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IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH VARIABLE MOMENTS

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We establish some existence results for the solutions of initial-value problems for fractional-order impulsive functional differential equations with neutral-delay at variable moments.

Встановлено деякі результати про існування розв'язків початкової задачі для імпульсивних фунцкціонально-диференціальних рівнянь з нейтральним запізненням в змінні моменти часу.

1. Introduction. We deal with the existence of solutions to the following initial value problem (IVP) for the neutral impulsive fractional differential equations with variable times:

$$D^\alpha\big[x(t)-g(t,x_t)\big]=f(t,x_t),\quad t\in J,\qquad t\neq\tau_k\big(x(t)\big),\quad 0<\alpha\leq 1, \tag{1.1}$$

$$x(t^+)=I_k\big(x(t)\big),\quad t=\tau_k\big(x(t)\big), \tag{1.2}$$

$$x(t)=\phi(t),\quad t\in[-\rho,0], \tag{1.3}$$

where D^α is Caputo fractional derivative, $J=[0,T]$, $0<\rho<\infty$, $\mathcal{U}=\{\psi\colon[-\rho,0]\to R^n$ is continuous everywhere except for a finite number of points s at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-)=\psi(s)\}$, and $\phi\in\mathcal{U}$, $f,g\colon J\times\mathcal{U}\to R^n$, $I_k\colon R^n\to R^n, \tau_k\colon R^n\to R$, $k=1,2,\ldots,p$, are given functions satisfying some hypotheses to be specified later. For any function x defined on $[-\rho,T]$ and any $t\in J$ we denote by x_t the element of $\mathcal{U}$ defined by $x_t=x(t+\theta)$, $\theta\in[-\rho,0]$.

As well as fractional calculus [1 – 8], impulsive differential equations [9 – 15] play an important role in mathematical modeling of many practical phenomena arising in engineering and various areas of science. That is why, many scientists and researchers have devoted a great deal of attention to the topic of impulsive fractional differential equations during the past decades [16 – 24].

Incidentally, we should note that impulsive effects for differential equations are classified as fixed moments ($t=t_k$) and variable moments ($t=\tau_k(x(t))$) in the mentioned literature above. What is more, as far as we know, whereas some authors have addressed the functional(delay or neutral) impulsive differential equations of integer orders with both fixed and variable moments [25 – 29] and those of fractional orders with fixed moments [30, 31], only one author has considered impulsive retarded functional differential equations of fractional order with variable moments up to now [32].

Hence we are in the position to continue on this way, that is, we will take into account a class of fractional order neutral functional impulsive differential equations with variable moments in (1.1) – (1.3) by generalizing the integer order functional impulsive differential equations with variable moments

$$\frac{d}{dt}[y(t)-g(t,y_t)]=f(t,y_t),\qquad t\in J=[0,T],\quad t\neq\tau_k(y(t)), \tag{1.4}$$

$$y(t^+)=I_k(y(t)),\qquad t=\tau_k(y(t)), \tag{1.5}$$

$$y(t) = \phi(t), \qquad t \in [-\rho, 0], \quad 0 < \rho < \infty, \tag{1.6}$$

in [25] to the fractional order ones.

Throughout this paper, in Section 2 we firstly introduce some notations, definitions and basic facts to be used this work. Then we will establish sufficient conditions for existence of solution to the initial value problem (1.1)–(1.3) by extending the appreciable results in [25] consisting of (1.4)–(1.6). At the end, we will present an effective example illustrating the main result.

2. Basic results and preliminaries. By $C(J, R^n)$, $C\big([-\rho, 0], R^n\big)$ and $C\big([-\rho, T], R^n\big)$ we denote the Banach space of all continuous functions from J into R^n with the norm

$$\|x\|_C := \sup\big\{|x(t)| : t \in J\big\},$$

the Banach space of all continuous functions from $[-\rho, 0]$ into R^n with the norm

$$\|\phi\|_{\mathcal{U}} := \sup\big\{\|\phi(\theta)\| : \theta \in l - \rho, 0]\big\}$$

and the Banach space of all continuous functions from $[-\rho, T]$ into R^n with the norm

$$\|x\| := \max\big\{\|x\|_C, \|\phi\|_{\mathcal{U}}\big\},$$

respectively.

In order to define the solutions of problem (1.1)–(1.3) we will consider the piecewise continuous spaces:

$\Omega = \{x : [-\rho, T] \to R^n$: there exists $0 = t_0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = T$ such that $t_k = \tau_k(x(t_k))$ and $x_{k+1} \in C((t_k, t_{k+1}], R^n)$, $k = 0, 1, 2, \ldots, p\}$. Also, there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k)$ for $k = 1, 2, \ldots, p$, and $x(t) = \phi(t)$, $t \leq t_0$, where x_{k+1} is the restriction of x over $(t_k, t_{k+1}]$ and denoted by $x_{k+1} := x|_{(t_k, t_{k+1}]}$, $k = 0, 1, 2, \ldots, p$.

The space Ω forms a Banach space with the norm

$$\|x\|_\Omega := \max\big\{\|x_{k+1}\|, k = 0, 1, \ldots, p\big\} + \|\phi\|_{\mathcal{U}}.$$

Definition 2.1 [1, 2]. *The fractional* (*arbitrary*) *order integral of the function* $h \in L^1(J, R)$ *of order* $\alpha \in R_+$ *is defined by*

$$I_{0^+}^\alpha h(t) = \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where $\Gamma(.)$ *is the Euler gamma function.*

Definition 2.2 [1, 2]. *For a function* h *given on the interval* J, *Caputo fractional derivative of order* $\alpha > 0$ *is defined by*

$$D_{0^+}^\alpha h(t) = \int\limits_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} h^{(n)}(s)ds, \quad n = [\alpha] + 1,$$

where the function $h(t)$ *has absolutely continuous derivatives up to order* $(n-1)$.

Theorem 2.1 [33]. *If* U *is closed*, *bounded*, *convex subset of a Banach space* X *and the mapping* $A : U \to U$ *is completely continuous*, *then* A *has a fixed point in* U.

Theorem 2.2 [34]. *If* $x(t) \in C^1[0, T]$, *then for* $\alpha_1, \alpha_2 \in R^+$ *and* $\alpha_1 + \alpha_2 \leq 1$ *we have* $D^{\alpha_1} D^{\alpha_2} x(t) = D^{\alpha_2} D^{\alpha_1} x(t) = D^{\alpha_1 + \alpha_2} x(t)$.

As a matter of convenience, we shall use: $J_1 = [t_1, T]$, $J_2 = [t_2, T], \ldots, J_k = [t_k, T]$, $1 \leq k \leq p$.

3. Main results.

Definition 3.1. *A function* $x \in \Omega$ *is said to be a solution of problem* (1.1)–(1.3) *if* x *satisfies the equation* (1.1) *and the conditions* (1.2) *and* (1.3) *are satisfied for* x.

Now, let us state the following assumptions in order to establish some existence results for the solutions of the IVP (1.1)–(1.3):

(A_1) The function $g\colon J \times \mathcal{U} \to R^n$ is completely continuous with the set $\{t \to g(t,u)\colon u \in S\}$ equicontinuous for any bounded set S in $C([-\rho, T], R^n)$ such that $|g(t,u)| \leq q(t)$ for all $t \in J$, $u \in \mathcal{U}$, where $q(t)$, $t \in J$, is a function with $q^0 = \sup\big\{|q(t)|\colon t \in J\big\}$.

(A_2) The function $f\colon J \times \mathcal{U} \to R^n$ and $\mathcal{I}_k\colon R^n \to R^n$, $k = 1, 2, \ldots, p$, are continuous and there exist a function $\kappa(t) \geq 0$, $t \in J$, with $\kappa^0 = \sup\big\{|\kappa(t)|\colon t \in J\big\}$ such that $\big|f(t,u)\big| \leq \kappa(t)$ for all $t \in J$, $u \in \mathcal{U}$.

(A_3) There exist the functions $\tau_k \in C^1(R^n, R)$ for $k = 1, 2, \ldots, p$ such that $0 < \tau_1(x) < < \tau_2(x) < \ldots < \tau_k(x) < T$ for all $x \in R^n$.

Lemma 3.1 [35]. *The function* $x(t) \in C\big([-\rho, T], R^n\big)$ *is a solution of the problem*

$$
\begin{gathered}
D^\alpha\big[x(t) - g(t, x_t)\big] = f(t, x_t), \qquad t \in J, \quad 0 < \alpha \leq 1, \\
x(t) = \phi(t), \quad t \in [-\rho, 0],
\end{gathered}
\tag{3.1}
$$

if and only if $x(t)$ *satisfies the following integral equation*:

$$
x(t) = \begin{cases} \phi(t), & t \in [-\rho, 0], \\[2ex] \phi(0) - g(0, \phi) + g(t, x_t) + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s)\, ds, & t \in J. \end{cases}
\tag{3.2}
$$

Theorem 3.1. *In addition to the assumptions* (A_1)–(A_3) *let the following ones be satisfied*:

(A_4) *Either* g *is a nonnegative function and* τ_k*is a nonincreasing function*, *or* g *is a nonpositive function and* τ_k *is a non-decreasing function.*

(A_5) *For all* $x \in R^n$, $\tau_k(x) < \tau_{k+1}(I_k(x))$, $k = 1, 2, \ldots, p$.

(A_6) *Let* $x \in \Omega$, *then for any* $t \in J$ *we have*

$$
\big\langle \tau_k'(x(t) - g(t, x_t)),\ D^{1-\alpha} f(t, x_t) \big\rangle \neq 1
$$

for $k = 1, 2, \ldots, p$, *where* $\langle .,. \rangle$ *denotes the scalar product in* R^n.

Then the IVP (1.1)–(1.3) *has at least one solutions on* J.

Proof. The proof will be carried out in several steps:

Step 1: Consider the following problem:

$$
D^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \qquad t \in J, \quad 0 < \alpha \leq 1, \tag{3.3}
$$

$$
x(t) = \phi(t), \quad t \in [-\rho, 0]. \tag{3.4}
$$

Let us transform the problem (3.3), (3.4) into a fixed point problem. In view of Lemma 3.1, consider the operator $F\colon C\big([-\rho, T], R^n\big) \to C\big([-\rho, T], R^n\big)$ defined by

$$
F(x)(t) = \begin{cases} \phi(t), & t \in [-\rho, 0], \\[2ex] \phi(0) - g(0, \phi) + g(t, x_t) + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s)\, ds, & t \in J. \end{cases}
$$

We will use Schauder's fixed point theorem in order to show that the operator F has fixed points giving the solution to problem (3.3), (3.4). First of all, we define the set $C_r = \{x(t) \in C([-\rho, T], R^n) : \|x\| \le r \text{ for } r > 0\}$ which is obviously closed, bounded and convex. Then, we will prove the completely continuity of F in order to satisfy the rest of conditions of the Schauder's fixed point theorem. To do this, it is enough to show that the operator

$$\widetilde{F}(x)(t) = \begin{cases} \phi(t), & t \in [-\rho, 0], \\ \phi(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s)\, ds, & t \in J, \end{cases}$$

is completely continuous.

To begin with, for each $t \in J$, the continuity of the functions ϕ and f implies that $\widetilde{F}$ is continuous. For the compactness of $\widetilde{F}$:

(i) There exists a constant $L > 0$ such that we have $\left\|\widetilde{F}x\right\| \le L$ for each $x \in C_r$. In view of (A_1) and (A_2) we have, for each $t \in J$,

$$\left|\widetilde{F}(x)(t)\right| \le |\phi(0)| + \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\kappa(s)| ds \le \left\|\phi(0)\right\| + \kappa^0 \frac{T^\alpha}{\Gamma(\alpha+1)} := L, \qquad \left\|\widetilde{F}(x)(t)\right\| \le L$$

which implies that the operator $\widetilde{F}$ is uniformly bounded.

(ii) Let $l_1, l_2 \in J,\ l_1 < l_2$ and $x \in C_r$. Then, for each $t \in J$, we obtain

$$\left|\widetilde{F}(x)(l_2) - \widetilde{F}(x)(l_1)\right| \le \int\limits_0^{l_1} \frac{\left[(l_2-s)^{\alpha-1} - (l_1-s)^{\alpha-1}\right]}{\Gamma(\alpha)} \left|\kappa(s)\right| ds + \int\limits_{l_1}^{l_2} \frac{(l_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left|\kappa(s)\right| ds,$$

$$\left\|\widetilde{F}(x)(l_2) - \widetilde{F}(x)(l_1)\right\| \le \frac{\kappa^0}{\Gamma(\alpha+1)} \left|2(l_2-l_1)^\alpha + l_1^\alpha - l_2^\alpha\right| := K,$$

implying that $\widetilde{F}$ is equicontinuous on J since the right-hand side of the inequality converges to zero as $l_1 \to l_2$.

Consequently, as a result of Arzela–Ascoli theorem, the operator $\widetilde{F}$ is compact and continuous, that is, it is completely continuous.

Therefore, thanks to Schauder's fixed point theorem, we deduce that F has a fixed point which is a solution of problem (3.3), (3.4). We note this solution by x_1.

Now we will discuss possible discontinuity moment the solution $x(t)$ may beat. Let us define the following function so that our discussion will become easier:

$$\sigma_{k,1}(t) = \tau_k(x_1(t)) - t, \quad t \ge 0.$$

From (A_3) we get

$$\sigma_{k,1}(0) = \tau_k(x_1(0)) \ne 0, \quad k = 1, 2, \ldots, p.$$

If $\sigma_{k,1}(t) \ne 0$, that is, $\tau_k(x_1(t)) \ne t$ on J for $k = 1, 2, \ldots, p$, then $x_1(t)$ is a solution of both (3.3), (3.4) and (1.1)–(1.3).

Now, we are in position to consider the case when

$$\sigma_{1,1}(t) = 0, \quad \text{i.e.,} \quad t = \tau_1(x_1(t)) \quad \text{for some} \quad t \in J.$$

Since $\sigma_{1,1}$ is continuous and $\sigma_{1,1}(0) \neq 0$ by (A_3), there exists $t_1 > 0$ such that

$$\sigma_{1,1}(t_1) = 0 \quad \text{and} \quad \sigma_{1,1}(t) \neq 0 \quad \text{for all} \quad t \in [0, t_1).$$

Hence by (A3) we obtain

$$\sigma_{k,1}(t) \neq 0 \quad \text{for all} \quad t \in [0, t_1) \quad \text{and} \quad k = 1, 2, \ldots, p.$$

Thus, we have formed the discontinuity point t_1 where the solution $x(t)$ beats.

Step 2: Consider the following problem:

$$D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \qquad t \in J_1, \quad 0 < \alpha \leq 1, \tag{3.5}$$

$$x(t_1^+) = I_1(x_1(t_1)), \tag{3.6}$$

$$x(t) = x_1(t), \quad t \in [t_1 - \rho, t_1]. \tag{3.7}$$

Let us transform the problem (3.5)–(3.7) into a fixed point problem by considering the operator $F_1 \colon C\big([t_1 - \rho, T], R^n\big) \to C\big([t_1 - \rho, T], R^n\big)$ defined by

$$F_1(x)(t) = \begin{cases} x_1(t), & t \in [t_1 - \rho, t_1], \\ I_1(x_1(t_1)) - g(t_1, x_{t_1}) + g(t, x_t) + \displaystyle\int_{t_1}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s)\, ds, & t \in J_1. \end{cases}$$

Pursuing the process in the 1^{st} $Step$, as a consequence of Schauder's fixed point theorem, one can conclude that F_1 has a fixed point which is a solution of the problem (3.5)–(3.7) on J_1 by proving the completely continuity of the operator

$$\widetilde{F}_1(x)(t) = \begin{cases} x_1(t), & t \in [t_1 - \rho, t_1], \\ I_1(x_1(t_1)) + \displaystyle\int_{t_1}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s)\, ds, & t \in J_1. \end{cases}$$

Let us indicate this solution as x_2.

Then we will investigate a possible discontinuity moment coming after t_1 that the solution $x(t)$ meets. Let us state the function

$$\sigma_{k,2}(t) = \tau_k(x_2(t)) - t, \quad t \geq t_1.$$

If $\sigma_{k,2}(t) \neq 0$, that is, $\tau_k(x_2(t)) \neq t$ on $(t_1, T]$ for $k = 1, 2, \ldots, p$, then $x_2(t)$ is a solution of problem (3.5)–(3.7). That is,

$$x(t) = \begin{cases} x_1(t), & t \in [t_0, t_1], \\ x_2(t), & t \in (t_1, T], \end{cases}$$

is a solution of problem (1.1)–(1.3).

Now, let us consider the following case:

$$\sigma_{2,2}(t) = 0, \quad \text{i.e.,} \quad t = \tau_2(x_2(t)) \quad \text{for some} \quad t \in (t_1, T].$$

Then from (A_5) we have

$$\sigma_{2,2}(t_1^+) = \tau_2(x_2(t_1^+)) - t_1 = \tau_2\big(I_1(x_1(t_1))\big) - t_1 > \tau_1\big(x_1(t_1)\big) - t_1 = \sigma_{1,1}(t_1) = 0.$$

Since $\sigma_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$\sigma_{2,2}(t_2) = 0 \qquad \text{and} \qquad \sigma_{2,2}(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2).$$

Hence by (A_3) we get

$$\sigma_{k,2}(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2) \qquad \text{and} \qquad k = 2, 3, \ldots, p.$$

Also, let us show that there does not exist any $\xi \in (t_1, t_2)$ such that $\sigma_{1,2}(\xi) = 0$. Now, assume that there exists $\xi \in (t_1, t_2)$ such that $\sigma_{1,2}(\xi) = 0$. Considering the function $\gamma_1(t) = \tau_1\big(x_2(t) - g(t, x_{2_t})\big) - t$, by ($A_4$) it follows that

$$\gamma_1(\xi) = \tau_1\big(x_2(\xi) - g(\xi, x_{2_\xi})\big) - \xi \geq \tau_1\big((x_2(\xi))\big) - \xi = \sigma_{1,2}(\xi) = 0.$$

Thus the function γ_1 gains a nonnegative maximum at some point $\eta \in (t_1, t_2]$. Moreover, from Theorem 2.2 and in view of the Eq. (3.5) and the function $x_2(t)$, since

$$\frac{d}{dt}\big[x(t) - g(t, x_{2_t})\big] = D^{1-\alpha} f(t, x_{2_t}),$$

we obtain that, for some point $\eta \in (t_1, t_2]$,

$$\gamma_1'(\eta) = \tau_1'\big(x_2(\eta) - g(\eta, x_{2_\eta})\big)\frac{d}{dt}\big[x_2(\eta) - g(\eta, x_{2_\eta})\big] =$$
$$= \tau_1'(x_2(\eta) - g(\eta, x_{2_\eta}))^C D^{1-\alpha} f(t, x_{2_\eta}) - 1 = 0,$$

that is,

$$\Big\langle \tau_1'(x_2(\eta) - g(\eta, x_{2_\eta})),\ D^{1-\alpha} f(t, x_{2_\eta})\Big\rangle = 1,$$

which contradicts (A_6).

Consequently, we have built a second discontinuity point $t_2 > t_1$ where the solution $x(t)$ meets in such a way that $\sigma_{2,2}(t_2) = 0$ and $\sigma_{k,2}(t) \neq 0$ for all $t \in (t_1, t_2)$ and $k = 1, 2, 3, \ldots, p$.

Step 3: Let us continue the procedure as in the previous steps by taking into consideration that $x_p := x|_{(t_{p-1}, T]}$ is a solution of the following problem:

$$D^\alpha\big[x(t) - g(t, x_t)\big] = f(t, x_t), \qquad t \in J_{p-1}, \quad 0 < \alpha \leq 1, \tag{3.8}$$

$$x(t_{p-1}^+) = I_{p-1}(x_{p-1}(t_{p-1})), \tag{3.9}$$

$$x(t) = x_{p-1}(t), \quad t \in \big[t_{p-1} - \rho, t_{p-1}\big]. \tag{3.10}$$

We transform the problem (3.8)–(3.10) into a fixed point problem by considering the operator $F_{p-1}\colon C\big([t_{p-1}-\rho,T],R^n\big)\to C([t_{p-1}-\rho,T],R^n)$ defined by

$$F_{p-1}(x)(t)=\begin{cases} x_{p-1}(t), & t\in[t_{p-1}-\rho,t_{p-1}],\\ I_{p-1}(x_{p-1}(t_{p-1}))-g(t_{p-1},x_{t_{p-1}})+ & \\ +g(t,x_t)+\displaystyle\int_{t_{p-1}}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_s)\,ds, & t\in J_{p-1}. \end{cases}$$

As in $Step1$, as a result of Schauder's fixed point theorem we can conclude that F_{p-1} has a fixed point which is a solution of problem (3.8)–(3.10) on J_{p-1}. Denote now this solution by x_p.

Then we will explore a possible discontinuity moment after the point t_{p-1} the solution $x(t)$ encounters by making use of the function

$$\sigma_{k,p}(t)=\tau_k(x_p(t))-t,\ t\geq t_{p-1}.$$

If $\sigma_{k,p}(t)\neq 0$, that is, $\tau_k(x_p(t))\neq t$ on $(t_{p-1},T]$ for $k=1,2,\ldots,p$, then $x_p(t)$ is a solution of problem (3.8)–(3.10). That is,

$$x(t)=\begin{cases} x_1(t), & t\in[t_0,t_1],\\ x_2(t), & t\in(t_1,t_2],\\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ x_p(t), & t\in(t_{p-1},T], \end{cases}$$

is a solution of problem (1.1)–(1.3).

Now we are in the position to focus on the circumstance when

$$\sigma_{p,p}(t)=0,\quad \text{i.e.,}\quad t=\tau_p(x_p(t))\quad \text{for some}\quad t\in(t_{p-1},T].$$

From (A_5) we have

$$\sigma_{p,p}(t_{p-1}^+)=\tau_p\big(x_p\big(t_{p-1}^+\big)\big)-t_{p-1}=$$
$$=\tau_p\big(I_{p-1}(x_{p-1}(t_{p-1}))\big)-t_{p-1}>\tau_{p-1}\big(x_{p-1}\big(t_{p-1}\big)\big)-t_{p-1}=\sigma_{p-1,p-1}(t_{p-1})=0.$$

Since $\sigma_{p,p}$ is continuous, there exists $t_p>t_{p-1}$ such that

$$\sigma_{p,p}(t_p)=0\qquad\text{and}\qquad \sigma_{p,p}(t)\neq 0\quad\text{for all}\quad t\in(t_{p-1},t_p).$$

Thus by (A_3) we get

$$\sigma_{k,p}(t)\neq 0\quad\text{for all}\quad t\in(t_{p-1},t_p)\qquad\text{and}\qquad k=3,4,\ldots,p.$$

Also, we need to show that there does not exist any $\overline{\xi}\in(t_{p-1},t_p)$ such that $\sigma_{p-1,p}(\overline{\xi})=0$. Suppose now that there exists $\overline{\xi}\in(t_{p-1},t_p)$ such that $\sigma_{p-1,p}(\overline{\xi})=0$. Considering the function $\gamma_{p-1}(t)=\tau_{p-1}(x_p(t)-g(t,x_{p_t}))-t$, by (A4) it follows that

$$\gamma_{p-1}(\overline{\xi})=\tau_{p-1}(x_p(\overline{\xi})-g(\overline{\xi},x_{p_{\overline{\xi}}}))-\overline{\xi}\geq\tau_{p-1}\big((x_p(\overline{\xi}))\big)-\overline{\xi}=\sigma_{p-1,p}(\overline{\xi})=0.$$

Therefore, the function γ_{p-1} attains a nonnegative greatest value at some point $\overline{\eta} \in (t_{p-1}, t_p]$. Furthermore, from Theorem 2.2 and in view of the Eq. (3.8) and the function $x_p(t)$, since

$$\frac{d}{dt}\big[x(t) - g(t, x_{p_t})\big] = D^{1-\alpha} f(t, x_{p_t}),$$

we find that, for some point $\overline{\eta} \in (t_{p-1}, t_p]$,

$$\gamma'_{p-1}(\overline{\eta}) = \tau'_{p-1}(x_p(\overline{\eta}) - g(\overline{\eta}, x_{p_{\overline{\eta}}}))\frac{d}{dt}\big[x_p(\overline{\eta}) - g(\overline{\eta}, x_{p_{\overline{\eta}}})\big] - 1 =$$
$$= \tau'_{p-1}(x_p(\overline{\eta}) - g(\overline{\eta}, x_{p_{\overline{\eta}}}))D^{1-\alpha} f(t, x_{p_{\overline{\eta}}}) - 1 = 0,$$

that is,

$$\left\langle \tau'_{p-1}(x_p(\overline{\eta}) - g(\overline{\eta}, x_{p_{\overline{\eta}}})),\ D^{1-\alpha} f(t, x_{p_{\overline{\eta}}}) \right\rangle = 1,$$

which implies a contradiction with (A_6).

As a result, we have constituted a p-th discontinuity point $t_p > t_{p-1} > \ldots > t_2 > t_1$, where the solution $x(t)$ beats in such a way that $\sigma_{p,p}(t_p) = 0$ and $\sigma_{k,p}(t) \neq 0$ for all $t \in (t_{p-1}, t_p)$ and $k = 1, 2, 3, \ldots, p$.

Finally, the solution x of problem (1.1)–(1.3) is defined by

$$x(t) = \begin{cases} x_1(t), & t \in [t_0, t_1], \\ x_2(t), & t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ x_p(t), & t \in (t_{p-1}, t_p], \\ x_{p+1}(t), & t \in (t_p, T]. \end{cases}$$

Theorem 3.1 is proved.

In the sequel, we shall give some sufficient conditions for the uniqueness of the solutions of IVP (1.1)–(1.3).

Theorem 3.2. *In addition to the assumptions* (A_3)–(A_6), *suppose that*

(A_7) *There exists constant* $c > 0$ *such that* $|g(t,u) - g(t,v)| \leq c|u - v|$ *for each* $t \in J$ *and* $u, v \in R^n$.

(A_8) *There exists constant* $d > 0$ *such that* $|f(t,u) - f(t,v)| \leq d|u - v|$ *for each* $t \in J$ *and* $u, v \in R^n$.

(A_9) *There exist constants* $d_k > 0,\ k = 1, 2, 3, \ldots, p$, *such that* $|I_k(u) - I_k(v)| \leq d_k|u - v|$ *for each* $u, v \in R^n$.

Further, *if the condition*

$$\Lambda := d_k + 2c + \frac{dT^\alpha}{\Gamma(\alpha+1)} < 1$$

is fulfilled, *then the IVP* (1.1)–(1.3) *has a unique solution on* J.

Proof. Taking the steps in Theorem 3.1 into consideration, we consider the following problem

$$^{C}D^\alpha\big[x(t) - g(t, x_t)\big] = f(t, x_t), \qquad t \in [t_k, t_{k+1}], \quad 0 < \alpha \leq 1, \tag{3.11}$$

$$x(t_k^+) = I_k(x_k(t_k)), \tag{3.12}$$

$$x(t) = x_k(t), \quad t \in [t_k - \rho, t_k], \tag{3.13}$$

whose solution is $x_{k+1} := x|_{(t_k, t_{k+1}]}$. We transform problem (3.11)–(3.13) into a fixed point problem in view of the operator $\mathcal{F}_k : C\big([t_k - \rho, t_{k+1}], R^n\big) \to C\big([t_k - \rho, t_{k+1}], R^n\big)$ defined by

$$\mathcal{F}_k(x)(t) = \begin{cases} x_k(t), & t \in [t_k - \rho, t_k], \\ I_k(x_k(t_k)) - g(t_k, x_{t_k}) + g(t, x_t) + \displaystyle\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s)\, ds, & t \in [t_k, t_{k+1}]. \end{cases}$$

Here, it suffices to show that the operator F_k is a contracting mapping in order to prove that $x(t)$ is a unique solution of the IVP (1.1)–(1.3) on $[t_k, t_{k+1}]$. Now, let $x, y \in C\big([t_k - \rho, t_{k+1}], R^n\big)$. Then, for each $t \in [t_k, t_{k+1}]$, it is obvious that $\mathcal{F}_k$ is a contraction since

$$\|\mathcal{F}_k(x) - \mathcal{F}_k(y)\| \le \Lambda \|x - y\|.$$

As a consequence of Banach's fixed point theorem, $\mathcal{F}_k$ has a fixed point. Therefore, it leads that the IVP (1.1)–(1.3) has a unique solution.

Theorem 3.2 is proved.

***Example* 3.1.** Consider the following IVP for impulsive neutral fractional differential equation at variable moments:

$$D^{1/2}\left[x(t) + \frac{\sin x\left(t - \dfrac{1}{5}\right)}{\left(t + \dfrac{1}{4}\right)^2}\right] =$$

$$= \frac{e^{-t}\left|x\left(t - \dfrac{1}{5}\right)\right|}{(e^t + 2)^3\left(1 + \left|x\left(t - \dfrac{1}{5}\right)\right|\right)}, \qquad t \in J, \quad t \ne \tau_k(x(t)), \tag{3.14}$$

$$x(t^+) = I_k(x(t)), \qquad t = \tau_k(x(t)), \quad k = 1, 2, \ldots, p, \tag{3.15}$$

$$x(s) = \phi(s), \quad s \in \left[-\frac{1}{2}, 0\right], \tag{3.16}$$

where $J = [0, 1]$ and

$$\tau_k(x) = 1 - \frac{1}{3^k(1 + x^2)}, \qquad I_k(x) = c_k x, \quad c_k \in \left(\frac{1}{\sqrt{3}}, 1\right], \quad c_k > 0, \quad k = 1, 2, \ldots, p.$$

Immediately, since g is completely continuous and f is continuous such that

$$\big|g(t, x_t)\big| = \left|-\frac{\sin x\left(t - \dfrac{1}{5}\right)}{\left(t + \dfrac{1}{4}\right)^2}\right| \le \frac{1}{\left(t + \dfrac{1}{4}\right)^2} =: q(t)$$

with $q^0 = 16$ and

$$\left|f(t,x_t)\right| = \left|\frac{e^{-t}\left|x\left(t-\dfrac{1}{5}\right)\right|}{(e^t+2)^3\left(1+\left|x\left(t-\dfrac{1}{5}\right)\right|\right)}\right| \leq \frac{e^{-t}}{(e^t+2)^3} =: \kappa(t)$$

with $\kappa^0 = \dfrac{1}{27}$. So, (A_1) and (A_2) are satisfied. Since

$$\tau_{k+1}(x) - \tau_k(x) = \frac{2}{3^{k+1}(1+x^2)} > 0 \quad \forall x \in R, \quad k = 1, 2, \ldots, p,$$

and

$$\tau_{k+1}(I_k(x)) - \tau_k(x) = \frac{2+\left(3c_k^2-1\right)x^2}{3^{k+1}(1+x^2)\left(1+c_k^2x^2\right)} > 0 \quad \forall x \in R,$$

the assumptions (A_3) and (A_5) are fulfilled. Also, in view of g and $\tau_k'(x)$, one can see that the condition (A_4) holds. Finally, it is clear that (A_6) is valid.

Consequently, since all assumptions of the Theorem 3.1 hold, the problem (3.14) – (3.16) has at least one solution.

Conclusion. We have investigated existence of solution to the IVP (1.1) – (1.3) consisting of a class of impulsive fractional neutral functional differential equations with variable moments. In this work, we have extended the notable results of Benchohra and Ouahab [25] considering a class of integer order neutral functional impulsive differential equations with variable times to a class of fractional order ones.

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