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**Jun He** (School Math. and Comput. Sci., China),
**Ting-Zhu Huang, Liang-Jian Deng** (School Math. Sci., Univ. Electron. Sci. and Technology, Chengdu, China)

# A MODIFIED NEWTON METHOD FOR A QUADRATIC VECTOR EQUATION ARISING IN MARKOVIAN BINARY TREES*

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We prove the existence of solution for the quadratic vector equation arising in Markovian binary trees. A modified Newton method for finding the minimal solution of the equation is presented. The monotone convergence of the modified Newton method is proved. Numerical experiments show the efficiency of our method.

Доведено існування розв'язку квадратного векторного рівняння, що виникає в марковських бінарних деревах. Запропоновано модифікований метод Ньютона для знаходження мінімального розв'язку цього рівняння. Встановлено монотонну збіжність модифікованого методу Ньютона. Числові експерименти підтверджують ефективність цього методу.

**1. Introduction.** We consider the following vector equation:

$$x = a + B(x \otimes x), \tag{1}$$

where $a, x \in \mathbb{R}^n$, $a, x \geq 0$, $B \in \mathbb{R}^{n^2 \times n}$ is a nonnegative matrix, and the vector $e = (1, 1, \ldots, 1)^T$ is always a solution of equation (1). The matrix $B$ can be represented by a tensor $\mathcal{B}_{ijk}$, and $\mathcal{B}_{ijk} \geq 0$, $(\mathcal{B}_{ijk} x^2)_i = \sum_{j,k=1}^{n} \mathcal{B}_{ijk} x_j x_k$. Then we can write the vector equation as:

$$x = a + \mathcal{B} x^2.$$

Equations of this type occur in the analysis of the extinction probability of a particular family of multitype branching processes which are called Markovian Binary Trees (MBTs). We can refer to the papers [2, 3] for a detailed description of MBTs. The MBTs is called subcritical, supercritical or critical if the spectral radius $\rho(R)$ of the matrix

$$R := b(e, \cdot) + b(\cdot, e)$$

is strictly less than one, strictly greater than one, or equal to one, where the map $b(u, v) := B(u \otimes v)$. In this paper, we just consider the supercritical case, whereas the minimal nonnegative solution $x_* \leq e$.

Bean, Kontoleon and Taylor [2, 13] introduce two linearly convergent algorithms to solve (1). The first one is named the depth algorithm, which can be seen as the following functional iteration:

$$x_{k+1} = a + b(x_k, x_k).$$

The second algorithm named the order algorithm can be written as:

$$x_{k+1} = a + b(x_{k+1}, x_k), \tag{2}$$

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or equivalently as

$$x_{k+1} = a + b(x_k, x_{k+1}). \tag{3}$$

The thicknesses algorithm, still linearly convergent, is proposed in [3] and consists in alternating iterations of equations (2) and (3). In [1, 14], the authors use another fix point iteration which have been called the Perron iteration to solve the system (1). In this paper, we do not compare the perron iteration with our modified Newton method. If we rewrite equation (1) as

$$F(x) = x - a - b(x, x) = 0, \tag{4}$$

in [5], the authors apply the Newton method to the equation (4) yields the iteration

$$x_{k+1} = (I - b(x_k, \cdot) - b(\cdot, x_k))^{-1}(a - b(x_k, x_k)).$$

the authors show that the resulting sequence is quadratically and globally convergent. A modified version has been proposed in [6].

In this paper, we apply the modified Newton method for computing the minimal solution of the quadratic vector equation arising in Markovian binary trees, and we show the monotone convergence of the modified Newton method.

The paper is organized as follows. In Section 2, some efforts of establishing the solution of the system (1) are made. The convergence dynamics of the modified Newton method is studied in Section 3. In Section 4, some numerical experiments are given to show that the algorithm is efficient.

**2. Theoretical analysis of the solution.** We first introduce the concept of irreducible tensors which is given in [11, 12].

**Definition 1.** *An $m$ th order n-dimensional tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that*

$$a_{i_1, i_2, \ldots, i_m} = 0 \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

*If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.*

In this paper, we are only interested in the case in which $0 \leq \overline{x} \leq e$. We will show the existence of the solution of system (1), and we find that, when the solution is not strictly positive, both functional iterations and Newton-type algorithms may break down [4], if we assume that $\mathcal{B}$ is irreducible or $\mathcal{B}$ is reducible for all $i \in I$, $a_i > 0$, then we can ensure that $\overline{x}$ be positive.

**Theorem 1.** *If $\mathcal{B}$ is a nonnegative tensor, then there exists a nonzero non-negative vector $0 \leq \leq \overline{x} \leq e$ such that $\overline{x} = a + \mathcal{B}\overline{x}^2$. In particular, if $\mathcal{B}$ is irreducible or $\mathcal{B}$ is reducible for all $i \in I$, $a_i > 0$, then $\overline{x}$ must be positive.*

***Proof.*** Let

$$\Omega = \left\{ (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \;\middle|\; 0 \leq x_i \leq 1,\ 1 \leq i \leq n \right\}.$$

It is clear that $\Omega$ is a closed and convex set. We define the following map $\Phi$:

$$(\Phi(x))_i = \left(\mathcal{B}x^2 + a\right)_i.$$

It is clear that $\Phi$ is well-defined and continuous. According to the Brouwer fixed point theorem, there exists $\overline{x} \in \Phi$ such that $\overline{x} = a + \mathcal{B}\overline{x}^2$.

Next we would like to show that $\overline{x}$ is positive when $\mathcal{B}$ is irreducible. Assume that $\overline{x}$ is not positive, i.e., there exist some entries of $\overline{x}$ are zero. Let

$$I = \{i \mid \overline{x}_i = 0\}.$$

It is obvious that $I$ is a proper subset of $\{1, 2, \ldots, n\}$. Let $\delta = \min\{\overline{x}_j \,|j \notin I\}$. We must have $\delta > 0$. Since $\overline{x}$ satisfies $\overline{x} = a + \mathcal{B}\overline{x}^2$, we obtain

$$\sum_{j,k=1}^{n} \mathcal{B}_{ijk}\overline{x}_j\overline{x}_k + a_i = \overline{x}_i = 0 \quad \forall i \in I.$$

It follows that

$$\delta^2 \sum_{j,k\notin I}^{n} \mathcal{B}_{ijk} + a_i \leq \sum_{j,k\notin I}^{n} \mathcal{B}_{ijk}\overline{x}_j\overline{x}_k + a_i = 0.$$

Then we can get the results easily.

Theorem 1 just show the existence of the solution of the system (1), you can return to the paper [4] for more about the minimal solution.

**3. The modified Newton method.** In this part, we first introduce the iterative scheme of the modified Newton method as following: for a given $m \geq 2$ and $k = 1, 2, \ldots,$

$$\begin{gathered} \tilde{x}_{k,1} = x_k + F^{'}(x_k)^{-1} F(x_k), \\ \tilde{x}_{k,p+1} = \tilde{x}_{k,p} - F^{'}(x_k)^{-1} F(\tilde{x}_{k,p}), \quad 1 \leq p \leq m-1, \\ x_{k+1} = \tilde{x}_{k,m}. \end{gathered} \tag{5}$$

And we can find that, if $m = 2$, the modified Newton method reduce to the version of Newton method introduced in [7], see more about the Newton method in [8 – 10]. First, we give some remarks for the modified Newton method.

***Remark* 1.** If $F^{'}(x_k)$ is nonsingular and $x_0$ is sufficiently near $x_*$, the modified Newton method iterates converge $q$-superlinearly to $x_*$ with $q$-order $m+1$. This means that there is $K_s > 0$, such that

$$\|x_{k+1} - x_*\| \leq K_s\|x_k - x_*\|^{m+1}. \tag{6}$$

In fact, from [7], we can get, when $p = 1$, there is $K_c > 0$, such that

$$\|\tilde{x}_{k,2} - x_*\| \leq K_c\|x_k - x_*\|^3.$$

From the quadratic convergence for Newton's method, we can get, there is $K_n > 0$, such that

$$\|\tilde{x}_{k,p+1} - x_*\| \leq K_n^{p-1}\|\tilde{x}_{k,2} - x_*\|^{p-1}.$$

If let $p = m-1$, $K_s = K_n^{m-2}K_c$, we can get (6).

***Remark* 2.** Let us assume that $N_R$ operations are required for the construction of the Jacobi matrix of vector function $F(x)$, while $N_c$ operations are required for the solution of each system of linear algebraic equations in (5), we can find that, $N_R$ and $N_c$ are the same as the operations in the Shamanskii method. That is to say, the optimal value of $p$ is also the same as the optimal value in the Shamanskii method, which have been given in [15] exactly.

In this paper, we assume that, for the minimal solution $x_*$, the matrix $I - b(x_*, \cdot) - b(\cdot, x_*)$ is a nonsingular irreducible $M$-matrix. Since the equation (1) is quadratic, the following expansion holds:

$$F(y) = F(x) + F^{'}(y - x) + \frac{1}{2}F^{''}(y - x, y - x), \tag{7}$$

in particular, if $y = x_*$, the minimal positive solution of the equation (1), and $\Omega(h, h) := F^{''}(h, h)$, we have

$$0 = F(x) + F^{'}(x_* - x) + \frac{1}{2}F^{''}(x_* - x, x_* - x).$$

Then we obtain

$$F(x) = F^{'}(x - x_*) - \frac{1}{2}\Omega(x - x_*, x - x_*), \tag{8}$$

$$F^{'}(x - x_*) = F(x) + \frac{1}{2}\Omega(x - x_*, x - x_*). \tag{9}$$

The following lemma provides a useful tool for analyzing the convergence of the modified Newton method for the the vector equation (1).

**Lemma 1.** *Let $x_*$ be the minimal positive solution of the vector equation* (1). *The sequence of the vector sets $\{x_k, \tilde{x}_{k,1}, \tilde{x}_{k,2}, \ldots, \tilde{x}_{k,m}\}$ generated by the modified Newton method* (5) *with the initial vector $x_0 = 0$ are well defined. If $x_k \leq x_*$, $F(x_k) \leq 0$, for all $k \geq 0$ and $1 \leq p \leq m$, we have*

(a) $F(x_{k+1}) \leq 0$ *and* $F(\tilde{x}_{k,p}) \leq 0$;

(b) $\tilde{x}_{k,1} \leq x_k \leq \tilde{x}_{k,2} \leq \ldots \leq \tilde{x}_{k,m} = x_{k+1} \leq x_*$.

***Proof.*** Firstly, for $p = 1$, we have $\tilde{x}_{k,1} = x_k + F^{'}(x_k)^{-1} F(x_k)$. Since $F(x_k) \leq 0$ and $F^{'}(x)$ is a nonsingular M-matrix, then $\tilde{x}_{k,1} \leq x_k$.

By (7), we have

$$F(\tilde{x}_{k,1}) = F(x_k + F^{'}(x_k)^{-1} F(x_k)) =$$

$$= F(x_k) + F^{'}(x_k)F^{'}(x_k)^{-1}F(x_k) + \frac{1}{2}\Omega(F^{'}(x_k)^{-1}F(x_k), F^{'}(x_k)^{-1}F(x_k)) =$$

$$= 2F(x_k) + \frac{1}{2}\Omega(F^{'}(x_k)^{-1}F(x_k), F^{'}(x_k)^{-1}F(x_k)) \leq 0. \tag{10}$$

For $p = 2$, we obtain

$$\tilde{x}_{k,2} = \tilde{x}_{k,1} - F^{'}(x_k)^{-1} F(\tilde{x_{k,1}}) = x_k + F^{'}(x_k)^{-1}(F(x_k) - F(\tilde{x_{k,1}})).$$

From (10), we can get

$$F(x_k) - F(\tilde{x_{k,1}}) = -F(x_k) - \frac{1}{2}\Omega(F^{'}(x_k)^{-1}F(x_k), F^{'}(x_k)^{-1}F(x_k)) \geq 0,$$

then we obtain $x_k \leq \tilde{x}_{k,2}$.

Let $\tilde{e}_{k,1} = \tilde{x}_{k,1} - x_*$, $\tilde{e}_{k,2} = \tilde{x}_{k,2} - x_*$, and $H(x_2 - x_1) = F^{'}(x_1) - F^{'}(x_2)$, then, by (8),

$$\tilde{e}_{k,2} = \tilde{e}_{k,1} - F^{'}(x_k)^{-1} F(\tilde{x_{k,1}}) =$$

$$= \tilde{e}_{k,1} - F^{'}(x_k)^{-1}(F^{'}(\tilde{x}_{k,1})\,\tilde{e}_{k,1} - \frac{1}{2}\Omega(\tilde{e}_{k,1}, \tilde{e}_{k,1})) =$$

$$= F^{'}(x_k)^{-1}(F^{'}(x_k) - F^{'}(\tilde{x}_{k,1}))\tilde{e}_{k,1} + \frac{1}{2}F^{'}(x_k)\,\Omega(\tilde{e}_{k,1}, \tilde{e}_{k,1}) =$$

$$= F^{'}(x_k)^{-1}\,H(\tilde{x}_{k,2} - \tilde{x}_{k,1})\tilde{e}_{k,1} + \frac{1}{2}F^{'}(x_k)\,\Omega(\tilde{e}_{k,1}, \tilde{e}_{k,1}) \le 0.$$

Thus, $\tilde{x}_{k,2} \le x_*$.

Assume the results are true for $1 \le p \le t$. Then, for $p = t + 1$, with the similar process when $p = 2$, we can get $\tilde{x}_{k,t} \le \tilde{x}_{k,t+1} \le x_*$ and $F(\tilde{x}_{k,t+1}) \le 0$.

Therefore, we have proved the results by induction.

**Theorem 2.** *Let $x_*$ be the minimal positive solution of the vector equation* (1). *The sequence of the vector sets $\{x_k, \tilde{x}_{k,1}, \tilde{x}_{k,2}, \ldots, \tilde{x}_{k,m}\}$ generated by the modified Newton method* (5) *with the initial vector $x_0 = 0$ are well defined. For all $k \ge 0$ and $1 \le p \le m$, we have*

(a) $F(x_k) \le 0$ *and* $F(\tilde{x}_{k,p}) \le 0$;

(b) $\tilde{x}_{0,1} \le x_0 \le \tilde{x}_{0,2} \le \ldots \le \tilde{x}_{0,m} = x_1 \le \tilde{x}_{1,2} \le \ldots \le \tilde{x}_{1,m} = x_2 \le \ldots \le \tilde{x}_{k-1,m} = x_k \le$ $\le \tilde{x}_{k,2} \le \ldots \le \tilde{x}_{k,m} = x_{k+1} \le \ldots \le x_*$;

(c) $\lim_{k\to\infty} x_k = x_*$.

***Proof.*** We prove the theorem by mathematical induction. Firstly, for $k = 0$, from [5], we have $F(x_0) \le 0$ and $F^{'}(x_0)$ is the identity operator, by Lemma 1, we have

$$\tilde{x}_{0,1} \le x_0 \le \tilde{x}_{0,2} \le \ldots \le \tilde{x}_{0,m} = x_1 \le x_*.$$

Assume that the statements (a) – (c) are true for $0 \le k \le i$, again, from Lemma 1, the statements (a) – (c) holds when $k = i + 1$.

Therefore, the statements (a) – (c) are true for $k \ge 0$. The sequence $x_k$ is monotonic and bounded from above by $x_*$, thus it converges. By passing the first equation of (5) to the limit, we see that, the limit of $x_k$ is the minimal positive solution $x_*$ of (1).

**4. Numerical experiments.** In this section, we present our numerical experiments of the modified Newton method to show its efficiency, and let NM and MNM denote the Newton method and the modified Newton method. All of our tests were conducted in MATLAB 7.0. The machine we have used is a PC-Intel(R), Core(TM)2 CPU T7200 2.0 GHz, with 1024 MB of RAM. In all of our runs we used a zero initial guess. The stopping criterion is

$$\|x - a + b(x, x)\| \le n\varepsilon,$$

with $\varepsilon = 10^{-13}$.

***Example* 1.** We use a small-size Markovian binary tree with branches of varying length, described in [5] (Example 2). The characterizing matrices are

$$D_0 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 1 & -10 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 9(1-p) & 0 & 4.5p & 0 & 4.5p \\ 9p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9(1-p) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a = (-D_0)^{-1}d$, and $B = a = (-D_0)^{-1}R$.

***Example* 2.** We use a random-generated MBT of larger size (N=100), the system is created by the following algorithm [14]:

1. **Input**: the size of the MBT and a parameter lambda$>0$
2. $e$=ones$(N,1)$
3. rand('state',0)
4. $K$=max$(b*$kron$(e,e))$+lambda
5. $a = K * e - b*$kron$(e,e)$
6. $a = a/K$
7. $b = b/K$

Table 1
MNM vs NM for Example 1($p = 0.5$) and Example 2($\lambda = 4500$)

| Example | $m$ | ITs | CPU | ERR |
|---|---|---|---|---|
| Example 1 | NM | 10 | 0.1063 | 2.6668e-015 |
| | MNM ($m = 2$) | 8 | 0.1092 | 1.1102e-016 |
| | MNM ($m = 3$) | 7 | 0.0855 | 2.4825e-016 |
| | MNM ($m = 5$) | 6 | 0.0980 | 4.7103e-016 |
| | MNM ($m = 10$) | 5 | 0.0848 | 8.6711e-016 |
| Example 2 | NM | 11 | 2.0022 | 9.0316e-014 |
| | MNM ($m = 2$) | 8 | 1.5054 | 5.4054e-014 |
| | MNM ($m = 3$) | 7 | 1.0215 | 4.0122e-015 |
| | MNM ($m = 5$) | 6 | 1.0061 | 2.4974e-015 |
| | MNM ($m = 10$) | 5 | 0.9262 | 4.0122e-015 |

Table 2
MNM vs SM for Example 2($\lambda = 4950$)

| Example | $m$ | ITs | CPU | ERR |
|---|---|---|---|---|
| Example 2 | NM | 13 | 0.6841 | 2.8029e-13 |
| | MNM ($m = 3$) | 8 | 0.4726 | 3.8391e-14 |
| | SM ($m = 3$) | 10 | 0.5793 | 1.2885e-13 |

In Table 1, we list the number of iterations(ITs), the CPU time (in seconds) and the relative errors(ERR) for the Newton method and the modified Newton method. We observe, for different choice of the $m$, the modified Newton method may iterate faster than the standard Newton method. This is very attractive and has consequential effect on applications. we just test the results when $p = 0.6$ in Example 1, and $\lambda = 4500$ in Example 2, then, the vector of quadratic equation is $(0.9960, 0.9938, 0.9994)^T$ in Example 1.

Figures 1 and 2 show a plot of the iteration count of the Newton method and the modified Newton method with different of the parameter $p$ or $\lambda$, Figures 3 and 4 show a similar plot, considering the CPU time instead of the iteration count. We find that, if $m \ll n$, according to the number of iterations and the CPU time with suitable value of $m$, the modified Newton method can work better than the standard Newton method.

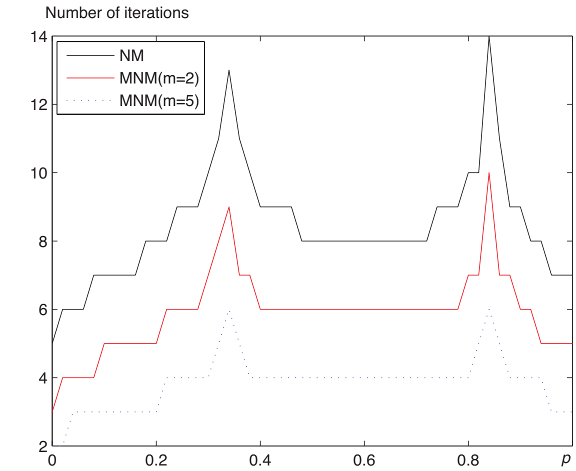


Fig. 1. Number of iterations needed with different $p$ for Example 1.

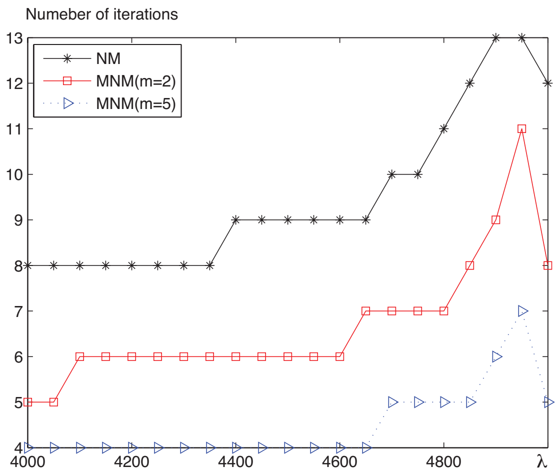


Fig. 2. Number of iterations needed with different $\lambda$ for Example 2.

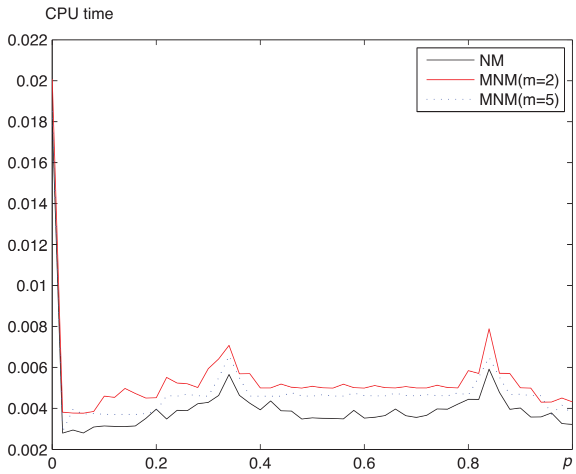


Fig. 3. CPU time needed with different $p$ for Example 1.

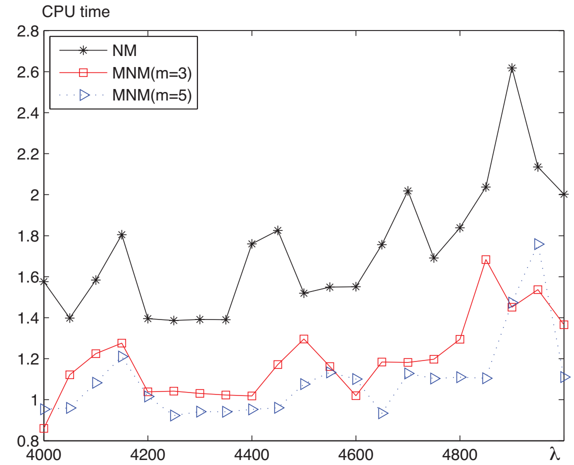


Fig. 4. CPU time needed with different $\lambda$ for Example 2.

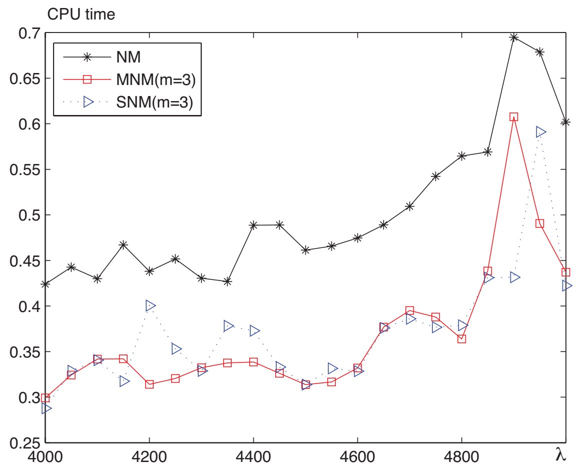


Fig. 5. MNM vs SM for Example 2: CPU time needed with different $\lambda$ for Example 2.

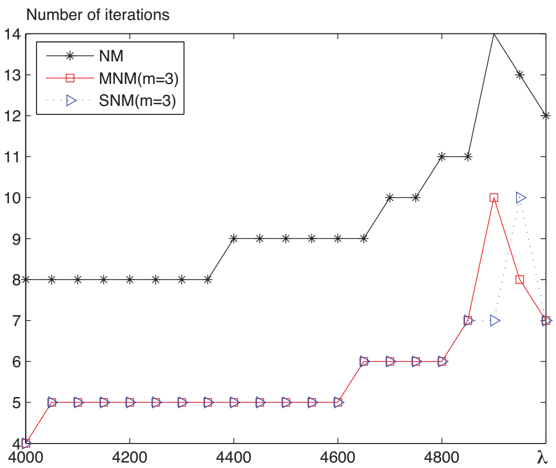


Fig. 6. MNM vs SM for Example 2: Number of iteration needed with different $\lambda$ for Example 2.

In Figures 5 and 6, we show a plot of the iteration count and the CPU time of the Newton method, the modified Newton method and the Shamanskii method (SM) with different of the parameter $p$ or $\lambda$. We observe, when $\lambda = 4950$ in Example 2, the modified Newton method can work better than the standard Newton method and the Shamanskii method. We list the number of iterations, the CPU time and the relative errors for the Newton method, the modified Newton method and the Shamanskii method in Table 2 when $m = 3$. But when $\lambda = 4900$ in Example 2, the Shamanskii method works better than the standard Newton method and the modified Newton method. Therefore, for different value of $\lambda$, we can different methods to solve the quadratic vector equation.

**5. Conclusion.** In this paper, we apply the modified Newton method for a quadratic vector equation arising in Markovian binary trees. The modified Newton method shows its computational advantage, for different choice of the $m$, the modified Newton method may have a better performance than the classical Newton method.

## References


1. *Meini B., Poloni F.* A Perron iteration for the solution of a quadratic vector equation arising in Markovian binary trees // SIAM. J. Matrix Anal. and Appl. – 2011. – **32**. – P. 248 – 261.
2. *Bean N. G., Kontoleon N., Taylor P. G.* Markovian trees: properties and algorithms // Ann. Oper. Res. – 2008. – **160**. – P. 31 – 50.
3. *Hautphenne S., Latouche G., Remiche M. A.* Algorithmic approach to the extinction probability of branching processes // Meth. and Comput. Appl. Probab. – 2009. – **26**. – P. 129 – 149.
4. *Poloni F.* Quadratic vector equations // Linear Algebra and Appl. – 2013. – **438**, Issue 4. – P. 1627 – 1644.
5. *Hautphenne S., Latouche G., Remiche M. A.* NewtonЎЇs iteration for the extinction probability of a Markovian binary tree // Linear Algebra and Appl. – 2008. – **428**. – P. 2791 – 2804.
6. *Hautphenne S., Van Houdt B.* On the link between Markovian trees and tree-structured Markov chains // Eur. J. Oper. Res. – 2010. – **201**. – P. 791 – 798.
7. *Kou J., Li Y., Wang X.* A modification of Newton method with third-order convergence // Appl. Math. and Comput. – 2006. – **181**. – P. 1106 – 1111.
8. *Shamanskii V. E.* A modification of Newtons method // Ukr. Math. J. – 1967. – **19**. – P. 133 – 138.
9. *Yiqun Lin, Liang B.* Convergence analysis of the Newton – Shamanskii method for a nonsymmetric algebraic Riccati equation // Numer. Linear Alg. and Appl. – 2008. – **15**. – P. 535 – 546.
10. *Yiqun Lin, Liang B.* A modified Newton method for solving non-symmetric algebraic Riccati equations arising in transport theory // IMA J. Numer. Anal. – 2008. – **28**. – P. 215 – 224.
11. *Wen Li, Ng M.* On the limiting probability distribution of a transition probability tensor. – Preprint. – 2011, http://www.math.hkbu.edu.hk/mng/tensor-research/highermarkov.pdf
12. *Chang K. C., Pearson K., Zhang T.* Perron-Frobenius theorem for nonnegative tensors // Commun. Math. Sci. – 2008. – **6**. – Issue 2. – P. 507 – 520.
13. *Kontoleon N.* The Markovian binary tree: a model of the macroevolutionary process: Ph.D. thesis. – Univ. Adelaide, 2005.
14. *Bini D. A., Meini B., Poloni F.* On the solution of a quadratic vector equation arising in Markovian binary trees // Numer. Linear Alg. and Appl. – 2011. – **18**. – P. 981 – 991.
15. *Brent R. P.* Some efficient algorithms for solving systems of nonlinear equations // SIAM J. Numer. Anal. – 1973. – **10**. – P. 327 – 344.


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