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TRIANGULAR MODELS OF COMMUTATIVE SYSTEMS OF LINEAR OPERATORS CLOSE TO UNITARY ONES

ТРИКУТНІ МОДЕЛІ КОМУТАТИВНИХ СИСТЕМ ЛІНІЙНИХ ОПЕРАТОРІВ, БЛИЗЬКИХ ДО УНІТАРНИХ

Triangular models are constructed for commutative systems of linear bounded operators close to unitary operators. The construction of these models is based on the continuation of basic relations for the characteristic function along the general chain of invariant subspaces.

Побудовано трикутні моделі комутативних систем лінійних обмежених операторів, близьких до унітарних. Побудову цих моделей засновано на подовженні основних співвідношень для характеристичної функції вздовж загального ланцюжка інваріантних просторів.

Introduction. It is common to consider [2, 3, 5] triangular or functional model as an analogue of spectral decomposition for nonself-adjoint and nonunitary operators. For the first time, a triangular model for nonself-adjoint operator has been built by M. S. Livšic, for nonunitary operator — by V. T. Polyatsky [2]. Triangular models for commutative systems of linear bounded nonself-adjoint operators has been built by V. A. Zolotarev [4], V. Vinnikov (part IV of [5]). These models are based on the basic idea of M. S. Livšic [5] of the spectral analysis of this class of operator systems. Reasonable constructions for the commutative systems of the nonunitary operators were built in [6, 7]. These constructions are the development of the method of M. S. Livšic [5]. This paper is dedicated to the construction of triangular models for commutative systems $\{T_1, T_2\}$ of linear operators close to unitary ones. Note that some results stated in this paper were announced in [1]. An analogue of the Hamilton — Cayley theorem is an important corollary of the constructed model representations, namely, it is proved that the polynomial $\mathbb{P}(z_1, z_2)$ with antiholomorphic involution with respect to the unit circle is such that $\mathbb{P}(T_1, T_2) = 0$. This result for commutative systems of nonself-adjoint operators has been obtained earlier by M. S. Livšic [5].

1. Let T be a linear bounded operator in a Hilbert space H. Let us recall that the set

$$\triangle = \left(\sigma;\ H \oplus E;\ V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix}; H \oplus \tilde{E};\ \tilde{\sigma}\right), \tag{1}$$

is called a unitary colligation [2, 3, 6] if the operator $V : H \oplus E \to H \oplus \tilde{E}$ has properties

$$V^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma} \end{bmatrix} V = \begin{bmatrix} I & 0 \\ 0 & \sigma \end{bmatrix}, \tag{2_1}$$

$$V \begin{bmatrix} I & 0 \\ 0 & \sigma^{-1} \end{bmatrix} V^* = \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma}^{-1} \end{bmatrix}, \tag{2_2}$$

where σ and $\tilde{\sigma}$ are self-adjoint invertible operators acting in the Hilbert spaces E and $\tilde{E}$ respectively. As it is known [2, 3, 6], for every bounded operator there always exists such unitary colligation Δ

(1), of which T is the main operator. The characteristic function $S_\Delta(z)$ of a colligation Δ,

$$S_\Delta(z) = K + \Psi\,(zI - T)^{-1}\,\Phi, \tag{3}$$

is the main analytic object in terms of which the spectral analysis of a operator T is realized [2, 3, 6].

Let $\dim E = \dim \tilde{E} = r < \infty$, then it is possible to suppose that $E = \tilde{E}$; suppose also that $\sigma = \sigma = J$ where J is an involution ($J = J = J^{-1}$). The well-known result of V. P. Potapov in this case (see [2, 3]) gives us the multiplicative decomposition of the characteristic function. Namely, the characteristic function $S_\Delta(z)$ (3) has a representation

$$S_\Delta(z) = U \int_0^{\overset{\frown}{l}} \exp\left\{\frac{e^{i\varphi_t} + z}{e^{i\varphi_t} - z} J dF_t\right\} \prod_{k=1}^{\overset{\frown}{N}} \Big(R_k - e^{i\varphi_k}\,(zI - \alpha_k)^{-1}\,J d_k\Big). \tag{4}$$

Moreover, U is a J-unitary matrix, φ_t is a nondecreasing function on $[0, l]$, $0 \le \varphi_t \le 2\pi$ $(0 \le l < < \infty)$, F_t is a nondecreasing matrix-function for which $\operatorname{tr} F_t \equiv t$, and the matrices α_k, R_k and d_k are such that

1) α_k are J-normal matrices, $\alpha_k \alpha_k^+ = \alpha_k^+\,\alpha_k$ $(\alpha_k^+ = j\alpha_k^* j)$;

2) $d_k = J - \alpha_k^* J\,\alpha_k = j\left(I - R_k^2\right) \ge 0$;

3) R_k is a J-module of a matrix α_k, $R_k^2 = \alpha_k^+\,\alpha_k$;

4) α_k, α_k^+ and R_k act on $d_k E^r$ (E^r is a Euclidean space with dimension r) as a multiplication by μ_k, $\bar{\mu}_k$, and $|\mu_k|$ respectively,

$$(\alpha_k - \mu_k I)\,d_k = \left(\alpha_k^+ - \bar{\mu}_k I\right) d_k = (R_k - |\mu_k|\,I)\,d_k = 0,$$

where $\mu_k \notin \mathbb{T}$, and $\varphi_k = \arg \mu_k$ $(N \le \infty)$.

An arrow in (4) signifies multiplicativity [2] of the integral and product

$$\prod_{k=1}^{\overset{\frown}{N}} a_k = a_N a_{N-1} \ldots a_1,$$

$$\int_0^{\overset{\frown}{l}} \exp\{N(t) dF(t)\} = \lim_{\substack{h\to\infty \\ d_n \to 0}} \exp\left\{N(\xi_n) F(\Delta_n)\right\} \ldots \exp\left\{N(\xi_1) F(\Delta_1)\right\},$$

where a_k, $N(t)$, $F(t)$ are matrices-functions and $0 = x_0 < x_1 < \ldots < x_n = l$, $\Delta x_k = x_k - x_{k-1}$, $x_{k-1} \le \xi_k \le x_k$, $d_n = \max\limits_k \Delta x_k$.

Hereinafter, we consider the case when the characteristic function $S_\Delta(z)$ is given by

$$S_\Delta(z) = \int_0^{\overset{\frown}{l}} \exp\left\{\frac{e^{i\varphi_t} + z}{e^{i\varphi_t} - z} J dF_t\right\}, \tag{5}$$

which means that the spectrum of T lies on the unit circle $\mathbb{T}$. We introduce the space

$$L^2_{r,l}(F_x) = \left\{ f(x) = (f_1(x), \ldots, f_r(x)) \in E^r : \int_0^l f(x) dF_x f^*(x) < \infty \right\}, \tag{6}$$

assuming that the respective factorization by the kernel of the metric has been done. Specify the operator $\mathring{T}$ in $L^2_{r,l}(F_x)$,

$$\left(\mathring{T}f\right)(x) = f(x)e^{i\varphi_x} - 2\int\limits_x^l f(t)dF_t\Phi_t^*\Phi_x^{*-1}Je^{i\varphi_x}, \tag{7}$$

where the matrix-function Φ_x is the solution of the equation

$$\Phi_x + \int\limits_0^x \Phi_t dF_t J = I. \tag{8}$$

Define now the operators

$$\begin{gathered}\mathring{\Psi}\colon L^2_{r,l}(F_x) \to E^r; \qquad \mathring{\Psi}f(x) = -\sqrt{2}\int\limits_0^l f(t)dF_t\Phi_t^*,\\ \mathring{\Phi}\colon E^r \to L^2_{r,l}(F_x); \qquad \mathring{\Phi}g(x) = g\sqrt{2}\Psi_x e^{i\varphi_x},\\ \mathring{K}\colon E^r \to E^r; \qquad \mathring{K} = S(\infty),\end{gathered} \tag{9}$$

in this case the matrix-function Ψ_x satisfies the equation

$$\Psi_x + \int\limits_x^l \Psi_t dF_t J = J. \tag{10}$$

It is easy to see [2] that the collection

$$\Delta_c = \left(J;\ L^2_{r,l}(F_x)\oplus E^r;\ \mathring{V}_T = \begin{bmatrix}\mathring{T} & \mathring{\Phi}\\ \mathring{\Psi} & \mathring{K}\end{bmatrix};\ L^2_{r,l}(F_x)\oplus E^r; J\right), \tag{11}$$

is a unitary colligation, the characteristic function of which coincides with $S(z)$ (5), and $\mathring{T}$, $\mathring{\Phi}$, $\mathring{\Psi}$, $\mathring{K}$ have the form of (7), (9).

Let us recall that the colligation Δ (1) is said to be simple [2, 3] if $H_1 = H$,

$$H_1 = \operatorname{span}\left\{T^n\Phi E + T^{*m}\Psi^*\tilde{E}\colon n, m \in \mathbb{Z}_+\right\}. \tag{12}$$

It is easy to show that the subspace $H_0 = H \ominus H_1$ reduces T and that the restriction T on H_0 induces the unitary operator [2, 3].

Theorem 1 [2]. *When the spectrum of the operator T lies on the circle T, the simple unitary colligation Δ (1) is unitarily equivalent to the simple part of the unitary colligation Δ_c (11).*

2. Following [6], we define an analogue of the unitary operator Δ (1) for the commutative system of linear bounded operators $\{T_1, T_2\}$, $[T_1, T_2] = 0$

Definition. *The collection*

$$\Delta = \left(\Gamma; \{N_s\}_1^2; \{\sigma_s\}_1^2; \{\tau_s\}_1^2; H \oplus E; \left\{V_s, \overset{+}{V}_s\right\}_1^2; H \oplus \tilde{E}; \{\tilde{\tau}_s\}_1^2; \{\tilde{\sigma}_s\}_1^2; \left\{\tilde{N}_s\right\}_1^2; \tilde{\Gamma}\right), \tag{13}$$

is said to be the commutative unitary colligation of the operator system $\{T_1, T_2\}$ *in* H $([T_1, T_2] = 0)$ *if in the Hilbert spaces* E *and* $\tilde{E}$ *there exist such operators* $\sigma_s, \tau_s, N_s, \Gamma$ *and* $\tilde{\sigma}_s, \tilde{\tau}_s, \tilde{N}_s, \tilde{\Gamma}$ *respectively* (σ_s, τ_s *and* $\tilde{\sigma}_s, \tilde{\tau}_s$ *are self-adjoint*, $s = 1, 2$), *such that the mappings*

$$V_s = \begin{bmatrix} T_s & \Phi N_s \\ \Psi & K \end{bmatrix} : H \oplus E \to H \oplus \tilde{E},$$
$$\overset{+}{V}_s = \begin{bmatrix} T_s^* & \Psi^* \tilde{N}_s^* \\ \Phi^* & K^* \end{bmatrix} : H \oplus \tilde{E} \to H \oplus E, \tag{14}$$

satisfy the following relations:

$$V_s^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma}_s \end{bmatrix} V_s = \begin{bmatrix} I & 0 \\ 0 & \sigma_s \end{bmatrix}, \quad s = 1, 2, \tag{15$_1$}$$

$$\overset{+}{V}_s^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\tau}_s \end{bmatrix} \overset{+}{V}_s = \begin{bmatrix} I & 0 \\ 0 & \tilde{\tau}_s \end{bmatrix}, \quad s = 1, 2, \tag{15$_2$}$$

$$T_2 \Phi N_1 - T_1 \Phi N_2 = \Phi \Gamma, \qquad \tilde{N}_1 \Psi T_2 - \tilde{N}_2 \Psi T_1 = \tilde{\Gamma} \Psi, \tag{15$_3$}$$

$$\tilde{N}_2 \Psi \Phi N_1 - \tilde{N}_1 \Psi \Phi N_1 = K\Gamma - \tilde{\Gamma} K, \tag{15$_4$}$$

$$K N_s = \tilde{N}_s K, \quad s = 1, 2. \tag{15$_5$}$$

It is easy to show [6] that for an arbitrary commutative system of bounded operators $\{T_1, T_2\}$ there always exists such an isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}$ (14) (the colligation Δ (13)) that conditions (15) hold. Denote by $S_1(z)$ the characteristic function of the colligation Δ (13) corresponding to the operator T_1,

$$S_1(z) = K + \Psi \left(zI - T_1\right)^{-1} \Phi N_1. \tag{16}$$

It is shown in [6] that in the case of invertibility of the operators N_1 and $\tilde{N}_1$ the totality

$$\left\{S_1(z); \sigma_s; \tau_s; N_s; \Gamma; \tilde{\sigma}_s; \tilde{\tau}_s; \tilde{N}_s; \tilde{\Gamma}\right\}_{s=1,2}, \tag{17}$$

is the total set of invariants of the commutative operator system $\{T_1, T_2\}$. The condition of invertibility of N_1 and $\tilde{N}_1$ is the additional restriction on the system of commutative operators T_1 and T_2. The last fact means that in the case of the simplicity of the colligation Δ (13) ($H = H_1$ (12)) set (17) defines the operator system $\{T_1, T_2\}$ up to the unitary equivalency [6].

We suppose that the operators σ_1 and $\tilde{\sigma}_1$ coincide with the involution J. Denote by N, $\tilde{N}$, γ, $\tilde{\gamma}$, σ, $\tilde{\sigma}$, τ, $\tilde{\tau}$ the operators corresponding to $N_1^{-1} N_2$, $\tilde{N}_1^{-1} \tilde{N}_2$, $N_1^{-1}\Gamma$, $\tilde{N}_1^{-1}\tilde{\Gamma}$, σ_2, $\tilde{\sigma}_2$, $N_1^{-1} \tilde{\tau}_2 \left(N_1^*\right)^{-1}$, $\tilde{N}_1^{-1} \tilde{\tau}_2 \big(\tilde{N}_1^*\big)^{-1}$ respectively. Then the characteristic function $S(z)$ of the operator T_1 (16) satisfies the following relations describing the commutative property of T_1 and T_2 in terms of the external parameters (17) [6, 7],

$$S(z)\left(\tilde{N}z+\tilde{\gamma}\right)=(Nz+\gamma)\,S(z), \tag{18_1}$$

$$S(z)JS^*(w)-J-(Nz+\gamma)\left\{S(z)JS^*(w)-J\right\}(N^*\bar{w}+\gamma^*)=$$
$$=(I-z\bar{w})\left\{S(z)\tilde{\sigma}S^*(w)-\sigma\right\}, \tag{18_2}$$

$$S^*(\bar{z})\,JS(\bar{w})-J-\left(\tilde{N}^*z+\tilde{\gamma}^*\right)\left\{S^*(\bar{z})\,JS(\bar{w})-J\right\}\left(\tilde{N}\bar{w}+\gamma\right)=$$
$$=(I-z\bar{w})\left\{S^*(\bar{z})\tau S(\bar{w})-\tilde{\tau}\right\}. \tag{18_3}$$

Simultaneous reduction of the commutative operator system $\{T_1,T_2\}$ to the triangular type means continuation and conservation of the main relations (18) along some chain of joint invariant subspaces for T_1 and T_2.

There is no corresponding parallel to the material of book [5] in the work. It is proposed to study the properties of the characteristic function $S_1(z)$ (16) in the presence of the colligation relations (15).

The main point is that the conditions (18) express the fact that the operator T_2 commutes with the operator T_1, this imposes the additional conditions on the characteristic function $S_1(z)$ (16) of the operator T_1.

3. Suppose that the characteristic function of the operator T_1 is given by (5),

$$S(z)=S_l(z),\qquad S_x(z)=\overset{\frown}{\int_0^x}\exp\left\{\frac{e^{i\varphi_t}+z}{e^{i\varphi_t}-z}JdF_t\right\}, \tag{19}$$

where $x\in[0;l]$ and φ_t, F_t have corresponding properties (see Section 1). First, let us study continuation of the condition (18_1) assuming that $dF_x=a_xdx$, where a_x is a nonnegative matrix-function on $[0,l]$ such that $\operatorname{tr}a_x\equiv 1$.

Theorem 2. $S_x(z)$ (19), *where* $dF_x=a_xdx$, *satisfies the intertwining condition*

$$S_x(z)\left(\tilde{N}z+\tilde{\gamma}\right)=(N_xz+\gamma_x)\,S_x(z), \tag{20}$$

then and only then when N_x *and* γ_x *are the solutions of the equations*

$$N_x'=-\left[Ja_x,N_x\right],\qquad N_0=\tilde{N}, \tag{21_1}$$

$$\gamma_x'=\left[Ja_x,\gamma_x\right],\qquad \gamma_0=\tilde{\gamma}, \tag{21_2}$$

moreover,

$$\left[Ja_x,\left(N_xe^{i\varphi_x}+\gamma_x\right)\right]=0, \tag{22}$$

which means that $\gamma_x'=e^{i\varphi_x}N_x'$ $\forall x\in[0,l]$.

Proof. Differentiate equality (20) assuming that $dF_x=a_xdx$, then

$$\frac{e^{i\varphi_x}+z}{e^{i\varphi_x}-z}Ja_xS_x(z)(\tilde{N}z+\tilde{\gamma})=\left\{zN_x'+\gamma_x'+(zN_x+\gamma_x)\,\frac{e^{i\varphi_x}+z}{e^{i\varphi_x}-z}Ja_x\right\}S_x(z).$$

Again taking into account (20) and the invertibility of $S_x(\infty)$, we obtain

$$\frac{e^{i\varphi_x}+z}{e^{i\varphi_x}-z}\left[Ja_x,(zN_x+\gamma_x)\right]=zN_x'+\gamma_x', \tag{23}$$

which, after equating the corresponding coefficients by z^k, $k = 0, 1, 2$), gives us (21_1), (21_2), and (22). And the initial conditions of the Cauchy problems (21) follow from (20) when $x = 0$.

To prove the sufficiency, we use (23), which is equivalent to (21_1), (21_2), and (22), and the equation for $S_x(z)$, then we have

$$\frac{d}{dx}\left\{(zN_x + \gamma_x)\, S_x(z)\right\} = \frac{e^{i\varphi_x} + z}{e^{i\varphi_x} - z} Ja_x\,(zN_x + \gamma_x)\, S_x(z). \tag{24}$$

Taking into account that the function $S_x(z)\left(z\tilde{N} + \tilde{\gamma}\right)$ satisfies similar equation, we get

$$\frac{d}{dx}\left\{S_x(z)\left(z\tilde{N} + \tilde{\gamma}\right) - (zN_x + \gamma_x)\, S_x(z)\right\} =$$

$$= \frac{e^{i\varphi_x} + z}{e^{i\varphi_x} - z} Ja_x \left\{S_x(z)\left(z\tilde{N} + \tilde{\gamma}\right) - (zN_x + \gamma_x)\, S_x(z)\right\},$$

which proves (20) in view of the uniqueness of the solution of the Cauchy problem since

$$\left\{S_x(z)\left(z\tilde{N} + \tilde{\gamma}\right) - (zN_x + \gamma_x)\, S_x(z)\right\} = 0, \quad \text{when} \quad x = 0.$$

Theorem 2 is proved.

Now let us consider similar continuation of relation (18_2) along the given chain of the invariant subspaces.

Theorem 3. *The matrix-function $S_x(z)$ (19) where $dF_x = a_x dx$ satisfies the relation*

$$S_x(z)JS_x^*(w) - J - (zN_x + \gamma_x)\left\{S_x(z)JS_x^*(w) - J\right\}(N_x^*\bar{w} + \gamma_x^*) =$$
$$= (I - z\bar{w})\left\{S_x(z)\tilde{\sigma}S_x^*(w) - \sigma_x\right\}, \tag{25}$$

under conditions (21_1)*,* (21_2) *and* (22)*, if and only if the relations*

$$\sigma_x' = 2N_xJa_xJN_x^* - Ja_x\sigma_x - \sigma_x a_x J, \quad \sigma_0 = \tilde{\sigma}, \tag{26_1}$$

$$\left\{N_xJ\left(N_x^* + e^{i\varphi_x}\gamma_x^*\right) - \sigma_x\right\} a_x = 0, \tag{26_2}$$

$$Ja_xJ + N_xJa_xJN_x^* - \gamma_xJa_xJ\gamma_x^* = \sigma_x a_x J + Ja_x\sigma_x, \tag{26_3}$$

hold for all $x \in [0; l]$.

Proof. To prove the necessity, we differentiate relation (25) and use the equalities (20) – (24), then we get

$$\left(\frac{e^{i\varphi_x} + z}{e^{i\varphi_x} - z} + \frac{e^{-i\varphi_x} + \bar{w}}{e^{-i\varphi_x} - \bar{w}}\right)\left\{Ja_xJ - (zN_x + \gamma_x)\, Ja_xJ\,(\bar{w}N_x^* + \gamma_x^*)\right\} =$$

$$= (I - z\bar{w})\left\{\frac{e^{i\varphi_x} + z}{e^{i\varphi_x} - z} Ja_x\sigma_x + \sigma_x a_x J \frac{e^{-i\varphi_x} + \bar{w}}{e^{-i\varphi_x} - \bar{w}} - \sigma_x'\right\}.$$

Now equating the coefficients by equal powers $z^k\bar{w}^{\,s}$, $k, s = 0, 1$, we obtain relations (26_1) – (26_3). To prove the sufficiency, we consider the following function:

$$\Psi_x(z, w) = S_x(z)JS_x^*(w) - J -$$

$$-(zN_x+\gamma_x)\left\{S_x(z)JS_x^*(w)-J\right\}(N_x^*\bar{w}+\gamma_x^*)+$$
$$+(z\bar{w}-I)\left\{S_x(z)\tilde{\sigma}S_x^*(w)-\sigma_x\right\}.$$

From (23), (26) it follows that $\Psi_x(z,w)$ satisfies the equation

$$\frac{d}{dx}\Psi_x(z,w)=\frac{e^{i\varphi_x}+z}{e^{i\varphi_x}-z}Ja_x\Psi_x(z,w)+\Psi_x(z,w)a_xJ\frac{e^{-i\varphi_x}+\bar{w}}{e^{-i\varphi_x}-\bar{w}},$$

and, taking into account that $\Psi_0(z,w)=0$, we obtain $\Psi_x(z,w)=0\ \forall x\in[0,l]$, this fact proves (25).

Theorem 3 is proved.

Continuation of relation (18_3) along the given chain of the invariant subspaces leads us to the following statement.

Theorem 4. *Suppose that the matrix-function* $S_x(z)$ (19), *where* $dF_x=a_xdx$, *satisfies the intertwining condition* (20), *then*

$$S_x^*(\bar{z})JS_x(\bar{w})-J-\left(\tilde{N}^*z+\tilde{\gamma}^*\right)\left\{S_x^*(\bar{z})JS_x(\bar{w})-J\right\}\left(\tilde{N}\tilde{w}+\tilde{\gamma}\right)=$$
$$=(I-z\bar{w})\left\{S_x^*(\bar{z})\tau_xS_x(\bar{w})-\tilde{\tau}\right\}, \tag{27}$$

is true then and only then, when

$$\tau_x'=\tau_xJa_x+a_xJ\tau_x-2N_x^*a_xN_x,\quad \tau_0=\tilde{\tau}, \tag{28_1}$$
$$\left\{N_x^*J\left(N_x+e^{-\varphi_x}\gamma_x\right)-\tau_x\right\}Ja_x=0, \tag{28_2}$$
$$a_x+N_x^*a_xN_x-\gamma_x^*a_x\gamma_x=\tau_xJa_x+a_xJ\tau_x, \tag{28_3}$$

for all $x\in[0,l]$.

Proof. Differentiating (27) and using (20), we have

$$\left(\frac{e^{-i\varphi_x}+z}{e^{-i\varphi_x}-z}+\frac{e^{i\varphi_x}+\bar{w}}{e^{i\varphi_x}-\bar{w}}\right)\left\{a_x-(zN_x^*+\gamma_x^*)\,a_x\,(\bar{w}N_x+\gamma_x)\right\}=$$
$$=(I-z\bar{w})\left\{\tau_xJa_x\frac{e^{i\varphi_x}+\bar{w}}{e^{i\varphi_x}-\bar{w}}+a_xJ\tau_x\frac{e^{-i\varphi_x}+z}{e^{-i\varphi_x}-z}+\tau_x'\right\}.$$

Now in order to obtain $(28_1)-(28_3)$ it is necessary to equate the corresponding coefficients by powers $z^k\bar{w}^s$, $k,s=0,1$. Now, if we consider the function

$$\Psi_x(z,w)=S_x^*(\bar{z})JS_x(\bar{w})-J-\left(\tilde{N}_x^*z+\tilde{\gamma}^*\right)\left\{S_x^*(\bar{z})JS_x(\bar{w})-J\right\}\times$$
$$\times\left(\tilde{N}\bar{w}+\tilde{\gamma}\right)(z\bar{w}-I)\left\{S_x^*\left(\bar{z}\right)\tau_xS_x(\bar{w})-\tilde{\tau}\right\},$$

then in view of the last equation and taking into account the intertwining relation (20) it is easy to verify that $\frac{d}{dx}\Psi_x(z,w)=0$, and using the initial data $\Psi_0(z,w)=0$, we obtain $\Psi_x(z,w)=0$, this proves the sufficiency of the statement.

Theorem 4 is proved.

4. Substituting equality (26_2) in relation (26_3) and using (22_3), we see that

$$\left(N_x e^{i\varphi_x} + \gamma_x\right) J a_x J \left(N_x e^{i\varphi_x} + \gamma_x\right)^* = J a_x J.$$

Thus the operator $J\left(N_x^* e^{-i\varphi_x} + \gamma_x^*\right) J$ is isometric in the metric given by the operator a_x. Therefore, this equality and relation (26_2) on the subspace $E_x^r = a_x E^r$ we can write in the following form:

$$\begin{gathered}\left\{\left(N_x e^{i\varphi_x} + \gamma_x\right) J \left(N_x^* e^{-i\varphi_x} + \gamma_x\right) - J\right\} f_x = 0,\\ \left\{N_x J\left(N_x^* + e^{-i\varphi_x}\gamma_x^*\right) - \sigma_x\right\} f_x = 0,\end{gathered} \tag{29}$$

where $f_x \in E_x^r$. Similarly, it follows from relations (28) that

$$\left(N_x e^{i\varphi_x} + \gamma_x\right)^* a_x \left(N_x e^{i\varphi_x} + \gamma_x\right) = a_x,$$

this means that $N_x e^{i\varphi_x} + \gamma_x$ is isometric in the metric a_x, thus

$$\begin{gathered}\left\{\left(N_x e^{i\varphi_x} + \gamma_x\right)^* J \left(N_x e^{i\varphi_x} + \gamma_x\right) J - I\right\} f_x = 0,\\ \left\{N_x^* J\left(N_x + e^{-i\varphi_x}\gamma_x\right) J - \tau_x J\right\} f_x = 0,\end{gathered} \tag{30}$$

where $f_x \in E_x^r$ in view of (28_2).

Let us turn to the solvability of the equation system (21), (22) which we can write in the form of (23),

$$\frac{d}{dx}\left(zN_x + \gamma_x\right) = \left[\frac{e^{i\varphi_x} + z}{e^{i\varphi_x} - z} J a_x, \left(zN_x + \gamma_x\right)\right]. \tag{31}$$

Following P. Lax [8], in order to integrate the equation $L_x' = [A_x, L_x]$ it is necessary to find an "isometric" operator-function V_x, such that $V_x' = A_x V_x$, where in this case V_x realizes the equivalency between L_x and L_0. In our case, $V_x = S_x(z)$ and the given equation $L_x' = [A_x, L_x]$ leads us to the intertwining condition (20),

$$zN_x + \gamma_x = S_x(z)\left(\tilde{N}z + \tilde{\gamma}\right) S_x^{-1}(z),$$

since, by our supposition, the matrix $S_x(z)$ is invertible when $|z| \gg 1$ (for example). Hence, eigenvalues of the matrices N_x and γ_x do not depend on x, and the root subspaces L_z of the bundle $\tilde{N}z_1 + \tilde{\gamma}$ corresponding to the number z_2 under the action of the matrix-function $S_x(z)$ passes into the root subspace $L_z(x) = S_x(z) L_z$ of the linear pencil $N_x z_1 + \gamma_x$.

Theorem 5. *Suppose that $S_x(z)$* (19) *is invertible at some point $z_0 \in \mathbb{C}$, then the solutions N_x and γ_x of the Cauchy problems* (21_1), (21_2) *exist, moreover, equality* (22) *holds.*

Invertibility of $S_x(z_0)$ for all $x \in [0, l]$ follows from the J-theory of V. P. Potapov [2] with the condition that $S(z_0)$ is an invertible matrix. Furthermore, since relations (26_1)–(26_3) and (28_1)–(28_3) are equivalent to the equalities (25) and (27) correspondingly with the condition that N_x and γ_x satisfy (21_1) and (21_2), then the existence of the matrix-functions σ_x and τ_x is obvious.

Theorem 6. *Suppose that N_x and γ_x are the solutions of the Cauchy problems* (21_1), (21_2) *for which* (22) *holds and the matrix-function $S_x(z)$ is invertible at one point $z_0 \in \mathbb{C}$ at least, then the solutions σ_x of the relations* (26) *and τ_x of* (28) *respectively exist and are unique.*

5. Now we turn to the construction of the triangular models of the systems of operators T_1, T_2. To do this, we consider in the Hilbert space $L^2_{r,l}(F_x)$ (6) the system of linear operators

$$\left(\overset{\circ}{T}_1 f\right)(x) = f(x)e^{i\varphi_x} - 2\int\limits_x^l f(t)dF_t\Phi_t^*\Phi_x^{*-1}Je^{i\varphi_x},$$
$$\left(\overset{\circ}{T}_2 f\right)(x) = f(x)J\left(N_xe^{i\varphi_x}+\gamma_x\right)J - 2\int\limits_x^l f(t)dF_t\Phi_t^*\Phi_x^{*-1}Je^{i\varphi_x}, \tag{32}$$

where Φ_x is the solution of the integral equation (7) and N_x, γ_x are the solutions of the Cauchy problems (21) respectively. Note that the operator $\overset{\circ}{T}_1$ in (32) coincides with $\overset{\circ}{T}$ (6). First of all, we make sure that the operators (32) are commutative. To do this, we consider

$$F(x)e^{i\varphi_x} = \left[\overset{\circ}{T}_1, \overset{\circ}{T}_2\right]f(x) = -2\int\limits_x^l f(t)J\left(N_te^{i\varphi_t}+\gamma_t\right)Ja_tdt\Phi_t^*\Phi_x^{*-1}Je^{i\varphi_x} +$$
$$+4\int\limits_x^l dt\int\limits_t^l f(s)a_sds\Phi_s^*\Phi_t^{*-1}N_tJe^{i\varphi_t}a_t\Phi_t^*\Phi_x^{*-1}Je^{i\varphi_x} +$$
$$+2\int\limits_x^l f(t)a_tdt\Phi_t^*\Phi_x^{*-1}\gamma_xJe^{i\varphi_x} + 2\int\limits_x^l f(t)e^{i\varphi_t}a_tdt\Phi_t^*\Phi_x^{*-1}N_xJe^{i\varphi_x} -$$
$$-4\int\limits_x^l dt\int\limits_t^l f(s)a_sds\Phi_s^*\Phi_t^{*-1}Je^{i\varphi_t}a_t\Phi_t^*\Phi_x^{*-1}N_xJe^{i\varphi_x}.$$

Using the equations (21_1), (21_2) and the fact that $\left(\Phi_x^{*-1}\right)' = \Phi_x^{*-1}Ja_x$, as a result of the elementary calculations using the equality (22), it is easy to see that the function F_x satisfies the differential equation $F_x' = F_xa_xJ$, and, since $F_l = 0$, then $F_x \equiv 0$, which proves the commutative properties of the operators $\overset{\circ}{T}_1$, $\overset{\circ}{T}_2$ (32).

Using equality (22), it is easy to show that

$$\left(\overset{\circ}{T}{}_1^* f\right)(x) = f(x)e^{i\varphi_x} - 2\int\limits_0^x f(t)e^{-i\varphi_t}dF_tJ\Phi_t^{-1}\Phi_x^*,$$
$$\left(\overset{\circ}{T}{}_2^* f\right)(x) = f(x)\left(N_xe^{i\varphi_x}+\gamma_x\right)^* - 2\int\limits_0^x f(t)e^{-i\varphi_t}dF_tJN_t^*\Phi_t^{-1}\Phi_x. \tag{33}$$

To construct the expansions V_s and $\overset{+}{V}_s$ (14), we have to calculate the defect operators $I - \overset{\circ}{T}{}_s^*\overset{\circ}{T}_s$ and $I - \overset{\circ}{T}_s\overset{\circ}{T}{}_s^*$, $s = 1, 2$. First of all, note the well-known fact [2] that

$$\left(I - \overset{\circ}{T}{}_1^*\overset{\circ}{T}_1\right)f(x) = 2\int\limits_0^l f(t)dF_t\Phi_t^*J\Phi_x.$$

We consider

$$\left(I - \mathring{T}_2^*\mathring{T}_2\right) f(x) = f(x)\left\{I - J\left(N_x e^{i\varphi_x} + \gamma_x\right) J \left(N_x e^{i\varphi_x} + \gamma_x\right)^*\right\} +$$

$$+ 2\int_0^x f(t) J\left(N_t e^{i\varphi_t} + \gamma_t\right) J e^{-i\varphi t} a_t J dt N^* \Phi_t^{-1}\Phi_x +$$

$$+ 2\int_x^l f(t) a_t dt \Phi_t^* \Phi_x^{*-1} N_x J e^{i\varphi_x}\left(N_x^* e^{-i\varphi_x} + \gamma_x^*\right) -$$

$$-4\int_0^x dt \int_t^l f(s) a_s ds \Phi_s^* \Phi_t^{*-1} N_t J a_t J \Phi_t^{-1}\Phi_x.$$

From equation (26_1) and the fact that $\left(\Phi_x^{-1}\right)' = a_x J \Phi_x^{-1}$, we conclude

$$\frac{d}{dx}\left(\Phi_x^{*-1}\sigma_x \Phi_x^{-1}\right) = 2\Phi_x^{*-1} N_x J a_x J N_x^* \Phi_x^{-1}.$$

Using relation (26_2), the equalities (29), and integration by parts, we obtain

$$\left(I - \mathring{T}_2^*\mathring{T}_2\right) f(x) = 2\int_0^l f(t) dF_t \Phi_t^* \tilde{\sigma} \Phi_x.$$

Hence, if we define the operator $\mathring{\Psi}\colon L^2_{r,l}(F_x) \to E^r$,

$$\mathring{\Psi} f(x) = -\sqrt{2}\int_0^l f(t) dF_t \Phi_t^*, \qquad \mathring{\Psi}^*\xi = -\sqrt{2}\xi\Phi_x, \tag{34}$$

where $\xi \in E^r$ and $f(x) \in L^2_{r,l}(F_x)$, then we get $\mathring{T}_1^*\mathring{T}_1 + \mathring{\Psi}^* J \mathring{\Psi} = I$ and $\mathring{T}_2^*\mathring{T}_2 + \mathring{\Psi}^*\tilde{\sigma}\mathring{\Psi} = I$.

Let us calculate other defect operators $I - \mathring{T}_s\mathring{T}_s^*$, $s = 1, 2$, then [2],

$$\left(I - \mathring{T}_1\mathring{T}_1^*\right) f(x) = 2\int_0^l f(t) e^{-i\varphi_t} dF_t \Psi_t^* J \Psi_x e^{i\varphi_x},$$

where Ψ_x is the solution of the integral equation (9) and $\Psi_x = \Phi_l^*\Phi_x^{*-1} J$. Next we consider

$$\left(I - \mathring{T}_2\mathring{T}_2^*\right) f(x) = f(x)\left\{I - \left(N_x e^{i\varphi_x} + \gamma_x\right)^* J \left(N_x e^{i\varphi_x} + \gamma_x\right) J\right\} +$$

$$+2\int_0^x f(t)\left(N_t e^{i\varphi_t} + \gamma_t\right)^* a_t \Phi_t^* dt \Phi_x^{-1} N_x J e^{i\varphi x} +$$

$$+2\int_0^x f(t) e^{-i\varphi_t} a_t dt J N_t^* \Phi_t^{-1}\Phi_x J\left(N_x^* e^{i\varphi_x} + \gamma_x\right) J -$$

$$-4\int\limits_x^l dt\int\limits_0^t f(s)e^{-i\varphi_s}a_s ds J N_s^*\Phi_s^{-1}\Phi_t a_t\Phi_t^*\Phi_x^{*-1}\Phi_x J e^{i\varphi_x},$$

since $(\Phi_x J\Phi_x^*)' = -2\Phi_x a_x\Phi_x$, then after the integration by parts in the last integral and taking into account (22) and (30), we obtain

$$\left(I-\mathring{T}_2\mathring{T}_2^*\right)f(x) = 2\int\limits_0^l f(t)e^{-i\varphi_t}a_t dt J N_t^* J\Psi_t^* J\Psi_x J N_x J e^{i\varphi_x}+$$

$$+2\int\limits_x^l f(t)a_t J\gamma_t^* J\Phi_t^* dt\Phi_x^{*-1}N_x J e^{i\varphi_x}+2\int\limits_0^x f(t)e^{-i\varphi_t}a_t J N_t^*\Phi_t^{-1}dt\Phi_x J\gamma_x J.$$

Representing the first integral as the sum of the integrals on the segments $[0,x]$ and $[x,l]$, we have

$$\left(I-\mathring{T}_2\mathring{T}_2^*\right)f(x) = 2\int\limits_0^x f(t)e^{-i\varphi_t}a_t J N_t^*\Phi_t^{-1}dt\left\{\Phi_l J\Phi_l^*\Phi_x^{*-1}N_x J e^{i\varphi_x}+\Phi_x J\gamma_x J\right\}+$$

$$+2\int\limits_x^l f(t)e^{-i\varphi_s}a_t\left\{J\gamma_t^*\Phi_l^*+J N_t^*\Phi_t^{-1}\Phi_l J\Phi_l^* e^{-i\varphi_t}\right\}dt\Phi_x^{*-1}N_x J e^{i\varphi_x}. \tag{35}$$

Since the above integrals are adjoint one to each other in terms of the metric $L^2_{r,l}(F_x)$, it is sufficient to calculate one of them, for example, the first. It follows from (21_1) that $\left(N_t^*\Phi_t^{-1}\right)' = a_t J N_t^*\Phi_t^{-1}$, therefore taking into account the initial data we get

$$N_t^*\Phi_t^{-1} = \Phi_t^{-1}\tilde{N}^* = J\Psi_t^*\Phi_l^{-1}\tilde{N}^*,$$

and hence the first of the integrals in (35) is equal to

$$2\int\limits_x^l f(t)e^{-i\varphi_s}a_t\Psi_t^* dt\Phi_l^{-1}\tilde{N}^*\left\{\Phi_l J\Phi_l^*\Phi_x^{*-1}N_x J e^{i\varphi_x}+\Phi_x J\gamma_x J\right\}.$$

Lemma. *Suppose that* τ_s *satisfies the relations* (28) *and the matrices* N_x *and* γ_x *are such that the equalities* (21) *are true and*, *moreover*, (29) *takes place*, *then*

$$\Phi_l^{-1}\tilde{N}^*\left\{\Phi_l J\Phi_l^*\Phi_x^{*-1}N_x J+e^{-i\varphi_x}\Phi_x J\gamma_x J\right\} = \tau_l\Psi_x, \tag{36}$$

where τ_l *is the value of the solution* τ_x *of problem* (28_1) *at the point* $x=l$.

Sum of the integrals (35) with (36) can be represented in the form of

$$\left(I-\mathring{T}_2\mathring{T}_2^*\right)f(x) = 2\int\limits_0^l f(t)e^{-i\varphi_t}dF_t\Psi_t^*\tau_l\Psi_x e^{i\varphi_x}.$$

So, if we define the operator $\mathring{\Phi}$ from E^r into $L^2_{r,l}(F_x)$ by the formula coinciding with (8),

$$\overset{\circ}{\Phi}\xi = \sqrt{2}\xi\Psi_x e^{i\varphi_x}, \qquad \left(\overset{\circ}{\Phi}{}^* f\right)(x) = \sqrt{2}\int\limits_0^l f(t)e^{-i\varphi_t}a_t\Psi_t^*\,dt, \tag{37}$$

where $\xi \in E^r$, then we get $\overset{\circ}{T}{}_1^*\overset{\circ}{T}{}_1^* + \overset{\circ}{\Phi}J\overset{\circ}{\Phi}{}^* = I$ and $\overset{\circ}{T}_2\overset{\circ}{T}{}_2^* + \overset{\circ}{\Phi}\tau_l\overset{\circ}{\Phi}{}^* = I$.

Proof. It follows from the Cauchy problem for N_x, (21_1) that $N_x = \Phi_x^*\tilde{N}\Phi_x^{-1}$ or $\Phi_x^{*-1}N_x = \tilde{N}\Phi_x^{*-1}$ and so

$$\Phi_l^{-1}\left\{\tilde{N}^*\Phi_l J\Phi_l^*\tilde{N}\Phi_x^{*-1}J + e^{-i\varphi_x}\tilde{N}^*\Phi_x J\gamma_x J\right\} = \tau_l\Psi_x.$$

Taking the limit as $z, \bar{w} \to \infty$ in equality (27) and assuming that $x = l$, we obtain that $\tilde{N}^*(\Phi_l^* J\Phi_l - J)\tilde{N} + \tilde{\tau} = \tau_l$. Therefore

$$\Phi_l^{-1}\left\{\left(\tilde{N}^*J\tilde{N} - \tilde{\tau}\right)\Phi_x^{*-1}J + e^{-i\varphi_x}\tilde{N}^*\Phi_x J\gamma_x J\right\} = 0,$$

and taking into account again that $\tilde{N}^*\Phi_x = \Phi_x N_x^*$ we find that

$$\tilde{N}^*J\tilde{N}\Phi_x^{*-1} - \tilde{\tau}\Phi_x^{*-1} + e^{-i\varphi_x}\Phi_x N^* J\gamma_x = 0.$$

And using (30) $N_x^* J\gamma_x e^{-i\varphi_x} = \tau_x - N_x^* J N_x$ we have

$$\tilde{N}^*J\tilde{N}\Phi_x^{*-1} - \tilde{\tau}\Phi_x^{*-1} + \Phi_x\left(\tau_x - N_x^* J N_x\right) = 0.$$

Thus

$$\tilde{N}^*J\tilde{N} - \tilde{\tau} + \Phi_x\tau_x\Phi_x^* - \tilde{N}^*\Phi_x J\Phi_x^*\tilde{N} = 0.$$

Now to complete the proof of the lemma, it is to be noted that the last equality follows from (27) after taking the limit as $z, \bar{w} \to \infty$, with fixed $x \in [0, l]$.

Lemma is proved.

Now we can construct the mappings

$$\overset{\circ}{V}_1 = \begin{bmatrix} \overset{\circ}{T}_1 & \overset{\circ}{\Phi} \\ \overset{\circ}{\Psi} & \overset{\circ}{K} \end{bmatrix}, \qquad \overset{\circ}{V}_2 = \begin{bmatrix} \overset{\circ}{T}_2 & \overset{\circ}{\Phi}N_l \\ \overset{\circ}{\Psi} & \overset{\circ}{K} \end{bmatrix}, \qquad \overset{\circ+}{V}_1 = \overset{\circ}{V}{}_1^*, \qquad \overset{\circ+}{V}_2 = \begin{bmatrix} \overset{\circ}{T}{}_2^* & \overset{\circ}{\Psi}{}^*\tilde{N}^* \\ \overset{\circ}{\Phi}{}^* & \overset{\circ}{K}{}^* \end{bmatrix}, \tag{38}$$

in $L^2_{r,l}(F_x) \oplus E^r$ where N_l is the value of the solution N_x (21_1) at the point $x = l$; $\overset{\circ}{K} = S_l(\infty) = \Phi_l^*$ and the operators $\overset{\circ}{\Phi}$, $\overset{\circ}{\Psi}$ are defined by the formulas (37) and (34) respectively. It is not difficult to see [2] that

$$\overset{\circ}{V}{}_1^*\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}\overset{\circ}{V}_1 = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \qquad \overset{\circ}{V}_1\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}\overset{\circ}{V}{}_1^* = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}.$$

In order to verify that the analogous relations are valid for $\overset{\circ}{V}_2$ and $\overset{\circ+}{V}_2$, it is necessary to show that

$$\overset{\circ}{T}{}_2^*\overset{\circ}{\Phi}N_l + \overset{\circ}{\Psi}{}^*\tilde{\sigma}\overset{\circ}{K} = 0, \qquad \overset{\circ}{T}{}_2^*\overset{\circ}{\Psi}{}^*\tilde{N}^* + \overset{\circ}{\Phi}\tau_l\overset{\circ}{K}{}^* = 0. \tag{39}$$

To prove the first relation in (39), we consider

$$\overset{\circ}{T}{}_2^*\xi N_l\Psi_x e^{i\varphi_x} = \xi N_l\Psi_x\left(N_x^* + e^{i\varphi_x}\gamma_x^*\right) - 2\int\limits_0^x \xi N_l\Psi_t a_t J N_t^*\Phi_t^{-1}dt\Phi_x,$$

where $\xi \in E^r$. Then taking into account that $\left(\Psi_t N_t^* \Phi_l^{-1}\right)' = 2\Psi_t a_t J N_t^* \Phi_l^{-1}$ we obtain that

$$\mathring{T}_2^* \xi N_l \Psi_x e^{i\varphi_x} = \xi N_l \Psi_x \gamma_x^* e^{i\varphi_x} + \xi N_l \Psi_0 \tilde{N}^* \Phi_x.$$

Since $\Psi_x = \Phi_l^* \Phi_x^{*-1} J$ and $N_x \Phi_x^* = \Phi_x^* \tilde{N}$, we have

$$\mathring{T}_2^* \xi N_l \Psi_x e^{i\varphi_x} = \xi \Psi_x J N_l J \gamma_x^* e^{i\varphi_x} + \xi N_l \Psi_0 \tilde{N}^* \Phi_x,$$

and in view of (29) we obtain the equality

$$\mathring{T}_2^* \xi N_l \Psi_x e^{i\varphi_x} = \xi \Psi_x J \left(\sigma_x - N_x J N_x^*\right) + \xi N_l \Psi_0 \tilde{N}^* \Phi_x.$$

Using the relation $N_x(\Phi_x^* J \Phi_x - J) N_x^* = \Phi_x^* \tilde{\sigma} \Phi_x - \sigma_x$, that follows from (25) as a result by taking the limit as $z, \bar{w} \to \infty$ and the fact that $N_x \Phi_x^* = \Phi_x^* \tilde{N}$, $\Psi_0 = \Psi_l^* J$, we finally get

$$\mathring{T}_2^* \xi N_l \Psi_x e^{i\varphi_x} = \xi \Phi_x^* \tilde{\sigma} \Phi_x,$$

this proves the necessity in view of the definition of the operators $\mathring{\Psi}$ (34), $\mathring{\Phi}$ (37) and $\mathring{K} = \Phi_l^*$.

Let us prove the second equality in (39),

$$\mathring{T}_2^* \xi \tilde{N}^* \Phi_x = \xi \tilde{N}^* \Phi_x J \left(N_x e^{i\varphi_x} + \gamma_x\right) J - 2 \int\limits_x^l \xi \tilde{N}^* \Phi_t a_t \Phi_t^* dt \Phi_x^{*-1} N_x J e^{i\varphi_x},$$

where $\xi \in E^r$, and since $(\Phi_t J \Phi_t^*)' = 2\Phi_t a_t \Phi_t^*$, then

$$\mathring{T}_2^* \xi \tilde{N}^* \Phi_x = \xi \tilde{N}^* \Phi_x J \gamma_x J + \xi \tilde{N}^* \Phi_l J \Phi_l^* \Phi_x^{*-1} N_x J e^{i\varphi_x}.$$

To prove the necessary equality $\mathring{T}_2^* \xi \tilde{N}^* \Phi_x = \xi \Phi_l \tau_l \Psi_x e^{i\varphi_x}$, we have to prove that

$$\tilde{N}^* \Phi_x J \gamma_x J e^{-i\varphi_x} + \tilde{N}^* \Phi_x J \Psi_x J N_x J = \Phi_l \tau_l \Psi_x.$$

Now using the fact that $\tilde{N}^* \Phi_x = \Phi_x N_x^*$ and in view of (30), we obtain

$$\Phi_x \left(\tau_x - N_x^* J N_x\right) J + \tilde{N}^* \Phi_x J \Psi_x J N_x J = \Phi_l \tau_l \Psi_x.$$

As a result of the multiplication from the right by $J\Phi_x^*$, we get

$$\Phi_x \left(\tau_x - N_x^* J N_x\right) \Phi_x^* + \tilde{N}^* \Phi_l J \Phi_l^* \tilde{N} = \Phi_l \tau_l \Phi_l^*.$$

Taking the limit as $z, \bar{w} \to \infty$ in equality (27), we have

$$\tilde{\tau} - \tilde{N}^* J \tilde{N} = \Phi_x \tau_x \Phi_x^* - \tilde{N}^* \Phi_x J \Phi_x^* \tilde{N}.$$

This means that

$$\tilde{\tau} - \tilde{N}^* J \tilde{N} + \tilde{N}^* \Phi_l \tau_l \Phi_l \tilde{N} = \Phi_l \tau_l \Phi_l^*.$$

This relation coincides with the same relation when $x = l$.

Simple test shows that $N_l^* \mathring{\Phi}^* \mathring{\Phi} N_l + \mathring{K}^* \tilde{\sigma} \mathring{K} = \sigma_l$ and $\tilde{N} \mathring{\Psi} \mathring{\Psi}^* \tilde{N}^* + \mathring{K} \tau_l \mathring{K}^* = \tilde{\tau}$, therefore ($15_1$), ($15_2$) are holding for the expansion $\overset{\circ}{V}_2$, $\overset{\circ +}{V}_2$ (38).

Theorem 7. *Suppose that the simple isometric commutative expansion* $\left\{V_s, \overset{+}{V_s}\right\}_1^2$ (14) *corresponding to the commutative system of operators* T_1 *and* T_2 *is such that*:

1) $\dim E = \dim \tilde{E} =< \infty$, *the operators* σ_1, $\tilde{\sigma}_1$, N_1, $\tilde{N}_1$ *are invertible, and the spectrum of the operator* T_1 *lies on the unit circle* $\mathbb{T}$;

2) *the characteristic function* $S_l(z)$ (19) *of the operator* T_1 *is invertible at least at one point* $z \in \mathbb{C}$, *the matrix-valued measure* a_t *and the function* φ_t *from the multiplicative representation of* $S_x(z)$ (19) *are such that the operators* N_x *and* γ_x *exist as the solutions of* (21), (22), *and there exist* σ_x, τ_x, *for which the relations* (26) *and* (28) *respectively are true.*

Then the expansion $\left\{V_s, \overset{+}{V_s}\right\}_1^2$ *is unitarily equivalent to the simple part of the isometric commutative expansion* $\left\{\overset{\circ}{V_s}, \overset{\circ+}{V_s}\right\}_1^2$ (38) *of the commutative operator system* (33) *where* $\overset{\circ}{\Psi}$ *and* $\overset{\circ}{\Phi}$ *are defined by the formulas* (34) *and* (37) *respectively.*

Proof. To prove that the relations $(15_3)-(15_5)$ are true for the expansions $\overset{\circ}{V_s}$, $\overset{\circ+}{V_s}$ (38), we note that equality (15_5), $NK = K\tilde{N}$, follows from the equality $N_x\Phi_x^* = \Phi_x^*\tilde{N}$ in an obvious manner when $x = l$ since $N = N_l$, $N_0 = \tilde{N}$ and $\overset{\circ}{K} = \Phi_l^*$. Moreover, relation (15_4) follows from the intertwining condition (20) when $x = l$ after proceeding to limit as $z \to \infty$. To prove the first of the relations in (15_3), we consider

$$\overset{\circ}{T}{}_2^*\xi\Psi_x e^{i\varphi_x} - \overset{\circ}{T}{}_1^*\xi N_l\Psi_x e^{i\varphi_x} = \xi\left\{\Psi_x J\left(N_x e^{i\varphi_x} + \gamma_x\right)Je^{i\varphi_x} - N_l\Psi_x e^{2i\varphi_x}\right\} -$$
$$-2\xi\int\limits_x^l \Psi_t a_t\Psi_t^* e^{-i\varphi_s}dt\Phi_l^{*-1}N_xJe^{i\varphi_x} + 2\xi\int\limits_x^l N_t\Psi_t a_t\Phi_t^* e^{-i\varphi_s}dt\Phi_l^{*-1}Je^{i\varphi_x}.$$

It is obvious that

$$\overset{\circ}{T}{}_2^*\xi\Psi_x e^{i\varphi_x} - \overset{\circ}{T}{}_1^*\xi N_l\Psi_x e^{i\varphi_x} = \xi F_x e^{i\varphi_x},$$

where the function F_x is given by

$$F_x = \Psi_x J\gamma_x J - 2\int\limits_x^l \Psi_t a_t\Phi_t^* e^{i\varphi_s}dt\Phi_l^{*-1}N_xJ + 2\int\limits_x^l N_t\Psi_t a_t\Phi_t^* e^{i\varphi_s}dt\Phi_l^{*-1}J,$$

since $N_l\Psi_x = \Psi_x JN_xJ$. Elementary calculations using (21_1), (21_2), and (22) show that $F_x' = F_x a_x J$. Taking into account that $F_l = \gamma_l J$, we obtain that $F_x = \gamma_l\Psi_x$. This proves the necessity.

To prove the second of the relations in (15_3), $T_2^*\Psi^* - T_1^*\Psi^*\tilde{N}^* = \Psi^*\tilde{\gamma}^*$, in view of (34), we denote $\overset{\circ}{T}{}_2^*\xi\Phi_x - \overset{\circ}{T}{}_1^*\xi\tilde{N}^*\Phi_x = \xi G_x$ where the operator-function G_x equals to

$$G_x = \Phi_x\gamma_x^* - 2\int\limits_0^x \Phi_t e^{-i\varphi_s}a_tJN_t^*\Phi_t^{-1}dt\Phi_x + 2\int\limits_0^x \Phi_t N_t^* e^{-i\varphi_s}a_tJ\Phi_t^{-1}dt\Phi_x.$$

As in the previous case, it is easy to verify that G_x satisfies the equation $G_x' + G_x a_x J = 0$, and since $G_0 = \tilde{\gamma}^*$, it is obvious that $G_x = \tilde{\gamma}^*\Phi_x$. So both of relations (15_3) are proved.

To use the theorem of the unitary equivalence [1], it is necessary to verify that the characteristic function $\overset{\circ}{S}(z) = \overset{\circ}{K} + \overset{\circ}{\Psi}\left(zI - \overset{\circ}{T}_1\right)^{-1}\overset{\circ}{\Phi}$ of the expansion $\overset{\circ}{V}_1$ (38) coincides with $S_l(z)$ (19). Consider the vector-function $f = \left(zI - \overset{\circ}{T}_1\right)^{-1}\overset{\circ}{\Phi}\xi$ which, obviously, is a solution of the equation

$$\left(z - e^{i\varphi_x}\right) f(x) + 2 \int\limits_x^l f(t) a_t \Phi_t^* dt \Phi_x^{*-1} J e^{i\varphi_x} = \sqrt{2} \xi \Psi_x e^{i\varphi_x}. \tag{40}$$

The function $F(x) = (ze^{-i\varphi_x} - 1) f(x) J$ satisfies the following equation:

$$F(x) + 2 \int\limits_x^l \frac{1}{ze^{-i\varphi_x} - 1} F(t) J a_t \Phi_t^* dt \Phi = \sqrt{2} \xi \Psi_x J.$$

It is obvious that $F(x)$ satisfies the Cauchy problem

$$\begin{aligned} F'(x) &= \frac{z + e^{-i\varphi_x}}{z - e^{-i\varphi_x}} F(x) J a_x, \\ F(l) &= \sqrt{2} \xi, \end{aligned} \tag{41}$$

whose solution is well-known [2] as

$$F(x) = \sqrt{2} \xi \int\limits_x^{\overset{\frown}{l}} \exp \left\{ \frac{e^{i\varphi_t} + z}{e^{i\varphi_t} - z} J dF_t \right\}. \tag{42}$$

Now it follows from (34) that

$$\mathring{\Psi} \left(zI - \mathring{T}_1 \right)^{-1} \mathring{\Phi} \xi = -\sqrt{2} \int\limits_x^l f(t) a_t \Phi_t^* \, dt,$$

then from equation (40), when $x = 0$, we have $\mathring{\Psi}\left(zI - \mathring{T}_1\right)^{-1} \mathring{\Phi} \xi = \dfrac{1}{\sqrt{2}} F(0) - \xi \Psi_0 J$ and since $\Psi_0 J = \Phi_l^* = \mathring{K}$ we finally obtain $\mathring{S}(z) \xi = \dfrac{1}{\sqrt{2}} F(0)$ and so $\mathring{S}(z) = S_l(z)$ (19).

Theorem 7 is proved.

In the case when the spectrum of the operator T_1 lies outside the unit circle $\mathbb{T}$ (this means that in representation (4) there are no multiplier of the form (5)), it's also possible to use the methods presented above to construct the triangular model of the commutative operator system $\{T_1, T_2\}$.

6. Since the simple component of the triangular model $\{\mathring{T}_1, \mathring{T}_2\}$ (32) in $L^2_{r,l}(F_x)$ (6) is given by [6, 7]

$$\mathring{H}_1 = \operatorname{span} \left\{ \mathring{R}_z \mathring{\Phi} g + \mathring{R}_w^* \mathring{\Psi}^* f : g, f \in E^r; z, w \in \mathbb{C} \right\}, \tag{43}$$

where z and w are points of regularity of the resolvent $\mathring{R}_z = \left(zI - \mathring{T}_1\right)$ of the operator $\mathring{T}_1$, consider the vector-function $g(x, z) \in L^2_{r,l}(F_x)$,

$$g(x, z) = \mathring{R}_z \mathring{\Phi} g, \tag{44}$$

where $g \in E^r$ and $z \notin \sigma\left(\mathring{T}_1\right)$. Then it follows from the triangular model (32) that

$$\left\{ \mathring{T}_2 - J N_x J \mathring{T}_1 - J \gamma_x J \right\} g(x, z) = 0. \tag{45}$$

Moreover, we suppose that the matrices-functions JN_xJ and $J\gamma_xJ$ acting as multiplication from the right by every function $f(x) \in L^2_{r,l}(F_x)$.

Denote by $f(x,z)$ the following vector-function from $L^2_{r,l}(F_x)$:

$$f(x,w) = \mathring{R}^*_w \mathring{\Psi}^* f, \tag{46}$$

where $f \in E^r$ and $w \notin \sigma\big(\mathring{T}_1\big)$. It is easy to see that

$$f(x.w) = \frac{\sqrt{2}}{e^{-i\varphi_x} - \bar{w}} f S^*_x(w), \tag{47}$$

where the operator-function $S_x(w)$ is given by (19).

Now we calculate how the model operators $\mathring{T}_1$ and $\mathring{T}_2$ (32) act on $f(x,w)$ (46). Since $\mathring{T}^*_1 \mathring{R}^*_w = \bar{w}\mathring{R}^*_w - I$, then

$$\mathring{T}^*_1 \mathring{R}^*_w \mathring{\Psi} = \mathring{T}_1 \frac{1}{\bar{w}} \left\{ \mathring{T}^*_1 \mathring{R}^*_w + I \right\} \mathring{\Psi}^* = \frac{1}{\bar{w}} \left\{ \mathring{R}^*_w \mathring{\Psi}^* - \mathring{\Phi} J S^*_\Delta(w) \right\},$$

in view of $\mathring{T}_1 \mathring{T}^*_1 + \mathring{\Phi} J \mathring{\Phi}^* = I$, $\mathring{T}_1 \mathring{\Psi}^* + \mathring{\Phi} J \mathring{K} = 0$. Therefore

$$\bar{w}\mathring{T}_1 \mathring{R}^*_w \mathring{\Psi}^* = \mathring{R}^*_w \mathring{\Psi}^* - \mathring{\Phi} J S^*_\Delta(w). \tag{48}$$

Now using the colligation relation (see (15_3), we obtain

$$\mathring{T}_2 \mathring{R}^*_w \mathring{\Psi} \tilde{N}^* = \frac{1}{\bar{w}} T_2 \left\{ \mathring{R}^*_w \mathring{T}^*_1 \mathring{\Psi}^* \tilde{N}^* + \mathring{\Psi} \tilde{N}^* \right\} =$$

$$= \frac{1}{\bar{w}} T_2 \mathring{R}^*_w \left(\mathring{T}^*_2 \mathring{\Psi}^* - \mathring{\Psi}^* \tilde{\gamma}^* \right) - \frac{1}{\bar{w}} \mathring{\Phi} \tau_l \mathring{K}^* = \frac{1}{\bar{w}} \mathring{R}^*_w \mathring{\Psi}^* - \frac{1}{\bar{w}} \mathring{T}_2 \mathring{R}^*_w \mathring{\Psi}^* \tilde{\gamma}^* - \frac{1}{\bar{w}} \mathring{\Phi} \tau_l S^*_\Delta(w),$$

since $\mathring{T}_2 \mathring{T}^*_2 + \mathring{\Phi} \tau_l \mathring{\Phi}^* = I$, $\mathring{T}_2 \mathring{\Psi}^* \tilde{N}^* + \mathring{\Phi} \tau_l \mathring{K}^* = 0$. Hence

$$\mathring{T}_2 \mathring{R}^*_w \mathring{\Psi}^* \left(\tilde{N}^* \bar{\Psi} + \tilde{\gamma}^* \right) = \mathring{R}^*_w \mathring{\Psi}^* - \mathring{\Phi} \tau_l S^*_\Delta(w). \tag{49}$$

Subtracting (49) from (48), we get

$$\bar{w}\mathring{T}_1 \mathring{R}^*_w \mathring{\Psi}^* f - \mathring{T}_2 \mathring{R}^*_w \mathring{\Psi}^* \left(\tilde{N}^* \bar{\Psi} + \tilde{\gamma}^* \right) f = \mathring{\Phi} \left(\tau_1 - J \right) S^*_\Delta(w) f, \tag{50}$$

where $f \in E^r$. It follows from (46), (47) and the intertwining condition (20) that

$$\mathring{R}^*_w \mathring{\Psi}^* \left(\tilde{N}^* \bar{\Psi} + \tilde{\gamma}^* \right) f = \frac{\sqrt{2}}{e^{-i\varphi_x} - \bar{w}} f \left(\tilde{N}^* \bar{w} + \tilde{\gamma}^* \right) S^*_\Delta(w) = f(x,w) \left(N^*_x \bar{w} + \tilde{\gamma}^* \right).$$

Therefore we can write (50) in the form

$$\bar{w}\mathring{T}_1 f(x.w) - \left(\tilde{N}^* \bar{\Psi} + \tilde{\gamma}^* \right) \mathring{T}_2 f(x,w) = \mathring{\Phi} \left(\tau_l - J \right) S^*_\Delta(w) f,$$

and in view of (46), (48) we finally obtain

$$\left\{ \mathring{T}_1 - N^*_x \mathring{T}_2 - \tilde{\gamma}^* \mathring{T}_1 \mathring{T}_2 \right\} f(x,w) = \frac{1}{\bar{w}} \left[\mathring{\Phi} \left(\tau_l - J \right) S^*_\Delta(w) + \tilde{\gamma}^* \mathring{T}_2 \mathring{\Phi} J S^*_\Delta(w) f \right]. \tag{51}$$

Let us show that the right-hand side of equality (51) belongs to the closed linear span generated by the functions $g(x,z)$ (44). From the equality $\mathring{T}_2\mathring{\Phi} = \mathring{T}_1\mathring{\Phi}N_l + \mathring{\Phi}\gamma_l$ (see (15_3)) it follows that it is sufficient to prove that the functions

$$\tilde{\gamma}^*\mathring{T}_1\mathring{\Phi}N_l JS_\Delta^*(w)f, \qquad \tilde{\gamma}^*\mathring{\Phi}\gamma_l JS_\Delta^*(w)f,$$

have this property.

These functions belong to the subspace generated by $g(x,z)$. The proofs are analogous. So, we exhibit the proof of the second only.

Really,

$$\gamma_x\mathring{\Phi}\gamma_l JS_\Delta^*(w)f = \sqrt{2}fS_\Delta^*(w)J\gamma_l\Psi_x\gamma_x^*e^{i\varphi_x} = \sqrt{2}fS_\Delta^*(w)J\gamma_l\gamma_l^*\Psi_x e^{i\varphi_x},$$

in view of (37) and the fact that $\Psi_x\gamma_x^* = \gamma_l^*\Psi_x$.

Since the right-hand side of equality (51) belongs to the subspace generated by the functions $g(x,z)$ (44), then it follows from (45) that

$$\left[\mathring{T}_2 - JN_xJ\mathring{T}_1 - J\gamma_x J\right]\left[\mathring{T}_1 - N_x^*\mathring{T}_2 - \gamma_x^*\mathring{T}_1\mathring{T}_2\right]f(x) = 0,$$

for all $f(x) \in \mathring{H}_1$ (42). From this it easily follows that $\mathbb{P}_x\big(\mathring{T}_1,\mathring{T}_2\big)f(x) = 0$ where

$$\mathbb{P}_x(z_1,z_2) = \det\left\{[N_xz_1 - z_2I + \gamma_x][z_1 - N_x^*z_2 - \gamma_x^*z_1z_2]\right\}.$$

It follows from the intertwining relation (20) that

$$\det[N_xz_1 - z_2I + \gamma_x] = \det\left[\tilde{N}_xz_1 - z_2I + \tilde{\gamma}\right] = \tilde{\mathbb{Q}}(z_1,z_2),$$

$$\det[N_x^*z_2 - z_1I + \gamma_x^*z_1z_2] = \det\left[\tilde{N}^*z_2 - z_1I + \tilde{\gamma}^*z_1z_2\right] = \tilde{\mathbb{Q}}^*(z_1,z_2),$$

so the polynomial $\mathbb{P}_x(z_1,z_2) = \mathbb{P}(z_1,z_2) = \tilde{\mathbb{Q}}(z_1,z_2)\tilde{\mathbb{Q}}^*(z_1,z_2)$ do not depend on x. Thus, the following theorem shows.

Theorem 8. *Suppose that the simple commutative isometric expansion* $\{V_s, \overset{+}{V}_s\}_1^2$(14) *corresponding to the commutative operator system* $\{T_1,T_2\}$ *is such that the suppositions of Theorem* 6 *are true. Then the* T_1, T_2 *operators annul the polynomial* $P(z_1,z_2)$,

$$\mathbb{P}(T_1,T_2) = 0,$$

where $P(z_1,z_2) = \tilde{Q}(z_1,z_2)\tilde{Q}^*(z_1,z_2)$.

This theorem represents an analogue of the Hamilton – Cayley theorem for the system of the commuting operators $\{T_1,T_2\}$ close to the unitary ones and contains several fundamental distinctions from the well-known result of M. S. Livšic [5] for the nonself-adjoint commutative operator systems. Note that the polynomial $\mathbb{P}(z_1,z_2)$ has the following symmetry relatively to the unit circle $\mathbb{T}$:

$$\overline{\mathbb{P}\left(\frac{1}{z_1},\frac{1}{z_2}\right)} = (z_1z_2)^{-2r}\,\mathbb{P}(z_1,z_2).$$

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