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$\mathcal{I}_\lambda$-DOUBLE STATISTICAL CONVERGENCE OF ORDER α IN TOPOLOGICAL GROUPS

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We introduce a new notion, namely, $\mathcal{I}_\lambda$-double statistical convergence of order α in topological groups. Consequently, we investigate some inclusion relations between $\mathcal{I}$-double statistical and $\mathcal{I}_\lambda$-double statistical convergence of order α in topological groups. We also study many other related concepts.

Введено нове поняття, а саме поняття $\mathcal{I}_\lambda$-подвійної статистичної збіжності порядку α в топологічних групах. Таким чином, вивчено деякі відношеняя включення між $\mathcal{I}$-подвійною статистичною збіжністю та $\mathcal{I}_\lambda$–подвійною статистичною збіжністю порядку α в топологічних групах. Також вивчено багато інших подібних понять.

1. Introduction. The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [9] and Schoenberg [30] and its topological consequences were studied first by Fridy [10], Šalát [19] (also later by Maddox [15]). Recently Di Maio and KoЎcinac [16] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence. This notion was used by Kolk in [11] to extended the statistical convergence to normed spaces. Also, in [2] Cakalli extended this notion to topological Hausdorff groups.

Later on Cakalli and Savaş studied the statistical convergence of double sequences to topological groups (see [3]). Also Savaş [28], introduced Lacunary statistical convergence of double sequences in topological groups. Quite recently, Savaş [29] introduced $\mathcal{I}_\lambda$-statistical convergence of double sequences in topological groups where more references can be found.

The notion of $\mathcal{I}$-convergence ($\mathcal{I}$ denotes the ideal of subsets of, the set of positive integers), which is a generalization of statistical convergence, was introduced by Kostyrko et al. [12] and Das et al. [4] continued with this study and extended these ideas from single to double sequences. Further it was studied by many other authors (see [1, 5, 8, 13, 21 – 25]).

In [7], we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence of order α which naturally extend the notions of mentioned convergence in [8].

The concept of statistical convergence depends on the density of subsets of the set N of natural numbers. If $K \subset \mathbb{N}$, then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \qquad \text{and} \qquad \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \dfrac{K(1,n)}{n}$.

A sequence (x_k) of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{k \in \mathbb{N} \colon |x_k - L| \geq \epsilon\}$ has natural density zero.

Throughout the paper, $\mathbb{N}$ will denote the set of all natural numbers.

By X, we will denote an Abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. In [2], a sequence $x = (x_k)$ in X is called to be statistically convergent to an element L of X if for each neighbourhood U of 0,

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k \le n : x_k - L \notin U\}\big| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in X is denoted by $st(X)$.

2. Preliminaries. We now quote the following definitions and notions which will be needed in the sequel.

Definition 1. *A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold*:

(a) $A, B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$,

(b) $A \in \mathcal{I}$, $B \subset A$ *implies* $B \in \mathcal{I}$.

Definition 2. *A nonempty family $F \subset 2^{\mathbb{N}}$ is said to be an filter of $\mathbb{N}$ if the following conditions hold*:

(a) $\phi \notin F$,

(b) $A, B \in F$ *implies* $A \cap B \in F$,

(c) $A \in F$, $A \subset B$ *implies* $B \in F$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 3. *A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.*

Definition 4 (see [12])**.** *Let $I \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. Then the sequence (x_k) of elements of real numbers is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \ge \epsilon\} \in \mathcal{I}$.*

By the convergence of a double sequence we mean the convergence in Pringsheims sense [18].

A double sequence $x = (x_{kl})$ of real numbers is said to be convergent in the Pringsheim's sense or P-convergent if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l \ge N$ and L is called Pringsheim limit (denoted by $\mathrm{P} - \lim x = L$).

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and $K_{m,n}$ be the numbers of (i, j) in K such that $i \le n$ and $j \le m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\left(\dfrac{K_{m,n}}{mn}\right)_{m,n=1,1}^{\infty,\infty}$ has a limit then we say that K has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \le P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e., the set K has double natural density zero).

The concept of statistical convergence of double sequences was first introduced by Mursaleen and Edeley (see [17]) who have given main definition for double sequences and proved some related results supporting by some interesting example.

Throughout $\mathcal{I}_2$ will stand for a proper admissible ideal in $\mathbb{N}\times\mathbb{N}$.

A double sequence $x=(x_{kl})$ of real number is said to be convergent to the number L with respect to the ideal $\mathcal{I}_2$, if for each $\epsilon>0$

$$A(\epsilon)=\big\{(k,l)\in\mathbb{N}\times\mathbb{N}\colon |x_{kl}-L|\geq\epsilon\big\}\in\mathcal{I}_2.$$

In this case we write $\mathcal{I}-\lim_{kl}x_{kl}=L$

Note that, if we take $I_d=\big\{A\subset\mathbb{N}\times\mathbb{N}\colon \delta_2(A)=0\big\}$, then $\mathcal{I}_d$-convergence becomes statistical convergence for double sequences. We now define the concept of double λ-density:

Let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ be two nondecreasing sequences of positive real numbers both of which tends to ∞ as m and n approach ∞, respectively. Also let $\lambda_{m+1}\leq\lambda_m+1,\ \lambda_1=0$ and $\mu_{n+1}\leq\mu_n+1,\ \mu_1=0$. The collection of such sequence (λ,μ) will be denoted by Δ.

Let $K\subseteq\mathbb{N}\times\mathbb{N}$. The number

$$\delta_\lambda(K)=\lim_{mn}\frac{1}{\lambda_{mn}}\big|\{k\in I_n, l\in J_m\colon (k,l)\in K\}\big|,$$

where $I_m=[m-\lambda_m+1,m]$ and $J_n=[n-\mu_n+1,n]$ and $\lambda_{mn}=\lambda_m\mu_n$, is said to be the λ-*density* of K, provided the limit exists.

Throughout this paper we shall denote $(k\in I_m, l\in J_n)$ by $(k,l)\in I_{mn}$.

In this paper, we introduce the concept of $\mathcal{I}_\lambda$-double statistical convergence of order α in topological groups and investigate some of its consequences.

3. $\mathcal{I}_\lambda$-double convergence of order α. The order of double statistical convergence of a sequence of real numbers was given by Savaş in [27] as follows: Let $\lambda=(\lambda_{mn})\in\Delta$ and $0<\alpha\leq1$ be given. The sequences $x=(x_{kl})$ is said to be λ-double statistically convergent of order α if there is a complex number L such that

$$\left\{n\in\mathbb{N}\colon \frac{1}{\lambda_{mn}^\alpha}\big|\{(k,l)\in I_{mn}\colon |x_{kl}-L|\geq\epsilon\}\big|=0\right\},$$

where λ_{mn}^α denote the α th power $(\lambda_{mn})^\alpha$ of λ_{mn}.

In this section, we introduce and study $\mathcal{I}_\lambda$-double statistical convergence of order α for sequence in topological groups.

We now have the following definition.

Definition 5. *A sequence $x=(x_{kl})$ of points in a topological group X is called to be statistically convergent of order α to L of X if for each neighbourhood U of 0,*

$$\lim_{m,n\to\infty}\frac{1}{(mn)^\alpha}\big|\{k\leq m\quad\textit{and}\quad l\leq m\colon x_{kl}-L\notin U\}\big|=0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The set of all double statistically convergent of order α sequences in X is denoted by $S^\alpha(X)_2$.

Also we define λ-double statistical convergence of order α in topological groups as follows:

Definition 6. *A sequence $x = (x_{kl})$ of points in a topological group X, is said to be $S_\lambda^\alpha(X)_2-$ convergent of order α to L (or λ-double statistically convergent of order α to L) if for each neighborhood U of* 0,

$$\lim_{mn} \frac{1}{\lambda_{mn}^\alpha} \big|(k,l) \in I_{mn} : x_{kl} - L \notin U\}\big| = 0.$$

In this case, we define

$$S_\lambda^\alpha(X)_2 = \left\{ x = (x_{kl}) : \text{for some } \ L, S_\lambda^\alpha(X)_2 - \lim_{k,l \to \infty} x_{kl} = L \right\}.$$

If we take $\lambda_{mn} = mn,$ $S_\lambda^\alpha(X)_2$ reduce to $S^\alpha(X)_2$.

We now introduce our main definitions. Throughout $\mathcal{I}_2$ will stand for a proper admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 7. *A double sequence $x = (x_{kl})$ of points in a topological group X, is said to be $\mathcal{I}_2$- double statistically convergent of order α to L or $S^\alpha(\mathcal{I}_2)$-convergent to L, where $0 < \alpha \leq 1$, if for each $\epsilon > 0$ and $\delta > 0$*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{(mn)^\alpha} \big|\{k \leq m \quad and \quad l \leq m : x_{kl} - L \notin U\}\big| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case we write $x_{kl} \to L(S^\alpha(\mathcal{I}_2))$. The set of all $\mathcal{I}_2$-double statistically convergent sequences will be denoted by simply $S^\alpha(\mathcal{I}_2)(X)$.

***Remark* 1.** For $\mathcal{I}_2 = \mathcal{I}_{2fin} = \{A \subset \mathbb{N} \times \mathbb{N}, A \text{ is a finite}\}$, then $S(\mathcal{I}_2)$-convergence coincides with double statistical convergence in a topological group X which is studied by Cakalli and Savaş [2].

Definition 8. *A sequences $x = (x_{kl})$ of points in a topological group X, is said to be $\mathcal{I}_2^\lambda$-double statistically convergent of order α to L or $S_\lambda^\alpha(\mathcal{I}_2)$-convergent of order α to L, where $0 < \alpha \leq 1$, if for any $\delta > 0$ and for each neighbourhood U of* 0,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}^\alpha} \big|\{(k,l) \in I_{mn} : x_{kl} - L \notin U\}\big| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write

$$S_\lambda^\alpha(\mathcal{I}_2)(X) = \left\{ x = x_{kl} : \text{for some } \ L, S_\lambda^\alpha(\mathcal{I}_2) - \lim_{k,l \to \infty} x_{kl} = L \right\}$$

and, if we take $\alpha = 1$, we have

$$S_\lambda(\mathcal{I}_2)(X) = \left\{ x = (x_{kl}) : \text{for some } \ L, S_\lambda(\mathcal{I}_2) - \lim_{k,l \to \infty} x_{kl} = L \right\}.$$

***Remark* 2.** For $\mathcal{I}_2 = \mathcal{I}_2 fin$, $\mathcal{I}_2$-double statistical convergence of order α becomes double statistical convergence of order α in topological groups which has not been study till now. Finally, for $\mathcal{I}_2 = \mathcal{I}_2 fin$, $\lambda_{mn} = mn$, and $\alpha = 1$ it becomes statistical convergence which is studied in [3].

It is obvious that every $\mathcal{I}_2^\lambda$- double statistically convergent sequence has only one limit, that is, if a sequence is $\mathcal{I}_2^\lambda$ double statistically convergent to L_1 and L_2, then $L_1 = L_2$.

4. Inclusion theorems. In this section we prove some inclusion theorems.

Theorem 1. *Let* $0<\alpha\leq\beta\leq 1$. *Then* $S_\lambda^\alpha(\mathcal{I}_2)(X)\subset S_\lambda^\beta(\mathcal{I}_2)(X)$.

Proof. Let $0<\alpha\leq\beta\leq 1$. Then

$$\frac{|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}|}{\lambda_{mn}^\beta}\leq\frac{|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}|}{\lambda_{mn}^\alpha}$$

and so for any $\delta>0$ and any neighbourhood U of 0

$$\left\{(m,n)\in N\times N: \frac{|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}|}{\lambda_{mn}^\beta}\geq\delta\right\}\subset$$
$$\subset\left\{(m,n)\in \mathbb{N}\times \mathbb{N}: \frac{|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}|}{\lambda_{mn}^\alpha}\geq\delta\right\}.$$

Hence if the set on the right-hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand-side also belongs to $\mathcal{I}$. This shows that $S_\lambda^\alpha(\mathcal{I}_2)(X)\subset S_\lambda^\beta(\mathcal{I}_2)(X)$.

Theorem 1 is proved.

***Corollary* 1.** *If a sequence is* $\mathcal{I}_2^\lambda$*-double statistically convergent of order* α *to* L *for some* $0<\alpha\leq 1$, *then it is* $\mathcal{I}_2^\lambda$*- double statistically convergent to* L, *i.e.,* $S_\lambda^\alpha(\mathcal{I}_2)\subset S_\lambda(\mathcal{I}_2)$.

Similarly we can show that the following theorem.

Theorem 2. *Let* $0<\alpha\leq\beta\leq 1$. *Then*

(i) $S^\alpha(\mathcal{I}_2)\subset S^\beta(\mathcal{I}_2)$,

(ii) *In particular* $S^\alpha(\mathcal{I}_2)\subset S(\mathcal{I}_2)$.

Theorem 3. $S(\mathcal{I}_2)(X)\subset S_\lambda^\alpha(\mathcal{I}_2)(X)$ *if* $\liminf\limits_{n\to\infty}\dfrac{\lambda_{mn}^\alpha}{mn}^\alpha>0$.

Proof. Let us take any neighbourhood U of 0. Then

$$\frac{1}{(mn)^\alpha}\big|\{k\leq m, l\leq n: x_{kl}-L\notin U\}\big|\geq\frac{1}{(mn)^\alpha}\big|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}\big|\geq$$
$$\geq\frac{\lambda_{mn}^\alpha}{(mn)^\alpha}\frac{1}{\lambda_{mn}^\alpha}\big|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}\big|.$$

If $\liminf\limits_{mn\to\infty}\dfrac{\lambda_{mn}^\alpha}{mn}^\alpha=a$, then from definition $\left\{(m,n)\in N\times N: \dfrac{\lambda_{mn}^\alpha}{mn}^\alpha<\dfrac{a}{2}\right\}$ is finite. For $\delta>0$, and any neighbourhood U of 0,

$$\left\{(m,n)\in \mathbb{N}\times \mathbb{N}: \frac{1}{\lambda_{mn}^\alpha}\big|\{(k,l)\in I_{mn}: x_{kl}-L\notin U\}\big|\geq\delta\right\}\subset$$
$$\subset\left\{(k,l)\in I_{mn}: \frac{1}{(mn)^\alpha}\big|\{k\leq m, l\leq n: x_{kl}-L\notin U\}\big|\geq\frac{a}{2}\delta\right\}\cup$$
$$\cup\left\{(m,n)\in \mathbb{N}\times \mathbb{N}: \frac{\lambda_{mn}^\alpha}{(mn)^\alpha}<\frac{a}{2}\right\}.$$

The set on the right-hand side belongs to $\mathcal{I}_2$ and this completed the proof.

Theorem 4. *Let* $\lambda=(\lambda_{mn})$ *and* $\mu=(\mu_{mn})$ *be two sequences in* $\triangle$ *such that* $\lambda_{mn}\leq\mu_{mn}$ *for all* $(m,n)\in\mathbb{N}\times\mathbb{N}$ *and let* α *and* β *be fixed real numbers such that* $0<\alpha\leq\beta\leq 1$.

(i) *If*

$$\lim_{mn\to\infty} \inf \frac{\lambda_{mn}^{\alpha}}{\mu_{mn}^{\beta}} > 0, \tag{4.1}$$

then $S_{\mu}^{\beta}(\mathcal{I}_2)(X) \subseteq S_{\lambda}^{\alpha}(\mathcal{I}_2)(X)$,

(ii) *If*

$$\lim_{mn\to\infty} \frac{\mu_{mn}}{\lambda_{mn}^{\beta}} = 1, \tag{4.2}$$

then $S_{\lambda}^{\alpha}(\mathcal{I}_2) \subseteq S_{\mu}^{\beta}(\mathcal{I}_2)(X)$.

Proof. (i) Suppose that $\lambda_{mn} \leq \mu_{mn}$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$ and let 4.1 be satisfied. For neighbourhood U of 0, we have

$$\big\{(k,l) \in J_{mn} : x_{kl} - L \notin U\big\} \supseteq \big\{(k,l) \in I_{mn} : x_{kl} - L \notin U\big\}.$$

Therefore we can write

$$\frac{1}{\mu_{mn}^{\beta}}\Big|\big\{(k,l) \in J_{mn} : x_{kl} - L \notin U\big\}\Big| \geq \frac{\lambda_{mn}^{\alpha}}{\mu_{mn}^{\beta}}\frac{1}{\lambda_{mn}^{\alpha}}\Big|\big\{(k,l) \in I_{mn} : x_{kl} - L \notin U\big\}\Big|$$

and so for all $(m,n) \in \mathbb{N}\times\mathbb{N}$ we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{\lambda_{mn}^{\alpha}}\left|\{(k,l) \in I_{mn} : x_{kl} - L \notin U\}\right| \geq \delta\right\} \subseteq$$

$$\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{\mu_{mn}}\Big|\big\{(k,l) \in J_{mn} : x_{kl} - L \notin U\big\}\Big| \geq \delta\frac{\lambda_{mn}^{\alpha}}{\mu_{mn}^{\beta}}\right\} \in \mathcal{I}.$$

Hence $S_{\mu}^{\beta}(\mathcal{I}_2)(X) \subseteq S_{\lambda}^{\alpha}(\mathcal{I}_2)(X)$.

(ii) Let $x = (x_{kl}) \in S_{\lambda}^{\alpha}(\mathcal{I}_2)(X)$ and (4.2) be satisfied. Since $I_{mn} \subset J_{mn}$, for neighbourhood U of 0, we may write

$$\frac{1}{\mu_{mn}^{\beta}}\left|\{(k,l) \in J_{mn} : x_{kl} - L \notin U\}\right| =$$

$$= \frac{1}{\mu_{mn}^{\beta}}\left|\{m - \mu_m + 1 < k \leq m - \lambda_m, n - \mu_n + 1 < k \leq n - \lambda_n : x_{kl} - L \notin U\}\right| +$$

$$+\frac{1}{\mu_{mn}^{\beta}}\left|\{(k,l) \in I_{mn} : x_{kl} - L \notin U\}\right| \leq$$

$$\leq \frac{\mu_{mn} - \lambda_{mn}}{\mu_{mn}^{\beta}} + \frac{1}{\lambda_{mn}^{\beta}}\left|\{(k,l) \in I_{mn} : x_{kl} - L \notin U\}\right| \leq$$

$$\leq \left(\frac{\mu_{mn} - \lambda_{mn}^{\beta}}{\lambda_{mn}^{\beta}}\right) + \frac{1}{\lambda_{mn}^{\alpha}}\left|\{(k,l) \in I_{mn} : x_{kl} - L \notin U\}\right| \leq$$

$$\leq \left(\frac{\mu_{mn}}{\lambda_{mn}^{\beta}} - 1\right) + \frac{1}{\lambda_{mn}^{\alpha}}\left|\{(k,l) \in I_{mn} : x_{kl} - L \notin U\}\right|$$

for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. Hence we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}\colon \frac{1}{\mu_{mn}^{\beta}} \left|\{(k,l) \in J_{mn}: x_{kl} - L \notin U\}\right| \geq \delta\right\} \subseteq$$

$$\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{mn}^{\alpha}} \left|\{(k,l) \in I_{mn}: x_{kl} - L \notin U\}\right| \geq \delta\right\} \in \mathcal{I}.$$

This implies that $S_\lambda^\alpha(\mathcal{I}_2)(X) \subseteq S_\mu^\beta(\mathcal{I}_2)(X)$.

Theorem 4 is proved.

From Theorem 2 we have the following corollary.

Corollary 2. *Let $\lambda = (\lambda_{mn})$ and $\mu = (\mu_{mn})$ be two sequences in $\triangle$ such that $\lambda_{mn} \leq \mu_{mn}$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. If* (4.1) *holds, then*

(i) $S_\mu^\alpha(\mathcal{I}_2)(X) \subseteq S_\lambda^\alpha(\mathcal{I}_2)(X)$,
(ii) $S_\mu(\mathcal{I}_2)(X) \subseteq S_\lambda^\alpha(\mathcal{I}_2)(X)$,
(iii) $S_\mu(\mathcal{I}_2)(X) \subseteq S_\lambda(\mathcal{I}_2)(X)$.

Corollary 3. *Let $\lambda = (\lambda_{mn})$ and $\mu = (\mu_{mn})$ be two sequences in Λ such that $\lambda_{mn} \leq \mu_{mn}$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. If* (4.2) *holds, then*

(i) $S_\lambda^\alpha(\mathcal{I}_2)(X) \subseteq S_\mu^\alpha(\mathcal{I}_2)(X)$,
(ii) $S_\lambda^\alpha(\mathcal{I}_2)(X) \subseteq S_\mu(\mathcal{I}_2)(X)$,
(iii) $S_\lambda(\mathcal{I}^2)(X) \subseteq S_\mu(\mathcal{I}_2)(X)$.

Theorem 5. *If $\lambda \in \triangle$ be such that $\lim_{m,n} \dfrac{\lambda_m n}{mn} = 1$, then $S_\lambda^\alpha(\mathcal{I}_2)(X) \subset S^\alpha(\mathcal{I}_2)(X)$.*

Proof. Let $\delta > 0$ be given. Since $\lim_{mn} \dfrac{\lambda_{mn}}{mn} = 1$, we can choose $(r,s) \in N\times N$ such that $\left|\dfrac{\lambda_{mn}}{mn} - 1\right| < \dfrac{\delta}{2}$, for all $m \geq r,\ n \geq s$. Let us take any neighbourhood U of 0. Now observe that,

$$\frac{1}{(mn)^\alpha}\left|\{k \leq m, l \leq n\colon x_{kl} - L \notin U\}\right| =$$

$$= \frac{1}{mn}\left|\{k \leq m-\lambda_m, l \leq n-\lambda_n\colon x_{kl} - L \notin U\}\right| + \frac{1}{(mn)^\alpha}\left|\{(k,l) \in I_{mn}\colon x_{kl} - L \notin U\}\right| \leq$$

$$\leq \frac{mn - \lambda_{mn}}{(mn)^\alpha} + \frac{1}{mn}\left|\{(k,l) \in I_{mn}\colon x_{kl} - L \notin U\}\right| =$$

$$= \frac{\delta}{2} + \frac{1}{(mn)^\alpha}\left|\{(k,l) \in I_{mn}\colon x_{kl} - L \notin U\}\right|$$

for all $m \geq r,\ n \geq s$. Hence, for $\delta > 0$ and any neighbourhood U of 0

$$\left\{n \in N\colon \frac{1}{(mn)^\alpha} \left|\{k \leq m, l \leq n\colon x_{kl} - L \notin U\}\right| \geq \delta\right\} \subset$$

$$\subset \left\{n \in N\colon \frac{1}{\lambda_{mn}^\alpha} \left|\{(k,l) \in I_{mn}\colon x_{kl} - L \notin U\}\right| \geq \frac{\delta}{2}\right\} \cup A,$$

where A is the union of the first m_0 rows and the first n_0 columns of the double sequence.

If $S_\lambda^\alpha(\mathcal{I}_2) - \lim x = L$, then the set on the right-hand side belongs to $\mathcal{I}_2$ and so the set on the left-hand side also belongs to $\mathcal{I}_2$. This shows that $x = (x_{kl})$ is $\mathcal{I}_2$- statistically convergent to L.

Theorem 5 is proved.

***Remark* 3.** We do not know whether the condition in Theorem 5 is necessary and leave it as an open problem.

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