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CHARACTERIZATION OF THE GROUP $G_2(5)$ BY THE PRIME GRAPH

ХАРАКТЕРИЗАЦІЯ ГРУПИ $G_2(5)$ ЗА ДОПОМОГОЮ ПРОСТОГО ГРАФА

Let G be a finite group. The prime graph of G is a graph $\Gamma(G)$ with vertex set $\pi(G)$ and the set of all prime divisors of $|G|$, where two distinct vertices p and q are adjacent by an edge if G has an element of order pq. We prove that if $\Gamma(G) = \Gamma(G_2(5))$, then G has a normal subgroup N such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.

Нехай G – скінченна група. Простим графом G називається граф $\Gamma(G)$ з множиною вершин $\pi(G)$ та множиною всіх простих дільників $|G|$, в якому дві різні вершини p і q сполучені ребром, якщо G містить елемент порядку pq. Доведено, що у випадку, коли $\Gamma(G) = \Gamma(G_2(5))$, група G містить нормальну підгрупу N таку, що $\pi(N) \subseteq \{2, 3, 5\}$ та $G/N \cong G_2(5)$.

1. Introduction. For $n \in N$, let $\pi(n)$ denote the set of all the prime divisors of n, and for a finite group G let us set $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of a finite group G is a simple graph with vertex set $\pi(G)$ in which two distinct vertices p and q are joined by an edge if and only if G has an element of order pq. It is clear that a knowledge of $w(G)$ determines $\Gamma(G)$ completely but not vise-versa in general. Given a finite group G, the number of nonisomorphic classes of finite groups H with $\Gamma(G) = \Gamma(H)$ is denoted by $h_\Gamma(G)$. If $h_\Gamma(G) = 1$, then G is said to be recognizable by prime graph. If $h_\Gamma(G) = k < \infty$, then G is called k-recognizable by prime graph, in case $h_\Gamma(G) = \infty$ the group G is called nonrecognizable by prime graph. Obviously a group recognizable by spectra need not to be recognizable by prime graph, for example A_5 is recognizable by spectra but $\Gamma(A_5) = \Gamma(A_6)$.

The number of connected components of $\Gamma(G)$ is denoted by $s(G)$. As a consequence of the classification of the finite simple groups it is proved in [19] and [9], that for any finite simple group G we have $s(G) \leq 6$. Let $\pi_i = \pi_i(G)$, $1 \leq i \leq s$, be the connected components of G. For a group of even order we let $2 \in \pi_1$. Recognizability of groups by prime graph was first studied in [5] where some sporadic simple groups were characterized by prime graph. As another concept we say that a non-Abelian simple group G is quasirecognizable by graph if every finite group whose prime graph is $\Gamma(G)$ has a unique non-Abelian composition factor isomorphic to G.

It is proved in [20] that the simple groups $G_2(7)$ and ${}^2G_2(q)$, $q = 3^{2m+1} > 3$, are recognizable by prime graph, where both groups have disconnected prime graphs. A series of interesting results concerning recognition of finite simple groups were obtained by B. Khosravi et al. In particular they have stablished quasirecognizability of the group $L_{10}(2)$ by graph and the recognizability of $L_{16}(2)$ by graph in [7] and [8], where both groups have connected prime graphs.

Next we introduce useful notation. Let p be a prime number. The set of all non-Abelian finite simple groups G such that $p \in \pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $\mathfrak{S}_p$. It is clear that the set of all non-Abelian finite simple groups is the disjoint union of the finite sets $\mathfrak{S}_p$ for all primes p. The sets $\mathfrak{S}_p$, where p is a prime less than 1000 is given in [21].

2. Preliminary results. Let G be a finite group with disconnected prime graph. The structure of G is given in [19] which is stated as a lemma here.

Lemma 2.1. *Let G be a finite group with disconnected prime graph. Then G satisfies one of the following conditions*:

(a) $s(G) = 2$, $G = KC$ *is a Frobenius group with kernel K and complement C, and the two connected components of G are $\Gamma(K)$ and $\Gamma(C)$. Moreover K is nilpotent, and here $\Gamma(K)$ is a complete graph.*

(b) $s(G) = 2$ *and G is a 2-Frobeuius group, i.e., $G = ABC$, where $A, AB \trianglelefteq G$, $B \trianglelefteq BC$, and AB, BC are Frobenius groups.*

(c) *There exists a non-Abelian simple group P such that $P \leq \overline{G} = G/N \leq \mathrm{Aut}(P)$ for some nilpotent normal $\pi_1(G)$-subgroup N of G and $\overline{G}/P$ is a $\pi_1(G)$-group. Moreover*, $\Gamma(P)$ *is disconnected and $s(P) \geq s(G)$.*

If a group G satisfies condition (c) of the above lemma we may write $P = B/N$, $B \leq G$, and $\overline{G}/P = G/B = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where N is a nilpotent normal $\pi_1(G)$-subgroup of G and A is a $\pi_1(G)$-group.

The above structure lemma was extended to groups with connected prime graphs satisfying certain conditions [17]. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$.

In the following we list some properties of the Frobenius group where some of its proof can be found in [15].

Lemma 2.2. *Let G be a Frobenius group with kernel K and complement H. Then*:

(a) *K is nilpotent and $|H| \mid (|K| - 1)$.*

(b) *The connected components of G are $\Gamma(K)$ and $\Gamma(H)$.*

(c) $|\mu(K)| = 1$ *and $\Gamma(K)$ is a complete graph.*

(d) *If $|H|$ is even, then K is Abelian.*

(e) *Every subgroup of H of order pq, p and q not necessary distinct primes, is cyclic. In particular if H is Abelian, then it would be cyclic.*

(f) *If H is nonsolvable, then there is a normal subgroup H_0 of H such that $[H : H_0] \leq 2$ and $H_0 \cong SL_2(5) \times Z$, where every Sylow subgroup of Z is cyclic and $|Z|$ is prime to $2, 3$ and 5.*

A Frobenius group with cyclic kernel of order m and cyclic complement of order n is denoted by $m : n$.

The following result is also used in this paper whose proof is included in [3].

Lemma 2.3. *Every 2-Frobenius group is solvable.*

Lemma 2.4 [6]**.** *Let G be a finite solvable group all of whose elements are of prime power order, then the order of G is divisible by at most two distinct primes.*

Lemma 2.5 [12]**.** *Let G be a finite group, $K \trianglelefteq G$, and let G/K be a Frobenius group with kernel F and cyclic complement C. If $(|F|, |K|) = 1$ and F dose not lie in $(K \cdot C_G(K))/K$, then $r \cdot |C| \in w(G)$ for some prime divisor r of $|K|$.*

Lemma 2.6 [18]**.** *$L_n(q)$ contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1} - 1)/(n, q - 1)$.*

Using [1], we can find $\mu(G_2(5)) = \{20, 21, 24, 25, 30, 31\}$. Therefore, the prime graph of $G_2(5)$ is as a follows.

Our main results are the followin theorem.

Theorem 2.1. *If G is a finite group such that $\Gamma(G) = \Gamma(G_2(5))$, then G has a normal subgroup N such that $\pi(N) \subseteq \{2, 3, 5\}$ and $G/N \cong G_2(5)$.*

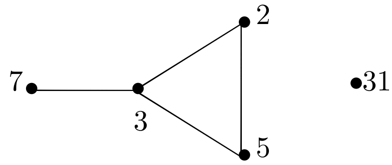

Fig. 1. The prime graph of $G_2(5)$.

3. Proof of the theorem. We assume G is a group with $\Gamma(G) = \Gamma(G_2(5))$. By Fig. 1, we have $s(G) = 2$, hence, G has disconnected prime graph and we can use Lemma 2.1 here:

(a) G is nonsolvable. If G is solvable, then consider a $\{5, 7, 31\}$-Hall subgroup of G and call it H. By Fig. 1, H dose not contain elements of order $5 \cdot 7,\ 7 \cdot 31,\ 5 \cdot 31$, and since it is solvable, by Lemma 2.4 we deduce $|t(H)| \leq 2$, a contradiction.

(b) G is neither a Frobenius nor a 2-Frobenius group. By (a) and Lemma 2.3, G is not a 2-Frobenius group. If G is a Frobenius group, then by Lemma 2.1, $G = KC$ with Frobenius kernel K and Frobenius complement C with connected components $\Gamma(K)$ and $\Gamma(C)$. Obviously $\Gamma(K)$ is a graph with vertex $\{31\}$ and $\Gamma(C)$ with vertex set $\{2, 3, 5, 7\}$. Since G is nonsolvable, by Lemma 2.2(a) C must be nonsolvable. Therefore, by Lemma 2.2(f) C has a subgroup isomorphic to H_0 and $[C : H_0] \leq 2$, where $H_0 \cong SL_2(5) \times Z$ with Z cyclic of order prime to $2, 3, 5$. But $\mu(SL_2(5)) = \{4, 6, 10\}$ from which we can observe that H_0 has no element of order 15. This implies that C has no element of order 15, contradicting Fig. 1.

Conditions (a) and (b) imply that case (c) of Lemma 2.1 holds for G. Hence, there is a non-Abelian simple group P such that $P \leq \overline{G} = G/N \leq \mathrm{Aut}(P)$ where N is a nilpotent normal $\pi_1(G)$-subgroup of G and $\overline{G}/P$ is a $\pi_1(G)$-group and $s(P) \geq 2$. We have $\pi_1(G) = \{2, 3, 5, 7\}$ and $\pi(G) = \{2, 3, 5, 7, 31\}$. Therefore, P is a simple group with $\pi(P) \subseteq \{2, 3, 5, 7, 31\}$, i.e., $P \in \mathfrak{S}_p$ where p is a prime number satisfying $p \leq 31, p \neq 11, 13, 17, 19, 23, 29$. Using [21] we list the possibilities for P in Table 1.

(c) $\{31\} \subseteq \pi(P)$. By Table 1, $|\mathrm{Out}(P)|$ is a number of the form $2^\alpha \cdot 3^\beta$, therefore, if $G/N = P \cdot S$, where $S \leq \mathrm{Out}(P)$, then $|P|_p = |G/N|_p/|S|_p$ for all $p \in \pi(G)$, where n_p denotes the p-part of the integer $n \in N$. Hence, $|N|_p = \dfrac{|G|_p}{|P|_p|S|_p}$, from which the claim follows because $\pi(N) \subseteq \{2, 3, 5, 7\}$.

Therefore only the following possibilities arise for P: $L_2(31)$, $L_5(2)$, $L_6(2)$, $L_3(5)$, $L_2(5^3)$ and $G_2(5)$.

(d) $P \cong G_2(5)$. By [4], we have $\mu(L_5(2)) = \{8, 12, 14, 15, 21, 31\}$ and $\mu(L_6(2)) = \{8, 12, 28, 30, 31, 63\}$. Therefore, if $P \cong L_5(2)$ or $L_6(2)$, then we have $2 \sim 7$ in $\Gamma(G)$, is a contradiction.

By [10], we have $\mu(L_2(5^3)) = \{5, 62, 63\}$. Therefore, if $P \cong L_2(5^3)$, then we have $2 \sim 31$ in $\Gamma(G)$, a contradiction.

By [1], we have $\mu(L_2(31)) = \{15, 16, 31\}$. Therefore, if $P \cong L_2(31)$, then $7 \in \pi(N)$. By Lemma 2.6, P has a Frobenius subgroup $31 : 15$, then, by Lemma 2.5, G has an element of order $5 \cdot 7$, a contradiction.

By [1], we have $\mu(L_3(5)) = \{20, 24, 31\}$. Therefore, if $P \cong L_3(5)$, then $7 \in \pi(N)$. By Lemma 2.6, P has a Frobenius subgroup $25 : 24$, then, by Lemma 2.5, G has an element of order $2 \cdot 7$, a contradiction. Therefore $P \cong G_2(5)$.

(e) $G/N \cong G_2(5)$. So far we proved that $P \leq G/N \leq \mathrm{Aut}(P)$ where $P \cong G_2(5)$. But $\mathrm{Aut}(G_2(5)) = G_2(5)$, therefore, $G/N \cong G_2(5)$.

Table 1. Simple groups in $\mathfrak{S}_p,\ p \leq 31,\ p \neq 11, 13, 17, 19, 23, 29$

P	$\|P\|$	$\|\mathrm{Out}(P)\|$	P	$\|P\|$	$\|\mathrm{Out}(P)\|$
A_5	$2^2 \cdot 3 \cdot 5$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$L_3(5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$L_2(5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	6
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$G_2(5)$	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	2

(f) $\pi(N) \subseteq \{2, 3, 5\}$. We know that N is a nilpotent normal $\{2, 3, 5, 7\}$-subgroup of G. Regarding Fig. 1 we obtain:

if $2, 5 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5\}$;

if $3 \mid |N|$, then $\pi(N) \subseteq \{2, 3, 5, 7\}$;

if $7 \mid |N|$, then $\pi(N) \subseteq \{3, 7\}$.

Now we observe that the group $G_2(5)$ contains Frobenius subgroup $31 : 5$. We may assume N is elementary Abelian p-group for $p \in \{2, 3, 5, 7\}$. Now if $7 \mid |N|$, then by Lemma 2.5, G has an element of order $5 \cdot 7$, a contradiction. Therefore, $\pi(N) \subseteq \{2, 3, 5\}$.

References

1. *Conway J. H., Curtis R. T., Norton S. P., Parker R. A., Wilson R. A.* Atlas of finite groups. – Oxford: Clarendon Press, 1985.
2. *Darafsheh M. R.* Order of elements in the groups related to the general linear groups // Finite Fields and Appl. – 2005. – **11**. – P. 738 – 747.
3. *Darafsheh M. R.* Pure characterization of the projective special linear groups // Ital. J. Pure and Appl. Math. – 2008. – № 23. – P. 229 – 244.
4. *Darafsheh M. R., Farjami Y.* Calculating the set of elements in the finite linear groups // J. Discrete Math. Sci. – 2007. – **10**, № 5. – P. 637 – 653.
5. *Hagie M.* The prime graph of a sporadic simple group // Communs Algebra. – 2003. – **31**, № 9. – P. 4405 – 4424.
6. *Higman G.* Finite groups in which every element has prime power order // J. London Math. Soc. – 1957. – **32**. – P. 335 – 342.
7. *Khosravi B.* Quasirecognition by prime graph of $L_{10}(2)$ // Sib. Math. J. – 2009. – **50**, № 2. – P. 355 – 359.
8. *Behrooz Khosravi, Bahman Khosravi, Behnam Khosravi.* A characterization of the finite simple group $L_{16}(2)$ by its prime graph // Manuscr. Math. – 2008. – **126**. – P. 49 – 58.
9. *Kondratiev A. S.* On prime graph components for finite simple groups // Math. Sb. – 1989. – **180**, № 6. – P. 787 – 797.
10. *Lucido M. S., Moghaddamfar A. R.* Groups with complete prime graph connected components // J. Group Theory. – 2004. – **31**. – P. 373 – 384.

11. *Mazurov V. D., Xu M. C., Cao H. P.* Recognition of finite simple groups $L_3(2^m)$ and $U_3(2^m)$ by their element orders // Algebra and Logika. – 2000. – **39**, № 5. – P. 567 – 585.
12. *Mazurov V. D.* Recognition of finite simple groups $S_4(q)$ by their element orders // Algebra and Logic. – 2002. – **41**, № 2. – P. 93 – 110.
13. *Mazurov V. D., Chen G. Y.* Recognisability of finite simple groups $L_4(2^m)$ and $U_4(2^m)$ by spectrum // Algebra and Logic. – 2008. – **47**, № 1. – P. 49 – 55.
14. *Mazurov V. D.* Characterization of finite groups by sets of element orders // Algebra and Logic. – 1997. – **36**, № 1. – P. 23 – 32.
15. *Passman D. S.* Permutation groups. – New York: W. A. Benjamin Inc., 1968.
16. *Shi W. J., Yang W. Z.* A new characterization of A_5 and the finite groups in which every non-identity element has prime order // J. Southwest China Teachers College. – 1984. – P. 9 – 36 (in Chinese).
17. *Vasilev A. V.* On connection between the structure of a finite group and the properties of its prime graph // Sib. Math. J. – 2005. – **46**, № 3. – P. 396 – 404.
18. *Vasilev A. V., Grechkoseeva M. A.* On recognition by spectrum of finite simple linear groups over fields of characteristic 2 // Sib. Math. J. – 2005. – **46**, № 4. – P. 593 – 600.
19. *Williams J. S.* Prime graph components of finite groups // J. Algebra. – 1981. – **69**, № 2. – P. 487 – 513.
20. *Zavarnitsine A. V.* Recognition of finite groups by the prime graph // Algebra and Logic. – 2006. – **45**, № 4.
21. *Zavarnitsine A. V.* Finite simple groups with narrow prime spectrum // Sib. Electron. Math. Rep. – 2009. – **6**. – P. 1 – 12.

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