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B. Koşar, C. Nebiyev (Ondokuz Mayıs Univ., Turkey)

T-RADICAL AND STRONGLY T-RADICAL SUPPLEMENTED MODULES

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We define (strongly) t-radical supplemented modules and investigate some properties of these modules. These modules lie between strongly radical supplemented and strongly $\oplus$-radical supplemented modules. We also study the relationship between these modules and present examples separating strongly t-radical supplemented modules, supplemented modules, and strongly $\oplus$-radical supplemented modules.

Визначено поняття (сильно) t-радикальних доповнених модулів та вивчено деякі властивості цих модулів. Тскі модулі лежать між сильно радикальними доповненими та сильно $\oplus$-радикальними доповненими модулями. Також вивчено співвідношеняя між цими модулями та наведено приклади, що відділяють сильно t-радикальні доповнені модулі, доповнені модулі та сильно $\oplus$-радикальні доповнені модулі.

1. Introduction. Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R0-$module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$ [14]. $\operatorname{Rad} M$ indicates the radical of M. A submodule N of M is called *radical* if N has no maximal submodules, i.e., $N = \operatorname{Rad} N$. M is called a *hollow* module if every proper submodule of M is small in M. M is called *local* module if M has a largest submodule, i.e., a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* [5, 9, 16] of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. A module M is called *amply supplemented* if V contains a supplement of U in M whenever $M = U + V$ [14]. Clearly every amply supplemented module is supplemented. M is called [7, 10, 11] $\oplus$-*supplemented module* if every submodule of M has a supplement that is a direct summand of M. Let M be an R-module and U, V be submodules of M. V is called a *generalized supplement* [2, 13] of U in M if $M = U + V$ and $U \cap V \leq \operatorname{Rad} V$. M is called *generalized supplemented* or briefly *GS-module* if every submodule of M has a generalized supplement and clearly that every supplement submodule is a generalized supplement. M is called a *generalized* $\oplus$-*supplemented* [6, 10, 11] module if every submodule of M has a generalized supplement that is a direct summand in M. A submodule N of an R-module M is called *cofinite* if M/N is finitely generated. Note that M is called π-*projective* if whenever $M = U + V$ then there exists a homomorphism $f : M \to M$ such that $f(M) \subseteq U$ and $(1 - f)(M) \subseteq V$ [14].

Lemma 1.1. *Let M be an R-module and N, K be submodules of M. If $N + K$ has a generalized supplement X in M and $N \cap (K + X)$ has a generalized supplement Y in N, then $X + Y$ is a generalized supplement of K in M.*

Proof. See [6], (Lemma 3.2).

Lemma 1.2. *If V is a supplement in a module M, then $\operatorname{Rad} V = V \cap \operatorname{Rad} M$.*

Proof. See [3] (Corollary 4.2].

Lemma 1.3. *Let M be a π-projective module and K, L be two submodules of M. If K and L are mutual supplements in M, then $K \cap L = 0$ and $M = K \oplus L$.*

Proof. See [14] (41.14(2)).

2. T-sum and T-summand.

Definition 2.1. *Let M be an R-module, U and V be two submodules of M. M is called t-sum of U and V if U and V are mutual supplements in M., i.e., $M = U + V$, $U \cap V \ll U$ and $U \cap V \ll V$. Having this property of M is called a t-decomposition of M, U and V are called t-summand of M.* (*see also* [8]).

Theorem 2.1. *Let M be an R-module. M is an amply supplemented module if and only if for every $U \leq M$ there exists a t-decomposition $M = X + Y$ of M such that $X \leq U$ and $U \cap Y \ll Y$.*

Proof. $(\Rightarrow)$ Let M be an amply supplemented module. Consider any submodule U of M. Since M is amply supplemented module, then M is supplemented module. So U has a supplement Y in M. In this case $M = U + Y$ and $U \cap Y \ll Y$. Since $M = U + Y$ and M is amply supplemented module, Y has a supplement X in M such that $X \leq U$. Therefore M is t-sum of X and Y.

$(\Leftarrow)$ Consider any submodule U of M and let $M = U + V$. By hypothesis, there exists a t-decomposition $M = X + Y$ of M such that $X \leq U \cap V$ and $U \cap V \cap Y \ll Y$. Since $M = X + Y$ and $X \leq U \cap V \leq V$, then the modular law, $V = X + V \cap Y$. So we have $M = U + V = U + X + V \cap Y = U + V \cap Y$. Also by hypothesis, there exists a t-decomposition $M = S + T$ of M such that $S \leq V \cap Y$ and $V \cap Y \cap T \ll T$. Since $S \leq V \cap Y$ and $M = S + T$, then by modular law, $V \cap Y = S + V \cap Y \cap T$. Moreover, since $V \cap Y \cap T \ll T$, we get $M = U + V \cap Y = U + S + V \cap Y \cap T = U + S$. In here, since $U \cap S \leq U \cap V \cap Y \ll Y$, then $U \cap S \ll M$. Since S is a supplement in M, then $U \cap S \ll S$. That is, U has a supplement S in M such that $S \leq V$. Therefore M is amply supplemented.

Definition 2.2. *Let M be an R-module and $\{U_i\}_{i \in I}$ be a collection of submodules of M. If for every $i \in I$, U_i and $\sum_{k \in I - \{i\}} U_k$ are mutual supplements in M, then M is called t-sum of the collection $\{U_i\}_{i \in I}$.* (*see also* [8]).

Lemma 2.1. *Let M be a π-projective R-module and a t-sum of U and V. Then $U \cap V = 0$ and $M = U \oplus V$.*

Proof. Clear from Lemma 1.3.

The following result generalizes Lemma 2.1 which is easly proved.

***Corollary* 2.1.** *Let M be an R-module and $\{U_i\}_{i \in I}$ be a collection of submodules of M. If M is π-projective and a t-sum of the collection $\{U_i\}_{i \in I}$, then $M = \oplus_{i \in I} U_i$.*

Proof. We take any $k \in I$. Hence U_k and $\sum_{i \in I - \{k\}} U_i$ are mutual supplements in M. By the Lemma 2.1, we have $U_k \cap \left(\sum_{i \in I - k} U_i \right) = 0$. Therefore $M = \oplus_{i \in I} U_i$.

Lemma 2.2. *Let M be an R-module and V be a supplement of U in M. T is a supplement of K in V with $K, T \leq V$ if and only if T is a supplement of $U + K$ in M.* (*see also* [8]).

Proof. $(\Rightarrow)$ Let T be a supplement of K in V. Consider any submodule T_1 of T with $U + K + T_1 = M$. Since $K, T \leq V$, $U + K + T_1 = M$ and V is a supplement of U in M, then we get $K + T_1 = V$. Since T is a supplement of K in V, then $T_1 = T$. So, T is a supplement of $U + K$ in M.

($\Leftarrow$) Let T be a supplement of $U+K$ in M. Consider any submodule T_1 of T with $K+T_1=V$. We get $M=U+V=U+K+T_1$. Since $T_1 \leq T$ and by the assumption, we can write $T_1=T$. Therefore T is a supplement of K in V.

Lemma 2.3. *Let M be a t-sum of U and V. If K is a supplement of S in U and L is a supplement of T in V, then $K+L$ is a supplement of $S+T$ in M (see also* [8]).

Proof. Since U is a supplement of V in M and K is a supplement of S in U, by Lemma 2.2, K is a supplement of $V+S$ in M. Hence $(V+S)\cap K \ll K$. Similarly, we can prove that $(U+T)\cap L \ll L$. Then $(S+T)\cap(K+L) \leq (S+T+K)\cap L+(S+T+L)\cap K=(U+T)\cap L+(V+S)\cap K \ll K+L$, and by $M=U+V=S+K+T+L=S+T+K+L$, $K+L$ is a supplement of $S+T$ in M.

Lemma 2.4. *Let M be a t-sum of U and V, and $L,T \leq V$. Then V is a t-sum of L and T if and only if M is a t-sum of $U+L$ and T, and M is a t-sum of $U+T$ and L (see also* [8]).

Proof. ($\Rightarrow$) Let V be a t-sum of L and T. Since T is a supplement of L in V and V is a supplement of U in M, then by Lemma 2.2, T is a supplement of $U+L$ in M. Then $(U+L)\cap T \ll T$. Similarly, we can prove that $(U+T)\cap L \ll L$. Then by $U\cap V \ll U$, $(U+L)\cap T \leq U\cap(L+T)+L\cap(U+T) = U\cap V+(U+T)\cap L \ll U+L$. Since $(U+L)\cap T \ll T, (U+L)\cap T \ll U+L$ and $M=U+V=U+L+T$, then by Definition 2.1 M is a t-sum of $U+L$ and T. Similarly, we can prove that M is a t-sum of $U+T$ and L.

($\Rightarrow$) Clear from Lemma 2.2.

***Corollary* 2.2.** *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If K_i is a supplement of T_i in U_i, $i = 1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n$ is a supplement of $T_1+T_2+\ldots+T_n$ in M (see also* [8]).

Proof. Clear from Lemma 2.7.

***Corollary* 2.3.** *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If U_i is a t-sum of K_i and T_i, $i = 1,2,\ldots,n$, then M is a t-sum of $K_1+K_2+\ldots+K_n$ and $T_1+T_2+\ldots+T_n$ (see also* [8]).

Proof. Clear from Corollary 2.2.

***Corollary* 2.4.** *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If K_i is a supplement in U_i, $i=1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n$ is a supplement in M (see also* [8]).

Proof. Clear from Corollary 2.9.

***Corollary* 2.5.** *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If K_i is a t-summand of U_i, $i=1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n$ is a t-summand of M (see also* [8]).

Proof. Clear from Lemma 2.4.

Let M be an R-module. We say that M is called *cofinitely t-generalized supplemented module* if every cofinite submodule of M has a generalized supplement such that it is a supplement in M.

Theorem 2.2. *Let M be a t-sum of collection of $\{U_i\}_{i\in I}$. If for every $i\in I$, U_i is cofinitely t-generalized supplemented, then M is also cofinitely t-generalized supplemented.*

Proof. Let K be any cofinite submodule of M. Since $M=\sum_{i\in I} U_i$, then there exist $i_1,i_2,\ldots,i_n \in I$ such that $M=K+U_{i_1}+U_{i_2}+\ldots+U_{i_n}$. By Lemma 1.1, clearly, K has a generalized supplement $V_{i_1}+V_{i_2}+\ldots+V_{i_n}$ in M such that V_{i_t} is a supplement in U_{i_t} for $1\leq t\leq n$. By Corollary 2.4, we get $V_{i_1}+V_{i_2}+\ldots+V_{i_n}$ is a supplement in M. Therefore M is a cofinitely t-generalized supplemented.

Lemma 2.5. *Let M be a t-sum of collection of $\{U_i\}_{i\in I}$. Then* $\operatorname{Rad} M = \sum_{i\in I}\operatorname{Rad} U_i$ (*see also* [8]).

Proof. Clearly $\sum_{i\in I}\operatorname{Rad} U_i \leq \operatorname{Rad} M$. Let $x\in \operatorname{Rad} M$. Since $x\in M=\sum_{i\in I}\operatorname{Rad} U_i$, there exist $i_1,i_2,\ldots,i_n \in I$ and $x_{i_t}\in U_{i_t}$, $t=1,2,\ldots,n$ such that $x=x_{i_1}+x_{i_2}+\ldots+x_{i_n}$. Suppose that some submodule S of U_{i_t} for $1\leq t\leq n$ with $Rx_{i_t}+S=U_{i_t}$. In here, we can show that $Rx_{i_t} + S + \sum_{i \in I - \{i_t\}} U_i = M$. Since $Rx \ll M$, we have $S + \sum_{i \in I - \{i_t\}} U_i = M$. Moreover, since $S \leq U_{i_t}$ and U_{i_t} is a supplement of $\sum_{i \in I - \{i_t\}} U_i$ in M, then we can write $S = U_{i_t}$. Hence $Rx_{i_t} \ll U_{i_t}$, then $x_{i_t} \in \operatorname{Rad} U_{i_t}$. Therefore $\operatorname{Rad} M \leq \sum_{i \in I} \operatorname{Rad} U_i$.

3. (Strongly) T-radical supplemented modules.

Definition 3.1. *Let M be an R-module. If the radical of M has a supplement such that is a t-summand in M, then M is called t-radical supplemented module, that is, there exist $K, L \leq M$ such that $M = \operatorname{Rad} M + K$, $\operatorname{Rad} M \cap K \ll K$ and $M = K + L$, $K \cap L \ll K$, $K \cap L \ll L$.*

Definition 3.2. *Let M be an R-module. If every submodule of M containing the radical of M has a supplement that is a t-summand in M, then M is called strongly t-radical supplemented module. That is, for every submodule K of M with* $\operatorname{Rad} M \subseteq K$, *there exists a t-summand L of M such that $M = K + L$, $K \cap L \ll L$.*

Lemma 3.1. *Every supplemented module is strongly t-radical supplemented.*

Proof. Let M be a supplemented module and let $\operatorname{Rad} M \leq U \leq M$. So U has a supplement V in M. Since M is supplemented, V has a supplement V' in M. Hence V and V' are mutual supplements in M. Therefore V is a t-summand of M. This means that M is strongly t-radical supplemented.

In the last of this section, we will give an example of a strongly t-radical supplemented module that is not supplemented.

Lemma 3.2. *Every radical module is* (*strongly*) *t-radical supplemented.*

Proof. Let M be a radical module. Clearly M has the trivial supplement 0 in M. Hence M is t-radical supplemented. Since M is the unique submodule containing the radical, M is a strongly t-radical supplemented.

By $P(M)$ we denote the sum of all radical submodules of a module M. It is clear that, for any module M, $P(M)$ is the largest radical submodule.

***Corollary* 3.1.** *For every R-module M, $P(M)$ is strongly t-radical supplemented.*

Proof. Since $\operatorname{Rad} P(M) = P(M)$, the proof is complete.

Lemma 3.3. *Let M be (strongly) t-radical supplemented module. Then M has a t-summand which is radical.*

Proof. By hypothesis, there exists $V, V' \leq M$ such that $M = \operatorname{Rad} M + V$, $\operatorname{Rad} M \cap V \ll V$, $M = V + V'$, $V \cap V' \ll V$ and $V \cap V' \ll V'$. Now we prove that $\operatorname{Rad} V' = V'$. Since $\operatorname{Rad} M \cap V = \operatorname{Rad} V$, $\operatorname{Rad} V \ll V$. Note that $\operatorname{Rad} M = \operatorname{Rad} V + \operatorname{Rad} V'$. So, $M = V + \operatorname{Rad} V'$. Applying the modular law, $V' = \operatorname{Rad} V' + \left(V \cap V'\right)$. Since $V \cap V' \ll V'$, then $\operatorname{Rad} V' = V'$. Therefore V' is a radical t-summand.

Recall that a module M is called reduced if $P(M) = 0$.

Lemma 3.4. *Let M be a reduced module. If M is* (*strongly*) *t-radical supplemented*, *then* $\operatorname{Rad} M \ll M$.

Proof. Since M is (strongly) t-radical supplemented, there exists $V, V' \leq M$, such that $M = \operatorname{Rad} M + V$, $\operatorname{Rad} M \cap V \ll V$ and $M = V + V'$, $V \cap V' \ll V$, $V \cap V' \ll V'$. Since $\operatorname{Rad} M \cap V = \operatorname{Rad} V$, $\operatorname{Rad} V \ll V$. By Lemma 3.3, we have $\operatorname{Rad} V' = V'$. Since M is reduced, $P(M) = 0$. Hence we get $M = V$.

Lemma 3.5. *Every module M with* $\operatorname{Rad} M \ll M$ *is t-radical supplemented.*

Proof. Let M be a module with $\operatorname{Rad} M \ll M$. We assume that $M = \operatorname{Rad} M + N$ for some submodule N of M. Since $\operatorname{Rad} M \ll M$, then $M = N$.

An R-module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Note that $\operatorname{Rad} M$ is small in M for every coatomic R-module M.

***Corollary* 3.2.** *Every coatomic module is t-radical supplemented.*

The module ${}_RR$ is a maximal module if every nonzero ideal contains a maximal submodule. ${}_RR$ is a left Bass module if every nonzero R-module has a maximal submodule; such rings are called *left Bass rings*. R is left Bass ring if and only if for every nonzero R-module M, $\operatorname{Rad} M \ll M$. Now, we obtain the following result.

***Corollary* 3.3.** *Every nonzero module over the left Bass ring is t-radical supplemented.*

By combining the Lemma 3.1 and definitions we have the following lemma.

Lemma 3.6. *Let M be an R-module with $\operatorname{Rad} M \ll M$. Then the following conditions are equivalent.*

(i) M is strongly t-radical supplemented,

(ii) M is strongly radical supplemented,

(iii) M is supplemented.

The factor modules of a strongly t-radical supplemented module need not be strongly t-radical supplemented in general. A module M is called *distributive* if for every submodules K, L, N of M, $N + (K \cap L) = (N + K) \cap (N + L)$ or equivalently $N \cap (K + L) = (N \cap K) + (N \cap L)$. For distributive modules we have the following fact.

Lemma 3.7. *Let M be a distributive strongly t-radical supplemented module and U be a submodule of M. Then M/U is strongly t-radical supplemented.*

Proof. Let V/U be any submodule of M/U with $\operatorname{Rad}(M/U) \subseteq V/U$. From canonical epimorphism $\pi\colon M \to M/U$, we have $(\operatorname{Rad} M + U)/U \subseteq \operatorname{Rad}(M/U)$. So $\operatorname{Rad} M \subseteq V$. Since M is a strongly t-radical supplemented module, then V has a supplement which is a t-summand in M. Hence there exists $T, T' \leq M$ such that $M = V + T$, $V \cap T \ll T$ and $M = T + T'$, $T \cap T' \ll T$, $T \cap T' \ll T'$. Since T is a supplement of V in M, then $(T + U)/U$ is a supplement of V/U in M/U. Now we show that $(T + U)/U$ is a t-summand in M/U. From $M = T + T'$, we get $M/U = (T + U)/U + (T' + U)/U$. Since M is distributive, we have $\big[(T + U) \cap (T' + U)\big]/U = \big(U + (T \cap T')\big)/U$. On the other hand, $\big(U + (T \cap T')\big)/U \ll (T + U)/U$ and $\big(U + (T \cap T')\big)/U \ll (T' + U)/U$. Therefore M/U is strongly t-radical supplemented.

Theorem 3.1. *Let M be t-sum of M_1 and M_2. If M_1 and M_2 are t-radical supplemented, then M is t-radical supplemented.*

Proof. Since M_1 is t-radical supplemented module, then $\operatorname{Rad} M_1$ has a supplement V_1 which is t-summand in M_1. Since M_2 is t-radical supplemented module, then $\operatorname{Rad} M_2$ has a supplement V_2 which is t-summand in M_2. From M, is a t-sum of M_1 and M_2, by Lemma 2.5, we have $\operatorname{Rad} M = \operatorname{Rad} M_1 + \operatorname{Rad} M_2$. By Lemma 2.3, $V_1 + V_2$ is a supplement of $\operatorname{Rad} M = \operatorname{Rad} M_1 + \operatorname{Rad} M_2$ in M. On the other hand, by Corollary 2.5 $V_1 + V_2$ is a t-summand in M.

***Corollary* 3.4.** *The finite t-sum of t-radical supplemented modules is t-radical supplemented.*

Lemma 3.8. *Let R be a nonlocal commutative domain and M be an injective $R-$module. Then M is (strongly) t-radical supplemented module.*

Proof. By our assumption, we can write $\operatorname{Rad} M = M$.

Over Dedekind domains, divisible modules coincide with injective modules as in Abelian groups. Note that for a module M over a Dedekind domain R, M is divisible if and only if $\operatorname{Rad} M = M$, and this holds if and only if M is injective; see for example [1] (Lemma 4.4).

***Corollary* 3.5.** *Every module over nonlocal Dedekind domain is a submodule of* (*strongly*) t-*radical supplemented module.*

Now we give examples for to separate the structure of strongly t-radical supplemented, supplemented and strongly $\oplus$-radical supplemented module.

***Example* 3.1.** Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\operatorname{Rad}\mathbb{Q} = \mathbb{Q}$, ıt follows that ${}_{\mathbb{Z}}\mathbb{Q}$ is strongly t-radical supplemented. But it is well known that ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented (see [7], Example 20.12).

***Example* 3.2.** Let R be a commutative local ring which is not a valuation ring. Let a and b be elements of R, where neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a) \cap (b) = 0$ and $am = bm = 0$, where m is the maximal ideal of R. Let F be a free R-module with generators x_1, x_2 and x_3, K be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus, $M = \dfrac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}$. Here M is not $\oplus$-supplemented. But $F = Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented [7].

Since F is completely $\oplus$-supplemented, F is supplemented. Since a factor module of a supplemented module is supplemented, we have M is supplemented. By Lemma 3.1 M is strongly t-radical supplemented module. But M is not strongly $\oplus$-radical supplemented.

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