subset AMIS\,, \quad HBVMS \subset AMIMS.$$

Moreover, Mohapatra and Szal in [7] showed that the following embedding relations are true:

$$AMDS \subset AMDMS$$

and

$$AMIS \subset AMIMS.$$

Taking into these inclusions, we will extend the some results given in [9] by weakening the monotonicity conditions. Furthermore we shall give the degree of approximation of functions by matrix means of their Fourier series under the new conditions. We see that the results obtained in this paper generalize the some results in [3, 5, 9].

2. Main result. The following theorem shall extend the some results of Leindler in [5] and the some of the their results by weakening the monotonicity conditions in results of Singh and Sonker [9] for some classes of numerical sequences that given by Mohapatra and Szal in [7]. Also it includes some result under a new condition. Accordingly, the main result is as follows:

Theorem 2.1. *Let v and ω be moduli of continuity such that $\dfrac{\omega(t)}{v(t)}$ is nondecreasing. Moreover, let the function*

$$\eta(t) := \eta(v, \omega, \varepsilon; t) := t^{-\varepsilon}\frac{\omega(t)}{v(t)}$$

be nonincreasing for some $0 < \varepsilon \le 1$ and $T := (a_{n,m})$ be a lower triangular infinite regular matrix. If one of the following additional conditions is satisfied:

(i) $(a_{n,m}) \in AMIMS$,

(ii) $(a_{n,m}) \in AMDMS$ *and* $(n+1)a_{n,0} \ll 1$,

(iii) $\sum_{m=0}^{n-1} |\Delta_m(A_{n,m})| \ll n^{-1}$,

(iv) $\sum_{m=1}^{n-1} m^{1-\varepsilon}|\Delta_m(a_{n,m})| \ll n^{-\varepsilon}$, $n|\Delta_m(a_{n,m})| \ll 1 (m = 0, 1)$ *and* $(n+1)a_{n,n} \ll 1$,

then for $f \in H_p^{(\omega)}$, $p \ge 1$

$$\|f(x) - \tau_n(f; x)\|_p^{(v)} \le \frac{w(1/n)}{v(1/n)} \log n \tag{2.1}$$

for all $n \ge 2$.

3. Known results. In this section we will give some known results to use in proof of the Theorem 2.1.

Lemma 3.1 [7]. *Let* (1.2) *and* (1.3) *hold. If $(a_{n,m}) \in AMIMS$ or $(a_{n,m}) \in AMDMS$ and $(n+1)a_{n,0} \ll 1$, then*

$$\sum_{m=0}^{n} (m+1)^{-\alpha} a_{n,m} \ll (n+1)^{-\alpha}$$

holds for $0 < \alpha < 1$.

Lemma 3.2 [5]. *If the conditions of Theorem* 1.2 *are satisfied with $0 < \varepsilon < 1$,*

$$\|\sigma_n - f\|_p^{(v)} \le \frac{w(1/n)}{v(1/n)} \log n, \quad n \ge 2, \tag{3.1}$$

and

$$\|\sigma_n - s_n\|_p^{(v)} \le \frac{w(1/n)}{v(1/n)} \log n, \quad n \ge 2. \tag{3.2}$$

4. Proof of Theorem 2.1. First of all, let us give the following inequalities will be used in proof of Theorem 2.1. Since $\eta(t)$ is nonincreasing, the sequence

$$\eta\left(\frac{1}{n}\right) = n^{\varepsilon}\frac{w(1/n)}{v(1/n)} \tag{4.1}$$

is nondecreasing. Accordingly,

$$\frac{w(1/n)}{v(1/n)} \gg \frac{1}{n^{\varepsilon}} \geq \frac{1}{n}, \quad 0 < \varepsilon \leq 1, \tag{4.2}$$

and based on the knowledge of nonincreasing of sequence $\left(\dfrac{w(1/n)}{v(1/n)}\right)$ we have

$$\frac{w(1/\tilde{n})}{v(1/\tilde{n})} \ll \frac{w(1/n)}{v(1/n)}, \tag{4.3}$$

where $\tilde{n}$ denotes integer part of $\dfrac{n}{2}$. The method of proof will be similar with some parts in [9].

***Proof Theorem* 2.1.** Let us prove the theorem for all the cases, respectively.

Case 1. Let $(a_{n,k}) \in AMIMS$. By the definition of $\tau_n(f,x)$, we have

$$\tau_n(f;x) - f(x) = \sum_{m=0}^{n} a_{n,m} \left\{ s_m(f;x) - f(x) \right\}.$$

So, from hypothesis and (1.1) we obtain

$$\|\tau_n(f;x) - f(x)\|_p^{(v)} \leq \left(\sum_{m=0}^{\tilde{n}} + \sum_{m=\tilde{n}+1}^{n} \right) a_{n,m} \|s_m(f;x) - f\|_p^{(v)} \ll$$

$$\ll \{a_{n,0} \|s_0(f;x) - f(x)\|_p^{(v)} + a_{n,1} \|s_1(f;x) - f(x)\|_p^{(v)}\} +$$

$$+ \sum_{m=2}^{\tilde{n}} a_{n,m} \frac{w(1/m)}{v(1/m)} \log m + \sum_{m=\tilde{n}+1}^{n} a_{n,m} \frac{w(1/m)}{v(1/m)} \log m =: I_1 + I_2 + I_3.$$

Now let's estimate I_1, I_2 and I_3, respectively: since $(a_{n,k}) \in AMIMS$, we get

$$a_{n,0} = C_0 \ll C_n = \frac{1}{n+1} \sum_{m=0}^{n} a_{n,m} = \frac{1}{n+1} \ll n^{-1}$$

and

$$C_1 = \frac{1}{2}\{a_{n,0} + a_{n,1}\} = \frac{1}{2}\{C_0 + a_{n,1}\} \Rightarrow a_{n,1} \ll n^{-1},$$

where $C_k := A_{n,k} = \dfrac{1}{k+1} \sum_{m=0}^{k} a_{n,m}$. Therefore, we have

$$I_1 \ll \frac{w(1/n)}{v(1/n)}$$

in view of (4.2) for all $n \geq 2$

$$I_2 = \sum_{m=1}^{\tilde{n}} a_{n,m} \frac{(m+1)^{\varepsilon}}{(m+1)^{\varepsilon}} \frac{w(1/m)}{v(1/m)} \log m \ll (\tilde{n})^{\varepsilon} \frac{w(1/\tilde{n})}{v(1/\tilde{n})} \log \tilde{n} \sum_{m=1}^{\tilde{n}} a_{n,m} (m+1)^{-\varepsilon}.$$

From Lemma 3.1 and (4.3), we obtain

$$I_2 \ll \frac{w(1/n)}{v(1/n)} \log n$$

for all $n \geq 2$. With a simple analysis, we get

$$I_3 = \sum_{m=\tilde{n}+1}^{n} a_{n,m} \frac{w(1/m)}{v(1/m)} \log m \ll \frac{w(1/\tilde{n})}{v(1/\tilde{n})} \log n \sum_{m=\tilde{n}+1}^{n} a_{n,m} \ll \frac{w(1/\tilde{n})}{v(1/\tilde{n})} \log n \sum_{m=0}^{n} a_{n,m}$$

by considering (4.3) for all $n \geq 2$. Therefore, we obtain (2.1) for the case (i).

Case 2. Let $(a_{n,k}) \in AMDMS$ and $(n+1)a_{n,0} \ll 1$. As the first case, we have

$$\|\tau_n(f;x) - f(x)\|_p^{(v)} \leq \left(\sum_{m=0}^{\tilde{n}} + \sum_{m=\tilde{n}+1}^{n} \right) a_{n,m} \|s_m(f;x) - f\|_p^{(v)} \ll$$

$$\ll \{a_{n,0} \|s_0(f;x) - f(x)\|_p^{(v)} + a_{n,1} \|s_1(f;x) - f(x)\|_p^{(v)}\} +$$

$$+ \left(\sum_{m=2}^{\tilde{n}} + \sum_{m=\tilde{n}+1}^{n} \right) a_{n,m} \frac{w(1/m)}{v(1/m)} \log m =: J_1 + J_2.$$

Taking into account (4.3) and $a_{n,0} = C_0 \gg C_1 = \dfrac{1}{2}\{a_{n,0} + a_{n,1}\} \Rightarrow a_{n,0} \gg a_{n,1}$, we get $J_1 \ll \ll \dfrac{w(1/n)}{v(1/n)} \log n$ for all $n \geq 2$. By using Lemma 3.1 and (4.3), the evaluation of J_2 is obtained similarly to the case (i). Therefore, (2.1) is proved.

Case 3. By applying two times Abel's transformation and using $\sum_{m=0}^{n} a_{n,m} = 1$, we have

$$\tau_n(f;x) - f(x) = \sum_{m=0}^{n} a_{n,m} \{s_m(f;x) - f(x)\} =$$

$$= \sum_{m=0}^{n-1} (s_m(f;x) - s_{m+1}(f;x)) \sum_{k=0}^{m} a_{n,k} + \{s_n(f;x) - f(x)\} =$$

$$= \{s_n(f;x) - f(x)\} - \sum_{m=0}^{n-1} (m+1) U_{m+1}(f;x) A_{n,m} =$$

$$= \{s_n(f;x) - f(x)\} - \sum_{m=0}^{n-2} (A_{n,m} - A_{n,m+1}) \sum_{k=0}^{m} (k+1) U_{k+1}(f;x) -$$

$$- \frac{1}{n} \sum_{k=0}^{n-1} a_{n,k} \sum_{m=0}^{n-1} (m+1) U_{m+1}(f;x).$$

Here

$$\|\tau_n(f;x) - f(x)\|_p^{(v)} \leq \|s_m(f;x) - f\|_p^{(v)} + \sum_{m=0}^{n-2} |A_{n,m} - A_{n,m+1}| \left\| \sum_{k=1}^{m+1} k U_k(f;x) \right\|_p^{(v)} +$$

$$+ \frac{1}{n} \left\| \sum_{m=1}^{n} m U_m(f;x) \right\|_p^{(v)} \ll$$

$$\ll \|s_m(f;x) - f\|_p^{(v)} + |A_{n,0} - A_{n,1}| + |A_{n,1} - A_{n,2}| +$$

$$+ \sum_{m=2}^{n-2} |A_{n,m} - A_{n,m+1}| \left\| \sum_{k=1}^{m+1} kU_k(f;x) \right\|_p^{(v)} + \frac{1}{n} \left\| \sum_{m=1}^{n} mU_m(f;x) \right\|_p^{(v)}. \tag{4.4}$$

Since

$$s_n(f;x) - \sigma_n(f;x) = \frac{1}{n+1} \sum_{k=1}^{n} kU_k(f;x),$$

by (3.2) we get

$$\left\| \sum_{k=1}^{n} kU_k(f;x) \right\|_p^{(v)} = (n+1) \|s_n(f;x) - \sigma_n(f;x)\|_p^{(v)} \ll n \frac{w(1/n)}{v(1/n)} \log n. \tag{4.5}$$

So, combining (1.1), (4.4) and (4.5), we get that

$$\|\tau_n(f;x) - f(x)\|_p^{(v)} \ll \frac{w(1/n)}{v(1/n)} \log n + \sum_{m=0}^{n-1} |A_{n,m} - A_{n,m+1}| +$$

$$+ n \frac{w(1/n)}{v(1/n)} \log n \sum_{m=0}^{n-1} |A_{n,m} - A_{n,m+1}|. \tag{4.6}$$

Taking into account (4.2), (4.6) and the condition (iii) of Theorem 2.1, then

$$\|\tau_n(f;x) - f(x)\|_p^{(v)} \ll \frac{w(1/n)}{v(1/n)} \log n$$

holds for all $n \geq 2$.

Case 4. By applying Abel's transformation, we obtain

$$\tau_n(f;x) - f(x) = \sum_{m=0}^{n} a_{n,m} \{s_m(f;x) - f(x)\} =$$

$$= \sum_{m=0}^{n-1} \Delta_m a_{n,m} \sum_{k=0}^{m} (s_k(f;x) - f(x)) + a_{n,n} \sum_{k=0}^{n} \{s_k(f;x) - f(x)\} =$$

$$= \sum_{m=0}^{n-1} (m+1)(\Delta_m a_{n,m})(\sigma_m(f;x) - f(x)) + (n+1) a_{n,n} (\sigma_n(f;x) - f(x)).$$

Thus, owing to (3.1), (4.1), (4.2) and the condition (iv) of Theorem 2.1, we have

$$\|\tau_n(f;x) - f(x)\|_p^{(v)} \leq \sum_{m=0}^{n-1} |(\Delta_m a_{n,m})| (m+1) \|\sigma_m(f;x) - f(x)\|_p^{(v)} +$$

$$+ (n+1) a_{n,n} \|\sigma_n(f;x) - f(x)\|_p^{(v)} \ll$$

$$\ll |(\Delta_m a_{n,0})| \|\sigma_0(f;x) - f(x)\|_p^{(v)} +$$

$$+2|(\Delta_m a_{n,1})|\|\sigma_1(f;x)-f(x)\|_p^{(v)}+$$

$$+\sum_{m=2}^{n-1} m|(\Delta_m a_{n,m})|\,\|\sigma_m(f;x)-f(x)\|_p^{(v)}+\frac{w(1/n)}{v(1/n)}\log n \ll$$

$$\ll \frac{1}{n}+\sum_{m=2}^{n-1} m^{1-\varepsilon}|(\Delta_m a_{n,m})|\, m^{\varepsilon}\frac{w(1/m)}{v(1/m)}\log m+\frac{w(1/n)}{v(1/n)}\log n \ll$$

$$\ll n^{\varepsilon}\frac{w(1/n)}{v(1/n)}\log n \sum_{m=1}^{n-1} m^{1-\varepsilon}|(\Delta_m a_{n,m})|+\frac{w(1/n)}{v(1/n)}\log n \ll$$

$$\ll \frac{w(1/n)}{v(1/n)}\log n.$$

Therefore, Theorem 2.1 is proved.

5. Corollaries and remarks. Theorem 2.1 gives us some corollaries and remarks which related with the results in [3, 5, 9].

***Corollary* 5.1.** *If the conditions of Theorem* 2.1 *and the following additional conditions are satisfied*:

(i) $(a_{n,m}) \in HBVMS$,

(ii) $(a_{n,m}) \in RBVMS$ *and* $(n+1)a_{n,0} \ll 1$,

then for $f \in H_p^{(\omega)}$, $p \geq 1$,

$$\|f(x)-\tau_n(f;x)\|_p^{(v)} \leq \frac{w(1/n)}{v(1/n)}\log n$$

for all $n \geq 2$.

Proof. Since $HBVMS \subset AMIMS$ and $RBVMS \subset AMDMS$, the proof is obvious.

Since $AMDS \subset AMDMS$ and $AMIS \subset AMIMS$, we can write the following result which coincide with the conditions (i) and (ii) of Theorem 6 in [9].

***Corollary* 5.2.** *If the conditions of Theorem* 2.1 *and the following additional conditions are satisfied*:

(i) $(a_{n,m}) \in AMIS$,

(ii) $(a_{n,m}) \in AMDS$ *and* $(n+1)a_{n,0} \ll 1$,

then for $f \in H_p^{(\omega)}$, $p \geq 1$,

$$\|f(x)-\tau_n(f;x)\|_p^{(v)} \leq \frac{w(1/n)}{v(1/n)}\log n$$

for all $n \geq 2$.

***Remark* 5.1.** Theorem 2.1 generalizes Theorem 6 in [9] under the conditions (i) and (ii).

***Remark* 5.2.** In addition, the above corollary is also true for the classes of sequence $HBVS$ and $RBVS$ owing to the inclusions $HBVS \subset AMIS$ and $RBVS \subset AMDS$.

***Remark* 5.3.** If $T \equiv (a_{n,m})$ is a Nörlund matrix, then the conditions (i) and (ii) of Corollary 5.2 is reduced to the conditions of Theorem 2 in [5] and to more general case of Theorem 3.1 in [3], respectively. Moreover, Corollary 5.1 generalizes Theorem 2 in [5].

***Remark* 5.4.** If we take $a_{n,m} = \dfrac{p_{n-m}q_m}{r_n}$, where $r_n = \sum_{m=0}^{n} p_{n-m}q_m$, then the condition (iv) of Theorem 2.1 in this studying is reduced the conditions of Theorem 3.2 in [3]. Therefore, Theorem 2.1 generalizes Theorem 3.2 in [3] under the condition (iv). Here, if $p_{n-m} = \dfrac{1}{\lambda_{n+1}}$, $0 \le m \le n$, and

$$q_m = \begin{cases} 0, & 0 \le m < n - \lambda_n, \\ 1, & n - \lambda_n \le m \le n, \end{cases}$$

then it gives us generalized de la Vallée Poussin means where (λ_n) is a nondecreasing sequence of positive integers with $\lambda_0 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$.

***Remark* 5.5.** If we take $a_{n,m} = \dfrac{p_{n-m}^{\alpha}}{P_n^{\alpha}}$, where $P_n^{\alpha} = \sum_{m=0}^{n} p_m^{\alpha}$; $p_m^{\alpha} = \sum_{i=0}^{m} A_{m-v}^{\alpha-1} p_v$ and $A_n^{\alpha} = \dbinom{\alpha + n}{n}$, $\alpha > -1$, $n = 1, 2, 3, \ldots$, then the conditions of Corollary 5.2 is reduced the conditions (i) and (ii) of Theorem 3.3 in [3]. Furthermore, Corollary 5.1 and Corollary 5.2 generalize Theorem 3.3 and Theorem 3.4 in [3] by taking into account Remark 5.2 under the conditions (i) and (ii).

***Remark* 5.6.** The condition (iv) of Theorem 2.1 is different from the condition (iv) of Theorem 6 in [9] with the additional condition $n|\Delta_m(a_{n,m})| \ll 1$, $m = 0, 1$.

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