UDC 517.9

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FIXED-POINT THEOREMS FOR INTEGRAL-TYPE CONTRACTIONS ON PARTIAL METRIC SPACES

ТЕОРЕМИ ПРО НЕРУХОМУ ТОЧКУ ДЛЯ СТИСКІВ ІНТЕГРАЛЬНОГО ТИПУ НА ЧАСТИННИХ МЕТРИЧНИХ ПРОСТОРАХ

We present some fixed-point results for single-valued mappings on partial metric spaces satisfying a contractive condition of integral type.

Наведено деякі теореми про нерухому точку для однозначних відображень на частинних метричних просторах, що задовольняють умову стиску інтегрального вигляду.

1. Introduction and preliminaries. The Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed-point theory and its significance lies in its vast applicability in a number of branches of mathematics. There are a lot of generalization of Banach contraction mapping principle in the literature. One of the most important and interesting generalization of it was given by Branciari [1]. He obtained the existence of fixed points for mappings $f\colon X \to X$ defined on a complete metric space (X,d) satisfying contractive condition of integral type as follows:

$$\int\limits_0^{d(fx,fy)} \varphi(t)dt \leq c \int\limits_0^{d(x,y)} \varphi(t)dt,$$

where $c \in [0,1)$ and $\varphi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, nonnegative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$.

After that in [2], the author proved two fixed-point theorems involving more general contractive condition of integral type.

Recently, there is a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order. In [3, 4] the Banach contraction principle was discussed in a metric space endowed with partial order. Also, existence of fixed point in partially ordered sets has been considered recently in [5 – 15]. The study on the existence of fixed points for single-valued increasing operators is successful, the results obtained are widely used to investigate the existence of solutions to the ordinary and partial differential equations (see [9, 12]). Recently Bhaskar and Lakshmikantham [6], Nieto and Lopez [3, 14], Lakshmikantham and Ciric [13], Ran and Reurings [4] and Agarwal, El-Gebeily and O'Regan [5] presented some new results for contraction in partially ordered metric spaces. Bhaskar and Lakshmikantham [6] noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary-value problem.

On the other hand, after the definitions of partial metric space by Matthews [16], fixed-point theory on this interesting space is rapidly developed. For example in [16 – 24], the authors presented some fixed-point theorems for generalized contractive type mappings on partial metric spaces. Then, Altun and Erduran [25] gave some fixed-point theorems on ordered partial metric spaces.

In this paper, we give fixed-point theorems for mappings satisfying a general contractive condition of integral type on ordered partial metric spaces.

First, we recall some definitions and properties of partial metric space. Further detailed information about partial metric spaces can be found in [16–19].

A partial metric on a nonempty set X is a function $\mathcal{P}\colon X\times X\to\mathbb{R}^+$ such that for all $x,y,z\in X$:

(p_1) $x=y\Leftrightarrow \mathcal{P}(x,x)=\mathcal{P}(x,y)=\mathcal{P}(y,y)$,

(p_2) $\mathcal{P}(x,x)\le\mathcal{P}(x,y)$,

(p_3) $\mathcal{P}(x,y)=\mathcal{P}(y,x)$,

(p_4) $\mathcal{P}(x,y)\le\mathcal{P}(x,z)+\mathcal{P}(z,y)-\mathcal{P}(z,z)$.

A partial metric space is a pair $(X,\mathcal{P})$ such that X is a nonempty set and $\mathcal{P}$ is a partial metric on X. It is clear that, if $\mathcal{P}(x,y)=0$, then from (p_1) and (p_2) $x=y$. But if $x=y$, $\mathcal{P}(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+,\mathcal{P})$, where $\mathcal{P}(x,y)=\max\{x,y\}$ for all $x,y\in\mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [16].

Each partial metric $\mathcal{P}$ on X generates a T_0 topology τ_P on X which has as a base the family of open $\mathcal{P}$-balls $\{B_p(x,\epsilon)\colon x\in X,\epsilon>0\}$, where $B_p(x,\epsilon)=\{y\in X : \mathcal{P}(x,y)<\mathcal{P}(x,x)+\epsilon\}$ for all $x\in X$ and $\epsilon>0$.

If $\mathcal{P}$ is a partial metric on X, then the function $\mathcal{P}^s\colon X\times X\to\mathbb{R}^+$ given by

$$\mathcal{P}^s(x,y)=2\mathcal{P}(x,y)-\mathcal{P}(x,x)-\mathcal{P}(y,y)$$

is a metric on X.

Let $(X,\mathcal{P})$ be a partial metric space. Then:

a sequence $\{x_n\}$ in a partial metric space $(X,\mathcal{P})$ converges to a point $x\in X$ if and only if $\mathcal{P}(x,x)=\lim_{n\to\infty}\mathcal{P}(x,x_n)$;

a sequence $\{x_n\}$ in a partial metric space $(X,\mathcal{P})$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty}\mathcal{P}(x_n,x_m)$;

a partial metric space $(X,\mathcal{P})$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$ such that $\mathcal{P}(x,x)=\lim_{n,m\to\infty}\mathcal{P}(x_n,x_m)$;

a mapping $f\colon X\to X$ is said to be continuous at $x_0\in X$, if for every $\epsilon>0$, there exists $\delta>0$ such that $F(B_p(x_0,\delta))\subseteq B_p(Fx_0,\epsilon)$.

Lemma 1. *Let $(X,\mathcal{P})$ be a partial metric space.*

(1) *$\{x_n\}$ is a Cauchy sequence in $(X,\mathcal{P})$ if and only if it is a Cauchy sequence in the metric space $(X,\mathcal{P}^s)$.*

(2) *A partial metric space $(X,\mathcal{P})$ is complete if and only if the metric space $(X,\mathcal{P}^s)$ is complete. Furthermore, $\lim_{n\to\infty}\mathcal{P}^s(x_n,x)=0$ if and only if*

$$\mathcal{P}(x,x)=\lim_{n\to\infty}\mathcal{P}(x_n,x)=\lim_{n,m\to\infty}\mathcal{P}(x_n,x_m).$$

Now, we give some notations, which will be used in this paper. Let $\varphi\colon [0,+\infty)\to[0,+\infty)$ and $\psi\colon [0,+\infty)\to[0,+\infty)$ be two functions. For convenience, we consider the following properties of these functions:

(φ_1) φ is nonincreasing on $[0,\infty)$;

(φ_2) φ is Lebesgue integrable;

(φ_3) for each $\epsilon>0$, $\int_0^\epsilon \varphi(t)dt>0$.

and

(ψ_1) ψ is nondecreasing on $[0,+\infty)$;

(ψ_2) $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t>0$.

Denote by Φ the family of all functions $\varphi\colon [0,+\infty)\to[0,+\infty)$ satisfying (φ_1), (φ_2) and denote by Ψ the family of all functions $\psi\colon [0,+\infty)\to[0,+\infty)$ satisfying (ψ_1), (ψ_2). Note that Ψ is called the family of (c)-comparison functions in the literature and if $\psi\in\Psi$, then we have $\psi(t)<t$ for each $t>0$.

2. Main results. Let $(X,\leq)$ be a partially ordered set and $T\colon X\to X$ be a mapping, then T is said to be nondecreasing if $x,y\in X$ with $x\leq y$ implies $Tx\leq Ty$ and T is said to be nondecreasing if $x,y\in X$ with $x\leq y$ implies $Ty\leq Tx$.

Theorem 1. *Let $(X,\leq)$ be a partially ordered set and suppose that there is a partial metric $\mathcal{P}$ on X such that $(X,\mathcal{P})$ is a complete partial metric space. Suppose $T\colon X\to X$ is a continuous and nondecreasing mapping such that*

$$\int\limits_0^{\mathcal{P}(Tx,Ty)}\varphi(t)dt\leq\psi\left(\int\limits_0^{M(x,y)}\varphi(t)dt\right) \tag{1}$$

for all $x,y\in X$ with $x\leq y$, where $\varphi\in\Phi$, $\psi\in\Psi$, and

$$M(x,y):=\max\left\{\mathcal{P}(x,y),\mathcal{P}(x,Tx),\mathcal{P}(y,Ty),\frac{[\mathcal{P}(x,Ty)+\mathcal{P}(y,Tx)]}{2}\right\}.$$

If there exists an $x_0\in X$ with $x_0\leq Tx_0$, then there exists $x\in X$ such that $\mathcal{P}(x,Tx)=0$, in particular, $x=Tx$ and $\mathcal{P}(x,x)=0$.

Proof. Let $x_n=Tx_{n-1}$ for $n=1,2,\ldots$. Suppose that $x_{n+1}=x_n$ for some n. Then $x_n=Tx_n$ and x_n is a fixed point, therefore the proof is finished. Now, assume that $x_n\neq x_{n+1}$ for all n. Since $x_0<Tx_0$ and T is a nondecreasing mapping, we obtain by induction that

$$x_0<x_1\leq x_2\leq\ldots\leq x_n\leq x_{n+1}\leq\ldots.$$

From (1) and, as the elements x_n and x_{n+1} are comparable, we get

$$\int\limits_0^{\mathcal{P}(x_n,x_{n+1})}\varphi(t)dt=\int\limits_0^{\mathcal{P}(Tx_{n-1},Tx_n)}\varphi(t)dt\leq\psi\left(\int\limits_0^{M(x_{n-1},x_n)}\varphi(t)dt\right), \tag{2}$$

where

$$M(x_{n-1},x_n)=\max\left\{\mathcal{P}(x_{n-1},x_n),\mathcal{P}(x_{n-1},x_n),\mathcal{P}(x_n,x_{n+1}),\frac{[\mathcal{P}(x_{n-1},x_{n+1})+\mathcal{P}(x_n,x_n)]}{2}\right\},$$

and, as

$$\frac{\mathcal{P}(x_{n-1},x_{n+1})+\mathcal{P}(x_n,x_n)}{2} \leq \frac{\mathcal{P}(x_{n-1},x_n)+\mathcal{P}(x_n,x_{n+1})}{2} \leq$$
$$\leq \max\{\mathcal{P}(x_{n-1},x_n),\mathcal{P}(x_n,x_{n+1})\},$$

we obtain

$$M(x_{n-1},x_n) = \max\{\mathcal{P}(x_n,x_{n-1}),\mathcal{P}(x_n,x_{n+1})\}.$$

Now, if

$$\max\{\mathcal{P}(x_n,x_{n-1}),\mathcal{P}(x_n,x_{n+1})\} = \mathcal{P}(x_n,x_{n+1}),$$

for some n, then using (2), (φ_3) and the condition $\mathcal{P}(x_n,x_{n+1})>0$, we have

$$\int\limits_0^{\mathcal{P}(x_n,x_{n+1})} \varphi(t)dt \leq \psi\left(\int\limits_0^{\mathcal{P}(x_n,x_{n+1})} \varphi(t)dt\right) < \int\limits_0^{\mathcal{P}(x_n,x_{n+1})} \varphi(t)dt,$$

that is a contradiction. Thus

$$M(x_{n-1},x_n) = \max\{\mathcal{P}(x_n,x_{n-1}),\mathcal{P}(x_n,x_{n+1})\} = \mathcal{P}(x_n,x_{n-1}) \quad \text{for all} \quad n.$$

Then taking into account (2) and (ψ_1), we get

$$\int\limits_0^{\mathcal{P}(x_n,x_{n+1})} \varphi(t)dt \leq \psi\left(\int\limits_0^{\mathcal{P}(x_{n-1},x_n)} \varphi(t)dt\right) \leq \ldots \leq \psi^n(d), \tag{3}$$

where

$$d := \int\limits_0^{\mathcal{P}(x_0,x_1)} \varphi(t)dt.$$

In what follows we will show that $\{x_n\}$ is a Cauchy sequence in the metric space $(X,\mathcal{P}^s)$. By definition of $\mathcal{P}^s$ we have

$$\mathcal{P}^s(x_n,x_{n+1}) = 2\mathcal{P}(x_n,x_{n+1}) - \mathcal{P}(x_n,x_n) - \mathcal{P}(x_{n+1},x_{n+1}) \leq 2\mathcal{P}(x_n,x_{n+1})$$

or, equivalently,

$$\frac{1}{2}\mathcal{P}^s(x_n,x_{n+1}) \leq \mathcal{P}(x_n,x_{n+1}). \tag{4}$$

Let $m,n \in \mathbb{N}$, $m>n$. Then using the triangle inequality and (4)

$$\frac{1}{2}\mathcal{P}^s(x_n,x_m) \leq \frac{1}{2}\sum_{i=n}^{m-1}\mathcal{P}^s(x_i,x_{i+1}) \leq \sum_{i=n}^{m-1}\mathcal{P}(x_i,x_{i+1}).$$

Since φ is nonincreasing we obtain

$$\int\limits_0^{a+b}\varphi(t)dt \le \int\limits_0^{a}\varphi(t)dt + \int\limits_0^{b}\varphi(t)dt \quad \text{for all} \quad a,b \ge 0.$$

Now, we can get

$$\int\limits_0^{\frac{1}{2}\mathcal{P}^s(x_n,x_m)}\varphi(t)dt \le \int\limits_0^{\sum_{i=n}^{m-1}\mathcal{P}(x_i,x_{i+1})}\varphi(t)dt \le \sum_{i=n}^{m-1}\int\limits_0^{\mathcal{P}(x_i,x_{i+1})}\varphi(t)dt. \tag{5}$$

Using (3) and (5),

$$\int\limits_0^{\frac{1}{2}\mathcal{P}^s(x_n,x_m)}\varphi(t)dt \le \sum_{i=n}^{m-1}\psi^i(d) \le \sum_{i=n}^{\infty}\psi^i(d).$$

By the convergence of the series $\sum_{i=1}^{\infty}\psi^i(d)$, passing to the limit as $n,m \to \infty$, we have $\int_0^{1/2\mathcal{P}^s(x_n,x_m)}\varphi(t)dt \to 0$. Using condition (φ_3), it follows that $\mathcal{P}^s(x_n,x_m)\to 0$, that is, $\{x_n\}$ is a Cauchy sequence in the metric space $(X,\mathcal{P}^s)$. Since $(X,\mathcal{P})$ is complete, by Lemma 1, $(X,\mathcal{P}^s)$ is complete and the sequence $\{x_n\}$ is convergent in X, say $\lim_{n\to\infty}\mathcal{P}^s(x_n,x)=0$. From Lemma 1, we get

$$\mathcal{P}(x,x) = \lim_{n\to\infty}\mathcal{P}(x_n,x) = \lim_{n,m\to\infty}\mathcal{P}(x_n,x_m).$$

Now we show that $\mathcal{P}(x,Tx)=0$. Suppose $\mathcal{P}(x,Tx)>0$. Since T is continuous, then given $\epsilon>0$, there exists $\delta>0$ such that $T(B_p(x,\delta))\subseteq B_p(Tx,\epsilon)$. Since $\mathcal{P}(x,x)=\lim_{n\to\infty}\mathcal{P}(x_n,x)$, then there exists $k\in\mathbb{N}$ such that $\mathcal{P}(x_n,x)<\mathcal{P}(x,x)+\delta$ for all $n\ge k$. Therefore, we have $x_n\in B_p(x,\delta)$ for all $n\ge k$. Thus $Tx_n\in T(B_p(x,\delta))\subseteq B_p(Tx,\epsilon)$ and so $\mathcal{P}(Tx_n,Tx)<\mathcal{P}(Tx,Tx)+\epsilon$ for all $n\ge k$. This shows that $\lim_{n\to\infty}\mathcal{P}(x_{n+1},Tx)=\mathcal{P}(Tx,Tx)$. Then we have

$$\mathcal{P}(x,Tx) \le \mathcal{P}(x,x_{n+1}) + \mathcal{P}(x_{n+1},Tx) - \mathcal{P}(x_{n+1},x_{n+1}),$$

and letting $n\to\infty$, we get

$$\mathcal{P}(x,Tx) \le \mathcal{P}(Tx,Tx).$$

Now, by using inequality (1) and (φ_3) and (ψ_2) we obtain

$$\int\limits_0^{\mathcal{P}(x,Tx)}\varphi(t)dt \le \int\limits_0^{\mathcal{P}(Tx,Tx)}\varphi(t)dt \le$$

$$\le \psi\left(\int\limits_0^{M(x,x)}\varphi(t)dt\right) =$$

$$= \psi\left(\int\limits_0^{\max\{\mathcal{P}(x,x),\mathcal{P}(x,Tx)\}}\varphi(t)dt\right) =$$

$$= \psi\left(\int_0^{\mathcal{P}(x,Tx)} \varphi(t)dt\right) < \int_0^{\mathcal{P}(x,Tx)} \varphi(t)dt,$$

that is a contradiction. Thus $\mathcal{P}(x, Tx) = 0$.

Theorem 1 is proved.

In what follows we will prove a variant of Theorem 1, where we will use the following properties of the space X instead of the continuity of T:

(H_1) if $\{x_n\}$ is a nondecreasing sequence in X such that $\mathcal{P}(x, x) = \lim_{n\to\infty} \mathcal{P}(x, x_n)$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $(X, \leq)$ be a partially ordered set and suppose $\mathcal{P}$ is a partial metric on X such that $(X, \mathcal{P})$ is a complete partial metric space. Assume that X satisfies* (H_1)*. Let $T\colon X \to X$ be a nondecreasing mapping such that*

$$\int_0^{\mathcal{P}(Tx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) \tag{6}$$

for all $x, y \in X$ with $x \leq y$, where $\varphi \in \Phi$, $\psi \in \Psi$, and

$$M(x, y) := \max\left\{\mathcal{P}(x, y), \mathcal{P}(x, Tx), \mathcal{P}(y, Ty), \frac{[\mathcal{P}(x, Ty) + \mathcal{P}(y, Tx)]}{2}\right\}.$$

If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$, then there exists $x \in X$ such that $\mathcal{P}(x, Tx) = \mathcal{P}(x, x)$.

***Remark* 1.** For the point x, which is the limit of iterative sequence of $\{x_n\}$, in Theorem 2, if $\mathcal{P}(x, x) = 0$, then x is a fixed point of T. We know that the condition $\mathcal{P}(x, x) = 0$ is not strong. For example, if $(X, \mathcal{P})$ is a 0-complete partial metric space, then $\mathcal{P}(x, x) = 0$, because x is the limit of iterative sequence of $\{x_n\}$. We can find some detailed information about 0-complete partial metric space in [23]. If $(X, \mathcal{P})$ is an ordinary metric, then $\mathcal{P}(x, x) = 0$ is also satisfied.

Proof. Following the proof of Theorem 1 we only have to check that $\mathcal{P}(x, Tx) = \mathcal{P}(x, x)$. Suppose that $\mathcal{P}(x, Tx) > \mathcal{P}(x, x)$, as $\mathcal{P}(x, x) = \lim_{n\to\infty} \mathcal{P}(x_n, x) = \lim_{n\to\infty} \mathcal{P}(x_n, x_{n+1})$, there exists $k \in \mathbb{N}$ such that

$$\mathcal{P}(x_n, x) < \mathcal{P}(x, x) + \frac{\mathcal{P}(x, Tx) - \mathcal{P}(x, x)}{2} = \frac{\mathcal{P}(x, Tx) + \mathcal{P}(x, x)}{2}$$

and

$$\mathcal{P}(x_n, x_{n+1}) < \frac{\mathcal{P}(x, Tx) + \mathcal{P}(x, x)}{2}$$

for all $n \geq k$. Then for $n \geq k$ we have

$$\frac{1}{2}[\mathcal{P}(x, Tx_n) + \mathcal{P}(x_n, Tx)] \leq$$

$$\leq \frac{1}{2}[\mathcal{P}(x, x_{n+1}) + \mathcal{P}(x_n, x) + \mathcal{P}(x, Tx) - \mathcal{P}(x, x)] < \mathcal{P}(x, Tx).$$

Now from (H_1), we get $x_n \leq x$ for all $n \in \mathbb{N}$ and so we can use the inequality (6). Therefore, for $n \geq k$ we obtain

$$\int_0^{\mathcal{P}(Tx,x_{n+1})} \varphi(t)dt = \int_0^{\mathcal{P}(Tx,Tx_n)} \varphi(t)dt \le \psi\left(\int_0^{M(x,x_n)} \varphi(t)dt\right), \tag{7}$$

where

$$M(x,x_n) = \max\left\{\mathcal{P}(x,x_n), \mathcal{P}(x,Tx), \mathcal{P}(x_n,Tx_n), \frac{1}{2}[\mathcal{P}(x,Tx_n)+\mathcal{P}(x_n,Tx)]\right\} = \mathcal{P}(x,Tx).$$

Therefore from (7) we get

$$\int_0^{\mathcal{P}(Tx,x_{n+1})} \varphi(t)dt \le \psi\left(\int_0^{\mathcal{P}(x,Tx)} \varphi(t)dt\right). \tag{8}$$

On the other hand

$$\mathcal{P}(x,Tx) \le \lim_{n\to\infty} \mathcal{P}(x,x_{n+1}) + \lim_{n\to\infty} \mathcal{P}(x_{n+1},Tx) - \lim_{n\to\infty} \mathcal{P}(x_{n+1},x_{n+1}) =$$
$$= \lim_{n\to\infty} \mathcal{P}(x_{n+1},Tx).$$

Therefore by (8) and $\psi(t) < t$, we have

$$\int_0^{\mathcal{P}(x,Tx)} \varphi(t)dt \le \lim_{n\to\infty}\left(\int_0^{\mathcal{P}(Tx,x_{n+1})} \varphi(t)dt\right) \le$$
$$\le \psi\left(\int_0^{\mathcal{P}(x,Tx)} \varphi(t)dt\right) < \int_0^{\mathcal{P}(x,Tx)} \varphi(t)dt,$$

that is a contradiction. Thus $\mathcal{P}(x,Tx) = \mathcal{P}(x,x)$.

Theorem 2 is proved.

***Remark* 2.** If X is a totally ordered set in Theorem 1, then the fixed point of T is unique.

Suppose that there exist $z,y \in X$ which are two fixed points of T. Since X is totally ordered set, then z is comparable to y, and so $T^n z = z$ is comparable to $T^n y = y$ for all $n \in \mathbb{N}$. Therefore from (1) we get $\mathcal{P}(z,y) = 0$ since $M(y,z) = \mathcal{P}(z,y)$.

***Remark* 3.** In the following theorem we present parallel results to Theorems 1 and 2 for nonincreasing functions.

Theorem 3. *Let $(X,\le)$ be a partially ordered set and suppose that there is a partial metric $\mathcal{P}$ on X such that $(X,\mathcal{P})$ is a complete partial metric space. Suppose $T\colon X \to X$ is a nonincreasing mapping such that*

$$\int_0^{\mathcal{P}(Tx,Ty)} \varphi(t)dt \le \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right)$$

for all $x,y \in X$ with $x \le y$, where $\varphi \in \Phi$, $\psi \in \Psi$, and

$$M(x,y) := \max\left\{\mathcal{P}(x,y), \mathcal{P}(x,Tx), \mathcal{P}(y,Ty), \frac{[\mathcal{P}(x,Ty)+\mathcal{P}(y,Tx)]}{2}\right\}.$$

If T is continuous and there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$ then the existence of a fixed point is proved. Suppose that $x_0 \neq Tx_0$, by our assumption $T^{n+1}x_0$ and $T^n x_0$ are comparable, for every $n = 0, 1, 2, \ldots$. Using the same argument that in Theorem 1 we prove that $\{T^n x_0\}$ is convergent to certain $x \in X$ and by the continuity of T, we see that x is fixed point of T.

Theorem 3 is proved.

We can consider the following condition instead of the continuity of T in Theorem 3:

(H_2) if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable and such that $\mathcal{P}(x,x) = \lim_{n\to\infty} \mathcal{P}(x,x_n)$, then there exists a subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to x for all $n \in \mathbb{N}$.

***Remark* 4.** Suppose that all conditions of Theorem 3 are satisfied except for the continuity of T. If the condition (H_2) hold, then there exists $x \in X$ such that $\mathcal{P}(x,Tx) = \mathcal{P}(x,x)$.

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Received 11.06.12,
after revision — 03.01.16