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A NEW APPLICATION OF QUASIMONOTONE SEQUENCES

НОВЕ ЗАСТОСУВАННЯ КВАЗІМОНОТОННИХ ПОСЛІДОВНОСТЕЙ

We prove a general theorem dealing with generalized absolute Cesàro summability factors of infinite series. This theorem also includes some new and known results.

Доведено загальну теорему про узагальнені абсолютні фактори сумовності Чезаро для нескінченних рядів. Ця теорема також включає ряд нових та відомих результатів.

1. Introduction. A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A sequence (d_n) is said to be δ-quasimonotone, if $d_n \to 0$, $d_n > 0$ ultimately, and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta > -1$, of the sequence (na_n), that is (see [8])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha} = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^{\alpha}). \tag{2}$$

Let $(\theta_n^{\alpha,\beta})$ be a sequence defined by (see [5])

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \quad \beta > -1, \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \quad \beta > -1. \end{cases} \tag{3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta|_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|t_n^{\alpha,\beta}\right|^k < \infty. \tag{4}$$

If we take $\beta = 0$, then $|C,\alpha,\beta|_k$ summability reduces to $|C,\alpha|_k$ summability (see [10]). Also, if we take $\beta = 0$ and $\alpha = 1$, then we have $|C,1|_k$ summability. In [6], we proved the following theorem dealing with $|C,\alpha,\beta|_k$ summability factors of infinite series.

Theorem A. *Let* $(\theta_n^{\alpha,\beta})$ *be a sequence defined as in* (3). *Let* (X_n) *be an almost increasing sequence such that* $|\Delta X_n| = O(X_n/n)$ *and* $\lambda_n \to 0$ *as* $n \to \infty$. *Suppose that there exists a sequence of numbers* (A_n) *such that it is* δ*-quasimonotone with* $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ *is convergent, and* $|\Delta\lambda_n| \le |A_n|$ *for all* n. *If the condition*

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{5}$$

satisfies, then the series $\sum a_n\lambda_n$ *is summable* $|C,\alpha,\beta|_k$, $0 < \alpha \le 1$, $\beta > -1$, $(\alpha+\beta) > 0$, *and* $k \ge 1$.

2. Main result. The aim of this paper is to prove Theorem A under weaker conditions. Now, we shall prove the following theorem.

Theorem. *Let* $(\theta_n^{\alpha,\beta})$ *be a sequence defined as in* (3). *Let* (X_n) *be an almost increasing sequence such that* $|\Delta X_n| = O(X_n/n)$ *and* $\lambda_n \to 0$ *as* $n \to \infty$. *Suppose that there exists a sequence of numbers* (A_n) *such that it is* δ*-quasimonotone with* $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ *is convergent, and* $|\Delta\lambda_n| \le |A_n|$ *for all* n. *If the condition*

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{nX_n^{k-1}} = O(X_m) \qquad as \qquad m \to \infty \tag{6}$$

satisfies, then the series $\sum a_n\lambda_n$ *is summable* $|C,\alpha,\beta|_k$, $0 < \alpha \le 1$, $\beta > -1$, $k \ge 1$, *and* $(\alpha+\beta-1) > 0$.

Remark. It should be noted that condition (6) is the same as condition (5) when $k = 1$. When $k > 1$ condition (6) is weaker than condition (5), but the converse is not true. In fact, as in [11], if (5) is satisfied, then we get

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n\, X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} = O(X_m) \qquad \text{as} \qquad m \to \infty.$$

To show that the converse is false when $k > 1$, similar as in [7], the following example is sufficient. We can take $X_n = n^\epsilon$, $0 < \epsilon < 1$, and then construct a sequence (u_n) such that

$$\frac{(\theta_n^{\alpha,\beta})^k}{n{X_n}^{k-1}} = X_n - X_{n-1},$$

whence

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n{X_n}^{k-1}} = X_m = m^\epsilon,$$

and so

$$\sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} = \sum_{n=1}^{m} (X_n - X_{n-1})X_n^{k-1} = \sum_{n=1}^{m} (n^\epsilon - (n-1)^\epsilon) n^{\epsilon(k-1)} \ge$$

$$\ge \epsilon \sum_{n=1}^{m} (n-1)^{\epsilon-1} n^{\epsilon(k-1)} =$$

$$= \epsilon \sum_{n=1}^{m} (n-1)^{\epsilon k - 1} \sim \frac{m^{\epsilon k}}{k} \quad \text{as} \quad m \to \infty.$$

It follows that

$$\frac{1}{X_m} \sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta})^k}{n} \to \infty \quad \text{as} \quad m \to \infty$$

provided $k > 1$. This shows that (5) implies (6) but not conversely.

We need the following lemmas for the proof of our theorem.

Lemma 1 [3]. *Under the conditions of the theorem, we have*

$$|\lambda_n| X_n = O(1) \qquad \textit{as} \qquad n \to \infty. \tag{7}$$

Lemma 2 [4]. *Under the conditions of the theorem, we have*

$$n A_n X_n = O(1) \qquad \textit{as} \qquad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} n X_n |\Delta A_n| < \infty. \tag{9}$$

Lemma 3 [5]. *If* $0 < \alpha \leq 1$, $\beta > -1$, *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \right|. \tag{10}$$

3. Proof of the theorem. Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(n a_n \lambda_n)$. Then, by (1), we obtain

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

First, applying Abel's transformation and then using Lemma 3, we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$|T_n^{\alpha,\beta}| \leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \right| + \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \right| \leq$$

$$\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} |\Delta\lambda_v| + |\lambda_n| \theta_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.$$

To complete the proof of the theorem, by Minkowski's inequality , it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}|T_{n,r}^{\alpha,\beta}|^k<\infty \qquad \text{for} \qquad r=1,2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\dfrac{1}{k}+\dfrac{1}{k'}=1$, we get

$$\sum_{n=2}^{m+1}\frac{1}{n}|T_{n,1}^{\alpha,\beta}|^k\le\sum_{n=2}^{m+1}\frac{1}{n}\left|\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}\theta_v^{\alpha,\beta}\Delta\lambda_v\right|^k=$$

$$=O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta)k}}\left\{\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}|A_v|^k(\theta_v^{\alpha,\beta})^k\right\}\times\left\{\sum_{v=1}^{n-1}1\right\}^{k-1}=$$

$$=O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}|A_v|^k(\theta_v^{\alpha,\beta})^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2+(\alpha+\beta-1)k}}=$$

$$=O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}|A_v||A_v|^{k-1}(\theta_v^{\alpha,\beta})^k\int\limits_v^{\infty}\frac{dx}{x^{2+(\alpha+\beta-1)k}}=$$

$$=O(1)\sum_{v=1}^{m}|A_v|v^{k-1}\frac{(\theta_v^{\alpha,\beta})^k}{v^{k-1}X_v^{k-1}}=O(1)\sum_{v=1}^{m}v|A_v|\frac{(\theta_v^{\alpha,\beta})^k}{vX_v^{k-1}}=$$

$$=O(1)\sum_{v=1}^{m-1}\Delta(v|A_v|)\sum_{p=1}^{v}\frac{(\theta_p^{\alpha,\beta})^k}{pX_p^{k-1}}+O(1)m|A_m|\sum_{v=1}^{m}\frac{(\theta_v^{\alpha,\beta})^k}{vX_v^{k-1}}=$$

$$=O(1)\sum_{v=1}^{m-1}|(v+1)\Delta|A_v|-|A_v||X_v+O(1)m|A_m|X_m=$$

$$=O(1)\sum_{v=1}^{m-1}v|\Delta A_v|X_v+O(1)\sum_{v=1}^{m-1}|A_v|X_v+O(1)m|A_m|X_m=$$

$$=O(1) \qquad \text{as} \qquad m\to\infty,$$

in view of hypotheses of the theorem and Lemma 2. Again, we obtain

$$\sum_{n=1}^{m}\frac{1}{n}|T_{n,2}^{\alpha,\beta}|^k=\sum_{n=1}^{m}|\lambda_n||\lambda_n|^{k-1}\frac{(\theta_n^{\alpha,\beta})^k}{n}=O(1)\sum_{n=1}^{m}|\lambda_n|\frac{(\theta_n^{\alpha,\beta})^k}{nX_n^{k-1}}=$$

$$=O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\sum_{v=1}^{n}\frac{(\theta_v^{\alpha,\beta})^k}{vX_v^{k-1}}+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{(\theta_n^{\alpha,\beta})^k}{nX_n^{k-1}}=$$

$$=O(1)\sum_{v=n}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m=$$

$$=O(1)\sum_{n=1}^{m-1}|A_n|X_n+O(1)|\lambda_m|X_m=O(1) \qquad \text{as} \qquad m\to\infty,$$

by virtue of the hypotheses of the theorem and Lemma 1.

The theorem is proved.

If we take $\beta = 0$, then we get a new result dealing with the $|C,\alpha|_k$ summability factors. Also if we take $\beta = 0$ and $\alpha = 1$, then we obtain a new result concerning the $|C,1|_k$ summability factors.

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