UDC 517.5

S. S. Dragomir (Coll. Eng. and Sci., Victoria Univ., Australia and School Comput. Sci. and Appl. Math., Univ. Witwatersrand, Johannesburg, South Africa)

INTEGRAL INEQUALITIES OF THE HERMITE – HADAMARD TYPE FOR K-BOUNDED NORM CONVEX MAPPINGS

ІНТЕГРАЛЬНІ НЕРІВНОСТІ ТИПУ ЕРМІТА – АДАМАРА ДЛЯ K-ОБМЕЖЕНИХ ВІДОБРАЖЕНЬ З ОПУКЛОЮ НОРМОЮ

We obtain some inequalities of the Hermite – Hadamard type for K-bounded norm convex mappings between two normed spaces. The applications for twice differentiable functions in Banach spaces and functions defined by power series in Banach algebras are presented. Some discrete Jensen-type inequalities are also obtained.

Отримано деякі нерівності типу Ерміта – Адамара для K-обмежених відображень з опуклою нормою між двома нормованими просторами. Наведено застосування до двічі диференційовних функцій у банахових просторах та функцій, що визначені степеневими рядами в банахових алгебрах. Отримано також деякі нерівності типу Джексона.

1. Introduction. Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5, 34] and the references therein.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e., there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L\|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [32, 33] and Kato in [39], the following inequality holds:

$$\||A| - |B|\| \leq \frac{2}{\pi}\|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right)$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert – Schmidt norm* $\|C\|_{HS} := (\mathrm{tr}C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]:

$$\||A| - |B|\|_{HS} \leq \sqrt{2}\|A - B\|_{HS}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be self-adjoint, then the best coefficient is 1.

It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\| \le a_1\|A-B\| + a_2\|A-B\|^2 + O\left(\|A-B\|^3\right),$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \quad \text{and} \quad a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [3] the author also obtained the *Lipschitz-type inequality*

$$\|f(A)-f(B)\| \le f'(a)\|A-B\|,$$

where f is an *operator monotone function* on $(0,\infty)$ and $A, B \ge aI_H > 0$.

Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two Banach spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. For any mapping $F\colon C \subset X \to Y$ we can consider the associated functions $\Phi_{F,x,y,\lambda}$, $\Psi_{F,x,y,\lambda}\colon [0,1]\to Y$, where $x,y\in C$, $\lambda\in[0,1]$, defined by [25]

$$\Phi_{F,x,y,\lambda}(t) := (1-\lambda)F[(1-t)((1-\lambda)x+\lambda y)+ty] + \lambda F[(1-t)x+t((1-\lambda)x+\lambda y)]$$

and

$$\Psi_{F,x,y,\lambda}(t) := (1-\lambda)F[(1-t)((1-\lambda)x+\lambda y)+ty] + \lambda F[tx+(1-t)((1-\lambda)x+\lambda y)].$$

We say that the mapping $F\colon B\subset X\to Y$ is *Lipschitzian* with the constant $L>0$ on the subset B of X if

$$\|F(x)-F(y)\|_Y \le L\|x-y\|_X \quad \text{for any} \quad x,y\in B.$$

The following result holds [25]:

Theorem 1.1. *Let $F\colon C\subset X\to Y$ be a Lipschitzian mapping with the constant $L>0$ on the convex subset C of X. If $x,y\in C$, then we have*

$$\left\|\Lambda_{F,x,y,\lambda}(t) - \int\limits_0^1 F[sy+(1-s)x]\,ds\right\|_Y \le$$

$$\le 2L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|x-y\|_X \tag{1.1}$$

for any $t\in[0,1]$ and $\lambda\in[0,1]$, where $\Lambda_{F,x,y,\lambda}=\Phi_{F,x,y,\lambda}$ or $\Lambda_{F,x,y,\lambda}=\Psi_{F,x,y,\lambda}$.

If we take in (1.1) $\Lambda_{F,x,y,\lambda}=\Phi_{F,x,y,\lambda}$, $\lambda=\dfrac{1}{2}$, then we get

$$\left\|\frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2}+ty\right]+F\left[(1-t)x+t\frac{x+y}{2}\right]\right)-\int\limits_0^1 F[sy+(1-s)x]ds\right\| \le$$

$$\le \frac{1}{2}L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\|_X$$

for any $x,y\in C$ and $t\in[0,1]$.

If we take in (1.1) $\Lambda_{F,x,y,\lambda}=\Psi_{F,x,y,\lambda}$, $\lambda=\dfrac{1}{2}$, then we obtain

$$\left\| \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2}+ty\right]+F\left[tx+(1-t)\frac{x+y}{2}\right]\right)-\int\limits_0^1 F[sy+(1-s)x]ds\right\|_Y \le$$

$$\le \frac{1}{2}L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\|_X$$

for any $t\in[0,1]$ and $x,y\in C$.

We also have the simpler inequalities

$$\left\| \frac{1}{2}\left[F\left(\frac{3x+y}{4}\right)+F\left(\frac{x+3y}{4}\right)\right]-\int\limits_0^1 F\left[sy+(1-s)x\right]ds\right\|_Y \le \frac{1}{8}L\|x-y\|_X, \tag{1.2}$$

$$\left\| F\left(\frac{x+y}{2}\right)-\int\limits_0^1 F[sy+(1-s)]ds\right\|_Y \le \frac{1}{4}L\|x-y\|_X \tag{1.3}$$

and

$$\left\| \frac{1}{2}[F(x)+F(y)]-\int\limits_0^1 F[sy+(1-s)x]ds\right\|_Y \le \frac{1}{4}L\|x-y\|_X \tag{1.4}$$

for any $x,y\in C$. The constants $\frac{1}{8}$ and $\frac{1}{4}$ are best possible.

The inequalities (1.3) and (1.4) are the corresponding versions of Hermite – Hadamard inequalities for Lipschitzian functions. The scalar cases were obtained in [12] and [43]. For Hermite – Hadamard's type inequalities, see for instance [10, 12, 13, 35, 37, 38, 40, 42, 43, 46 – 50] and the references therein.

From (1.1) we also have the Ostrowski's inequality

$$\left\| F[ty+(1-t)x]-\int\limits_0^1 F[sy+(1-s)x]ds\right\|_Y \le L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\|_X$$

for any $t\in[0,1]$ and $x,y\in C$. For Ostrowski's type inequalities for the Lebesgue integral, see [1, 8, 9, 15 – 30]. Inequalities for the Riemann – Stieltjes integral may be found in [17, 19] while the generalization for isotonic functionals was provided in [20]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [23].

Motivated by the above results, we introduce here a class of functions that extends the concept of Lipschitzian function to power two of norm difference and called them K-bounded norm convex functions. Comprehensive examples of such functions are given. Integral inequalities of Hermite – Hadamard type are obtained and applications for discrete inequalities of Jensen type are provided as well.

2. *K*-bounded norm convex mappings. Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. We consider the following class of functions:

Definition 2.1. *A mapping $F: C \subset X \to Y$ is called K-bounded norm convex, for some given $K > 0$ if it satisfies the condition*

$$\|(1-\lambda)F(x) + \lambda F(y) - F((1-\lambda)x + \lambda y)\|_Y \leq \frac{1}{2} K\lambda(1-\lambda)\|x-y\|_X^2 \qquad (2.1)$$

for any $x, y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $F \in \mathcal{BN}_K(C)$.

We have from (2.1) for $\lambda = \dfrac{1}{2}$ the Jensen's inequality

$$\left\|\frac{F(x)+F(y)}{2} - F\left(\frac{x+y}{2}\right)\right\|_Y \leq \frac{1}{8} K\|x-y\|_X^2 \qquad (2.2)$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

We observe also that, by the triangle inequality, we have

$$\|F((1-\lambda)x + \lambda y)\|_Y - \|(1-\lambda)F(x) + \lambda F(y)\|_Y \leq$$
$$\leq \|(1-\lambda)F(x) + \lambda F(y) - F((1-\lambda)x + \lambda y)\|_Y$$

which, by the triangle inequality, gives

$$\|F((1-\lambda)x + \lambda y)\|_Y \leq \frac{1}{2} K\lambda(1-\lambda)\|x-y\|_X^2 + (1-\lambda)\|F(x)\|_Y + \lambda\|F(y)\|_Y \qquad (2.3)$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Now, if the function $t \mapsto \|F((1-\lambda)x + \lambda y)\|_Y$, for some $x, y \in C$, is Lebesgue integrable on $[0,1]$, then by taking the integral in (2.3) we get

$$\int_0^1 \|F((1-\lambda)x + \lambda y)\|_Y d\lambda \leq \frac{1}{12} K\|x-y\|_X^2 + \frac{1}{2}\left[\|F(x)\|_Y + \|F(y)\|_Y\right]. \qquad (2.4)$$

If we assume continuity for the function F on C in the norm topology of $(X; \|\cdot\|_X)$, then the inequality (2.4) holds for any $x, y \in C$. Moreover, if we assume that $(Y; \|\cdot\|_Y)$ is a Banach space and F is continuos on C, then we have the generalized triangle inequality

$$\left\|\int_0^1 F((1-\lambda)x + \lambda y)d\lambda\right\|_Y \leq \int_0^1 \|F((1-\lambda)x + \lambda y)\|_Y d\lambda,$$

and by (2.4) we obtain

$$\left\|\int_0^1 F((1-\lambda)x + \lambda y)d\lambda\right\|_Y \leq \frac{1}{12} K\|x-y\|_X^2 + \frac{1}{2}\left[\|F(x)\|_Y + \|F(y)\|_Y\right]$$

for any $x, y \in C$.

We can improve this result as follows.

Theorem 2.1. *$(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F: C \subset X \to Y$ is continuous on the convex set C in the norm topology. If $F \in \mathcal{BN}_K(C)$ for some $K > 0$, then we have*

$$\left\| \frac{F(x)+F(y)}{2} - \int\limits_0^1 F((1-\lambda)x+\lambda y)d\lambda \right\|_Y \leq \frac{1}{12}\,K\|x-y\|_X^2 \tag{2.5}$$

and

$$\left\| \int\limits_0^1 F((1-\lambda)x+\lambda y)d\lambda - F\left(\frac{x+y}{2}\right) \right\|_Y \leq \frac{1}{24}\,K\|x-y\|_X^2 \tag{2.6}$$

for any $x, y \in C$. The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

Proof. From (2.1) we get successively

$$\left\| \int\limits_0^1 [(1-\lambda)F(x)+\lambda F(y) - F((1-\lambda)x+\lambda y)]d\lambda \right\|_Y \leq$$

$$\leq \int\limits_0^1 \|(1-\lambda)F(x)+\lambda F(y) - F((1-\lambda)x+\lambda y)\|_Y d\lambda \leq$$

$$\leq \frac{1}{2}\,K\|x-y\|_X^2 \int\limits_0^1 \lambda(1-\lambda)d\lambda,$$

which produces the desired result (2.5).

Utilising (2.2) we have

$$\left\| \frac{F((1-\lambda)x+\lambda y)+F(\lambda x+(1-\lambda)y)}{2} - F\left(\frac{x+y}{2}\right) \right\|_Y \leq$$

$$\leq \frac{1}{8}\,K\|(1-\lambda)x+\lambda y-\lambda x-(1-\lambda)y\|_X^2 =$$

$$= \frac{1}{8}\,K(1-2\lambda)^2\|x-y\|_X^2 = \frac{1}{2}\,K\left(\lambda-\frac{1}{2}\right)^2\|x-y\|_X^2 \tag{2.7}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Integrating in (2.7) we obtain

$$\left\| \int\limits_0^1 \left[\frac{F((1-\lambda)x+\lambda y)+F(\lambda x+(1-\lambda)y)}{2} - F\left(\frac{x+y}{2}\right)\right] d\lambda \right\|_Y \leq$$

$$\leq \int\limits_0^1 \left\| \frac{F((1-\lambda)x+\lambda y)+F(\lambda x+(1-\lambda)y)}{2} - F\left(\frac{x+y}{2}\right) \right\|_Y d\lambda \leq$$

$$\leq \frac{1}{2}\,K\|x-y\|_X^2 \int\limits_0^1 \left(\lambda-\frac{1}{2}\right)^2 d\lambda = \frac{1}{24}\,K\|x-y|_X^2 \tag{2.8}$$

and since

$$\int_0^1 F((1-\lambda)x+\lambda y)d\lambda = \int_0^1 F(\lambda x+(1-\lambda)y)d\lambda,$$

then from (2.8) we get (2.6).

Now, consider the function $F_0 \colon H \to \mathbb{R}$, $F_0(x) = \|x\|^2$, where $(H, \langle .,. \rangle)$ is a complex inner product space. If $x, y \in H$ and $\lambda \in [0,1]$, then

$$(1-\lambda)F_0(x)+\lambda F_0(y)-F_0((1-\lambda)x+\lambda y)=$$
$$=(1-\lambda)\|x\|^2+\lambda\|y\|^2-\|(1-\lambda)x+\lambda y\|^2=$$
$$=(1-\lambda)\|x\|^2+\lambda\|y\|^2-(1-\lambda)^2\|x\|^2-2(1-\lambda)\lambda\mathrm{Re}\,\langle x,y\rangle-\lambda^2\|y\|^2=$$
$$=(1-\lambda)\lambda\left[\|x\|^2-2\mathrm{Re}\,\langle x,y\rangle+\|y\|^2\right]=(1-\lambda)\lambda\|x-y\|^2$$

showing that F_0 is continuous and K-bounded norm convex with $K=2$ on H.

We have

$$\int_0^1 F_0((1-\lambda)x+\lambda y)d\lambda = \int_0^1 \|(1-\lambda)x+\lambda y\|^2 d\lambda =$$
$$=\int_0^1 \left[(1-\lambda)^2\|x\|^2+2(1-\lambda)\lambda\mathrm{Re}\,\langle x,y\rangle+\lambda^2\|y\|^2\right]d\lambda=$$
$$=\frac{1}{3}\left[\|x\|^2+\mathrm{Re}\,\langle x,y\rangle+\|y\|^2\right]$$

for any $x, y \in H$.

Therefore

$$\frac{F_0(x)+F_0(y)}{2}-\int_0^1 F_0((1-\lambda)x+\lambda y)d\lambda=$$
$$=\frac{1}{2}\left[\|x\|^2+\|y\|^2\right]-\frac{1}{3}\left[\|x\|^2+\mathrm{Re}\,\langle x,y\rangle+\|y\|^2\right]=\frac{1}{6}\|x-y\|^2$$

showing that we have the same quantity $\frac{1}{6}\|x-y\|^2$ in both sides of (2.5).

We also have

$$\int_0^1 F_0((1-\lambda)x+\lambda y)d\lambda - F_0\left(\frac{x+y}{2}\right)=$$
$$=\frac{1}{3}\left[\|x\|^2+\mathrm{Re}\,\langle x,y\rangle+\|y\|^2\right]-\frac{1}{4}\left[\|x\|^2+2\mathrm{Re}\,\langle x,y\rangle+\|y\|^2\right]=\frac{1}{12}\|x-y\|^2$$

showing that we have the same quantity $\frac{1}{12}\|x-y\|^2$ in both sides of (2.6).

Theorem 2.1 is proved.

3. Some examples in Banach algebras. Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| \colon \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further,

$$\|ab\| \leq \|a\|\|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*.

We assume that the Banach algebra is *unital*, this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}\mathcal{B}$. If $a, b \in \operatorname{Inv}\mathcal{B}$ then $ab \in \operatorname{Inv}\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have

(i) if $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}\mathcal{B}$;

(ii) $\{a \in \mathcal{B} \colon \|1 - b\| < 1\} \subset \operatorname{Inv}\mathcal{B}$;

(iii) $\operatorname{Inv}\mathcal{B}$ is an *open subset* of $\mathcal{B}$;

(iv) the map $\operatorname{Inv}\mathcal{B} \ni a \longmapsto a^{-1} \in \operatorname{Inv}\mathcal{B}$ is continuous.

The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} \colon z1 - a \in \operatorname{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a \colon \rho(a) \to \operatorname{Inv}\mathcal{B}$,

$$R_a(z) := (z1 - a)^{-1}.$$

For each $z, w \in \rho(a)$ we get the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also obtain

$$\sigma(a) \subset \{z \in \mathbb{C} \colon \|z\| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{\|z\| : z \in \sigma(a)\}.$$

If a, b are *commuting* elements in $\mathcal{B}$, i.e., $ab = ba$, then

$$\nu(ab) \leq \nu(a)\nu(b) \quad \text{and} \quad \nu(a + b) \leq \nu(a) + \nu(b).$$

Let $\mathcal{B}$ be a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) the resolvent set $\rho(a)$ is open in $\mathbb{C}$;

(ii) for any *bounded linear functionals* $\lambda \colon \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;

(iii) the spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;

(iv) we have

$$\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series*

$$f(z) := \sum_{j=0}^{\infty} \alpha_j z^j \quad (\|z\| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \|a^j\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted $\exp$ and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!}\, a^j \quad \text{for each} \quad a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a + b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e., the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1 - a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}\,(1 - a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [31] and [45].

Now, by the help of power series $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_{abs}(\lambda) := \sum_{n=0}^{\infty} \|\alpha_n\| \lambda^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_{abs} = f$.

The following result provides a class of functions that are K-bounded norm convex on closed balls from Banach algebras.

Theorem 3.1. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \leq M < R$, $M > 0$ we have that*

$$\|(1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)\| \leq \frac{1}{2}\,\lambda\,(1 - \lambda)\, f''_{abs}(M)\|x - y\|^2 \tag{3.1}$$

for any $\lambda \in [0, 1]$.

In other words the function $f\colon \overline{B}(0, M) \subset \mathcal{B} \to \mathcal{B}$, where $\overline{B}(0, M)$ is the closed ball $\{x \in \in \mathcal{B}, \|x\| \leq M\}$ defined by $f(x) = \sum_{n=0}^{\infty} \alpha_n x^n$, $x \in \overline{B}(0, M)$ is K-bounded norm convex with $K = f''_{abs}(M)$.

Proof. We use the identity (see, for instance, [6, p. 254])

$$a^n - b^n = \sum_{j=0}^{n-1} a^{n-1-j}(a - b)b^j \tag{3.2}$$

that holds for any $a, b \in \mathcal{B}$ and $n \geq 1$.

Let $x, y \in \mathcal{B}$. By (3.2) we have

$$[(1 - \lambda)x + \lambda y]^n - x^n = \lambda \sum_{j=0}^{n-1} [(1 - \lambda)x + \lambda y]^{n-1-j}(y - x)x^j \tag{3.3}$$

and

$$[(1-\lambda)x+\lambda y]^n - y^n = -(1-\lambda)\sum_{j=0}^{n-1}[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)y^j \tag{3.4}$$

for $n \geq 1$ and $\lambda \in [0,1]$.

Multiply (3.3) by $1-\lambda$ and (3.4) by λ and add the obtained equalities to get

$$[(1-\lambda)x+\lambda y]^n - (1-\lambda)x^n - \lambda y^n =$$

$$= \lambda(1-\lambda)\sum_{j=0}^{n-1}[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)\left(x^j - y^j\right) =$$

$$= \lambda(1-\lambda)\sum_{j=1}^{n-1}[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)\left(x^j - y^j\right) \tag{3.5}$$

for $n \geq 2$ and $\lambda \in [0,1]$.

If $j \geq 1$ we also obtain

$$x^j - y^j = \sum_{\ell=0}^{j-1} x^{j-1-\ell}(x-y)\,y^\ell$$

and by (3.5) we have

$$(1-\lambda)x^n + \lambda y^n - [(1-\lambda)x+\lambda y]^n =$$

$$= \lambda(1-\lambda)\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1}[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^\ell \tag{3.6}$$

for $n \geq 2$ and $\lambda \in [0,1]$, which is an equality of interest in itself.

Let $m \geq 2$ and $x, y \in \mathcal{B}$. Then, by utilizing (3.6), we get

$$(1-\lambda)\sum_{n=0}^{m} a_n x^n + \lambda\sum_{n=0}^{m} a_n y^n - \sum_{n=0}^{m} a_n[(1-\lambda)x+\lambda y]^n =$$

$$= \sum_{n=0}^{m} a_n[(1-\lambda)x^n + \lambda y^n - [(1-\lambda)x+\lambda y]^n] =$$

$$= \sum_{n=2}^{m} a_n[(1-\lambda)x^n + \lambda y^n - [(1-\lambda)x+\lambda y]^n] =$$

$$= \lambda(1-\lambda)\sum_{n=2}^{m} a_n\left(\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1}[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^\ell\right) \tag{3.7}$$

for all $m \geq 2$, $x, y \in \mathcal{B}$ and $\lambda \in [0,1]$.

Taking the norm in (3.7) and using repeatedly the generalized triangle inequality we have

$$\left\|(1-\lambda)\sum_{n=0}^{m}a_nx^n+\lambda\sum_{n=0}^{m}a_ny^n-\sum_{n=0}^{m}a_n[(1-\lambda)x+\lambda y]^n\right\|\le$$

$$\le\lambda(1-\lambda)\sum_{n=2}^{m}|a_n|\left(\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1}\left\|[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^{\ell}\right\|\right). \tag{3.8}$$

If $\|x\|,\|y\|\le M<R$, then $\|(1-\lambda)x+\lambda y\|\le M$ for $\lambda\in[0,1]$ and using the Banach algebra properties we obtain

$$\left\|[(1-\lambda)x+\lambda y]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^{\ell}\right\|\le$$

$$\le\|(1-\lambda)x+\lambda y\|^{n-1-j}\|y-x\|\|x\|^{j-1-\ell}\|x-y\|\|y\|^{\ell}=$$

$$=\|y-x\|^2\|(1-\lambda)x+\lambda y\|^{n-1-j}\|x\|^{j-1-\ell}\|y\|^{\ell}\le$$

$$\le\|y-x\|^2M^{n-1-j}M^{j-1-\ell}M^{\ell}=\|y-x\|^2M^{n-2} \tag{3.9}$$

for $n\ge 2$.

Therefore, by (3.8) and (3.9) we get

$$\left\|(1-\lambda)\sum_{n=0}^{m}a_nx^n+\lambda\sum_{n=0}^{m}a_ny^n-\sum_{n=0}^{m}a_n[(1-\lambda)x+\lambda y]^n\right\|\le$$

$$\le\lambda(1-\lambda)\sum_{n=2}^{m}|a_n|\left(\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1}\|y-x\|^2M^{n-2}\right)=$$

$$=\lambda(1-\lambda)\|y-x\|^2\sum_{n=2}^{m}\|a_n\|M^{n-2}\sum_{j=1}^{n-1}j=$$

$$=\frac{1}{2}\lambda(1-\lambda)\|y-x\|^2\sum_{n=2}^{m}n(n-1)\|a_n\|M^{n-2} \tag{3.10}$$

for any $\|x\|,\|y\|\le M<R$, $m\ge 2$ and $\lambda\in[0,1]$.

Since the series whose partial sums involved in (3.10) are convergent and

$$\sum_{n=0}^{\infty}a_nx^n=f(x),\quad \sum_{n=0}^{\infty}a_ny^n=f(y),$$

$$\sum_{n=0}^{m}a_n[(1-\lambda)x+\lambda y]^n=f((1-\lambda)x+\lambda y),$$

$$\sum_{n=2}^{\infty}n(n-1)\|a_n\|M^{n-2}=f''_{abs}(M),$$

then by letting $m\to\infty$ in (3.10) we deduce the desired result (3.1).

Theorem 3.1 is proved.

***Corollary* 3.1.** *With the assumptions from Theorem* 3.1 *we have the inequalities*

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-\lambda)x+\lambda y)d\lambda \right\| \le \frac{1}{12} f''_{abs}(M)\|x-y\|^2 \tag{3.11}$$

and

$$\left\| \int_0^1 f((1-\lambda)x+\lambda y)d\lambda - f\left(\frac{x+y}{2}\right) \right\| \le \frac{1}{24} f''_{abs}(M)\|x-y\|^2 \tag{3.12}$$

for any $x, y \in \mathcal{B}$ *with* $\|x\|, \|y\| \le M < R,\ M > 0.$

The constants $\frac{1}{12}$ *and* $\frac{1}{24}$ *are best possible.*

It is known that if x and y are commuting, i.e., $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if z is invertible and $a, b \in \mathbb{R}$ with $a < b$, then

$$\int_a^b \exp(tz)dt = z^{-1}[\exp(bz) - \exp(az)].$$

Therefore, if x and y are commuting and $y - x$ is invertible, then

$$\int_0^1 \exp((1-s)x+sy)ds = \int_0^1 \exp(s(y-x))\exp(x)ds =$$

$$= \left(\int_0^1 \exp(s(y-x))ds\right)\exp(x) =$$

$$= (y-x)^{-1}[\exp(y-x)-1]\exp(x) = (y-x)^{-1}[\exp(y)-\exp(x)],$$

and by (3.11) and (3.12) we get

$$\left\| \frac{\exp(x)+\exp(y)}{2} - (y-x)^{-1}[\exp(y)-\exp(x)] \right\| \le \frac{1}{12}\exp(M)\|x-y\|^2$$

and

$$\left\| (y-x)^{-1}[\exp(y)-\exp(x)] - \exp\left(\frac{x+y}{2}\right) \right\| \le \frac{1}{24}\exp(M)\|x-y\|^2$$

provided $\|x\|, \|y\| \le M,\ M > 0.$

4. Case of twice differentiable mappings in normed spaces. We first recall some results concerning Taylor's formula for differentiable mappings between two normed spaces, see for instance [11] for the basic definitions and results.

Lemma 4.1 (Taylor's formula, Lagrange's remainder [11, p. 110, 111])**.** *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed linear spaces,* Ω *an open subset of* X *and* $f\colon \Omega \to Y$ *a* $(k+1)$*-differentiable mapping on* Ω *with* $k \ge 0$. *Suppose that* $x, y \in \Omega$ *are such that the segment* $[x, y] :=$ $:= \{(1-\lambda)x + \lambda y, \lambda \in [0,1]\}$ *is contained in* Ω. *Then*

$$f(y) = f(x) + f^{(1)}(x)(y-x) + \frac{1}{2!} f^{(2)}(x)(y-x, y-x) + \ldots$$
$$\ldots + \frac{1}{k!} f^{(k)}(x)(y-x, \ldots, y-x) + R_k(x,y), \tag{4.1}$$

where

$$\|R_k(x,y)\|_Y \le \frac{1}{(k+1)!} \|y-x\|_X^{k+1} \sup_{\lambda\in[0,1]} \left\| f^{(k+1)}((1-\lambda)x + \lambda y) \right\|_{\mathcal{L}(X^{k+1};Y)}.$$

We observe that if Ω is open and convex, then the equality (4.1) holds for any $x, y \in \Omega$. In this case we also have the bound

$$\|R_k(x,y)\|_Y \le \frac{1}{(k+1)!} \|y-x\|_X^{k+1} \sup_{z\in\Omega} \left\| f^{(k+1)}(z) \right\|_{\mathcal{L}(X^{k+1};Y)}$$

for any $x, y \in \Omega$.

We can prove the following result:

Theorem 4.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $f: C \to Y$ a twice-differentiable mapping on C. Then for any $x, y \in C$ and $\lambda \in [0,1]$ we have*

$$\|(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)\|_Y \le \frac{1}{2} K\lambda(1-\lambda)\|y-x\|_X^2, \tag{4.2}$$

where

$$K := \sup_{z\in C} \left\| f''(z) \right\|_{\mathcal{L}(X^2;Y)} \tag{4.3}$$

is assumed to be finite.

Proof. Using the above Lemma 4.1 we can state that

$$\left\| f(u) - f(v) - f'(v)(u-v) \right\|_F \le \frac{1}{2} K\|u-v\|_X^2 \tag{4.4}$$

for any $u, v \in C$, where K is given by (4.3).

Let $x, y \in C$ and $\lambda \in [0,1]$. By (4.4) we have

$$\left\| f(x) - f((1-\lambda)x + \lambda y) - \lambda f'((1-\lambda)x + \lambda y)(x-y) \right\|_Y \le \frac{1}{2} K\lambda^2\|y-x\|_X^2 \tag{4.5}$$

and

$$\left\| f(y) - f((1-\lambda)x + \lambda y) - (1-\lambda) f'((1-\lambda)x + \lambda y)(y-x) \right\|_Y \le$$
$$\le \frac{1}{2} K(1-\lambda)^2\|y-x\|_X^2. \tag{4.6}$$

Multiply (4.5) by $1-\lambda$ and (4.6) by λ and add the obtained inequalities to get

$$(1-\lambda) \left\| f(x) - f((1-\lambda)x + \lambda y) - \lambda f'((1-\lambda)x + \lambda y)(x-y) \right\|_Y +$$
$$+\lambda \left\| f(y) - f((1-\lambda)x + \lambda y) + (1-\lambda) f'((1-\lambda)x + \lambda y)(x-y) \right\|_Y \le$$

$$\le \frac{1}{2}K\lambda^2(1-\lambda)\|y-x\|_X^2 + \frac{1}{2}K(1-\lambda)^2\lambda\|y-x\|_X^2 = \frac{1}{2}K\lambda(1-\lambda)\|y-x\|_X^2. \tag{4.7}$$

By the triangle inequality we also obtain

$$\|(1-\lambda)f(x)+\lambda f(y)-f((1-\lambda)x+\lambda y)\|_Y \le$$
$$\le (1-\lambda)\left\|f(x)-f((1-\lambda)x+\lambda y)-\lambda f'((1-\lambda)x+\lambda y)(x-y)\right\|_Y +$$
$$+\lambda\left\|f(y)-f((1-\lambda)x+\lambda y)+(1-\lambda)f'((1-\lambda)x+\lambda y)(x-y)\right\|_Y \tag{4.8}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Making use of (4.7) and (4.8) we deduce the desired result (4.2).

Theorem 4.1 is proved.

***Corollary* 4.1.** *With the assumptions from Theorem* 4.1 *we have the inequalities*

$$\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-\lambda)x+\lambda y)d\lambda\right\|_Y \le \frac{1}{12}\sup_{z\in C}\left\|f''(z)\right\|_{\mathcal{L}(X^2;Y)}\|x-y\|_X^2$$

and

$$\left\|\int_0^1 f((1-\lambda)x+\lambda y)d\lambda - f\left(\frac{x+y}{2}\right)\right\|_Y \le \frac{1}{24}\sup_{z\in C}\left\|f''(z)\right\|_{\mathcal{L}(X^2;Y)}\|x-y\|_X^2$$

for any $x, y \in C$.

The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

5. Related inequalities. We have the following result as well:

Theorem 5.1. *Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F\colon C\subset X\to Y$ is continuous on the convex set C in the norm topology. If $F\in\mathcal{BN}_K(C)$ for some $K>0$, then we have*

$$\left\|\int_0^1 F(uy+(1-u)x)du - \frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} F(sx+(1-s)y)ds\right\|_F \le$$
$$\le \frac{1}{6}K\lambda(1-\lambda)\|y-x\|_X^2 \tag{5.1}$$

for any $\lambda\in[0,1]$, $\lambda\ne\frac{1}{2}$ and $x, y\in C$.

Proof. Since $F\in\mathcal{BN}_K(C)$ for $K>0$, then

$$\|(1-\lambda)F(u)+\lambda F(v)-F((1-\lambda)u+\lambda v)\|_Y \le \frac{1}{2}K\lambda(1-\lambda)\|u-v\|_X^2 \tag{5.2}$$

for any $u, v\in C$ and $\lambda\in[0,1]$.

Let $t\in[0,1]$ and for $x, y\in C$, take

$$u=(1-t)((1-\lambda)x+\lambda y)+ty, \quad v=tx+(1-t)((1-\lambda)x+\lambda y)\in C$$

in (5.2) to get

$$\|(1-\lambda)F((1-t)((1-\lambda)x+\lambda y)+ty)+\lambda F(tx+(1-t)((1-\lambda)x+\lambda y)t)-$$
$$-F((1-\lambda)[(1-t)((1-\lambda)x+\lambda y)+ty]+\lambda[tx+(1-t)((1-\lambda)x+\lambda y)])\|_Y\leq$$
$$\leq\frac{1}{2}K\lambda(1-\lambda)\|(1-t)((1-\lambda)x+\lambda y)+ty-[tx+(1-t)((1-\lambda)x+\lambda y)]\|_X^2. \tag{5.3}$$

Observe that

$$(1-\lambda)[(1-t)((1-\lambda)x+\lambda y)+ty]+\lambda[tx+(1-t)((1-\lambda)x+\lambda y)]=$$
$$=(1-\lambda)(1-t)((1-\lambda)x+\lambda y)+(1-\lambda)ty+\lambda tx+\lambda(1-t)((1-\lambda)x+\lambda y)=$$
$$=(1-t)((1-\lambda)x+\lambda y)+(1-\lambda)ty+\lambda tx=$$
$$=[(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y$$

and

$$(1-t)((1-\lambda)x+\lambda y)+ty-[tx+(1-t)((1-\lambda)x+\lambda y)]=$$
$$=(1-t)(1-\lambda)x+(1-t)\lambda y+ty-tx-(1-t)(1-\lambda)x-(1-t)\lambda y=t(y-x).$$

Then by (5.3) we have

$$\|(1-\lambda)F((1-t)((1-\lambda)x+\lambda y)+ty)+\lambda F(tx+(1-t)((1-\lambda)x+\lambda y))-$$
$$-F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)\|_Y\leq\frac{1}{2}K\,\lambda(1-\lambda)t^2\|y-x\|_X^2 \tag{5.4}$$

for any $t,\lambda\in[0,1]$ and $x,y\in C$.

Integrating the inequality (5.4) over t on $[0,1]$ and using the generalized triangle inequality for norms and integrals, we get

$$\left\|(1-\lambda)\int_0^1 F((1-t)((1-\lambda)x+\lambda y)+ty)dt+\right.$$
$$+\lambda\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))dt-$$
$$\left.-\int_0^1 F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)dt\right\|_Y\leq$$
$$\leq\frac{1}{6}K\lambda(1-\lambda)\|y-x\|_X^2 \tag{5.5}$$

for any $\lambda\in[0,1]$ and $x,y\in C$.

Observe that

$$\int_0^1 F[(1-t)(\lambda y+(1-\lambda)x)+ty]dt=\int_0^1 F[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]dt$$

and

$$\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))dt=$$

$$=\int_0^1 F((1-t)x+t((1-\lambda)x+\lambda y))dt=\int_0^1 F[t\lambda y+(1-\lambda t)x]dt.$$

If we make the change of variable $u:=(1-t)\lambda+t$, then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\int_0^1 F[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]dt=\frac{1}{1-\lambda}\int_\lambda^1 F[uy+(1-u)x]du.$$

If we make the change of variable $u:=\lambda t$, then we have $du=\lambda dt$ and

$$\int_0^1 F[t\lambda y+(1-\lambda t)x]dt=\frac{1}{\lambda}\int_0^\lambda F[uy+(1-u)x]du.$$

Therefore

$$(1-\lambda)\int_0^1 F[(1-t)(\lambda y+(1-\lambda)x)+ty]dt+$$

$$+\lambda\int_0^1 F[t(\lambda y+(1-\lambda)x)+(1-t)x]dt=$$

$$=\int_\lambda^1 F[uy+(1-u)x]du+\int_0^\lambda F[uy+(1-u)x]du=\int_0^1 F[uy+(1-u)x]du,$$

and we have the simple equality

$$(1-\lambda)\int_0^1 F((1-t)((1-\lambda)x+\lambda y)+ty)dt+$$

$$+\lambda\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))dt=\int_0^1 F[uy+(1-u)x]du$$

for any $\lambda\in[0,1]$ and $x,y\in C$.

Consider now the integral

$$\int_0^1 F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)dt.$$

Put

$$s = (1-t)(1-\lambda) + \lambda t = 1 - \lambda + (2\lambda - 1)t.$$

Then

$$1 - s = (1-t)\lambda + (1-\lambda)t.$$

If $\lambda \neq \frac{1}{2}$, then $s = 1 - \lambda + (2\lambda - 1)t$ is a change of variable with $dt = \frac{1}{2\lambda - 1}$ and we have

$$\int_0^1 F([(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y)dt =$$

$$= \frac{1}{2\lambda - 1} \int_{1-\lambda}^{\lambda} F(sx + (1-s)y)ds.$$

Now, making use of (5.5) we get the desired result (5.1).

Theorem 5.1 is proved.

***Remark* 5.1.** We observe that for $\lambda \to \frac{1}{2}$ we recapture from (5.1) the inequality (2.8).

If we take in (5.1) $\lambda = \frac{3}{4}$, then we get

$$\left\| \int_0^1 F[uy + (1-u)x]du - 2 \int_{1/4}^{3/4} F(sx + (1-s)y)ds \right\|_F \leq \frac{1}{32} K \|y - x\|_X^2.$$

Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. For any x, y in the Banach algebra $\mathcal{B}$ with $\|x\|, \|y\| \leq M < R$, $M > 0$ we have

$$\left\| \int_0^1 f(uy + (1-u)x)du - \frac{1}{2\lambda - 1} \int_{1-\lambda}^{\lambda} f(sx + (1-s)y)ds \right\| \leq$$

$$\leq \frac{1}{6} f''_{abs}(M)\lambda(1-\lambda)\|y - x\|^2$$

for any $\lambda \in [0,1]$, $\lambda \neq \frac{1}{2}$.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, with Y complete, C an open convex subset of X and $f : C \to Y$ a twice-differentiable mapping on C. Then for any $x, y \in C$ and $\lambda \in [0,1]$, $\lambda \neq \frac{1}{2}$, we obtain

$$\left\| \int_0^1 f(uy + (1-u)x)du - \frac{1}{2\lambda - 1} \int_{1-\lambda}^{\lambda} f(sx + (1-s)y)ds \right\| \leq$$

$$\leq \frac{1}{6} \sup_{z \in C} \left\| f''(z) \right\|_{\mathcal{L}(X^2;Y)} \lambda(1-\lambda)\|y - x\|^2.$$

References

1. *Anastassiou G. A.* Univariate Ostrowski inequalities, revisited // Monatsh. Math. – 2002. – **135**, № 3. – S. 175 – 189.
2. *Araki H., Yamagami S.* An inequality for Hilbert – Schmidt norm // Communs Math. Phys. – 1981. – **81**. – P. 89 – 96.
3. *Bhatia R.* First and second order perturbation bounds for the operator absolute value // Linear Algebra and Appl. – 1994. – **208/209**. – P. 367 – 376.
4. *Bhatia R.* Perturbation bounds for the operator absolute value // Linear Algebra and Appl. – 1995. – **226/228**. – P. 639 – 645.
5. *Bhatia R., Singh D., Sinha K. B.* Differentiation of operator functions and perturbation bounds // Communs Math. Phys. – 1998. – **191**, № 3. – P. 603 – 611.
6. *Bhatia R.* Matrix analysis. – Springer-Verlag, 1997.
7. *Cerone P., Dragomir S. S.* Midpoint-type rules from an inequalities point of view // Handb. Anal.-Comput. Meth. Appl. Math. /Ed. G. A. Anastassiou. – New York: CRC Press, 2003. – P. 135 – 200.
8. *Cerone P., Dragomir S. S.* New bounds for the three-point rule involving the Riemann – Stieltjes integrals // Adv. Statist. Combin. and Relat. Areas. – World Sci. Publ., 2002. – P. 53 – 62.
9. *Cerone P., Dragomir S. S., Roumeliotis J.* Some Ostrowski type inequalities for n-time differentiable mappings and applications // Demonstr. Math. – 1999. – **32**, № 2. – P. 697 – 712.
10. *Ciurdariu L.* A note concerning several Hermite – Hadamard inequalities for different types of convex functions // Int. J. Math. Anal. – 2012. – **6**, № 33 – 36. – P. 1623 – 1639.
11. *Coleman R.* Calculus on normed vector spaces. – New York etc.: Springer, 2012.
12. *Dragomir S. S., Cho Y. J., Kim S. S.* Inequalities of Hadamard's type for Lipschitzian mappings and their applications // J. Math. Anal. and Appl. – 2000. – **245**, № 2. – P. 489 – 501.
13. *Dragomir S. S., Pearce C. E. M.* Selected topics on Hermite – Hadamard inequalities and applications // RGMIA Monogr. – 2000 [Online http://rgmia.org/monographs/hermite_hadamard.html].
14. *Dragomir S. S.* Semi-inner products and applications. – Hauppauge, NY: Nova Sci. Publ., Inc. – 2004. – x+222 p.
15. *Dragomir S. S.* Ostrowski's inequality for monotonous mappings and applications // J. KSIAM. – 1999. – **3**, № 1. – P. 127 – 135.
16. *Dragomir S. S.* The Ostrowski's integral inequality for Lipschitzian mappings and applications // Comput. Math. and Appl. – 1999. – **38**. – P. 33 – 37.
17. *Dragomir S. S.* On the Ostrowski's inequality for Riemann – Stieltjes integral // Korean J. Appl. Math. – 2000. – **7**. – P. 477 – 485.
18. *Dragomir S. S.* On the Ostrowski's inequality for mappings of bounded variation and applications // Math. Inequal. and Appl. – 2001. – **4**, № 1. – P. 33 – 40.
19. *Dragomir S. S.* On the Ostrowski inequality for Riemann – Stieltjes integral $\int_a^b f(t)du(t)$ where f is of Hölder type and u is of bounded variation and applications // J. KSIAM. – 2001. – **5**, № 1. – P. 35 – 45.
20. *Dragomir S. S.* Ostrowski type inequalities for isotonic linear functionals // J. Inequal. Pure and Appl. Math. – 2002. – **3**, № 5. – Art. 68.
21. *Dragomir S. S.* An inequality improving the first Hermite – Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products // J. Inequal. Pure and Appl. Math. – 2002. – **3**, № 2. – Art. 31. – 8 p.
22. *Dragomir S. S.* An Ostrowski like inequality for convex functions and applications // Rev. Math. Comput. – 2003. – **16**, № 2. – P. 373 – 382.
23. *Dragomir S. S.* Operator inequalities of Ostrowski and trapezoidal type // Springer Briefs in Math. – New York: Springer, 2012. – x+112 p.
24. *Dragomir S. S.* Inequalities for power series in Banach algebras // SUT J. Math. – 2014. – **50**, № 1. – P. 25 – 45.
25. *Dragomir S. S.* Integral inequalities for Lipschitzian mappings between two Banach spaces and applications // Kodai Math. J. – 2016. – **39**. – P. 227 – 251.
26. *Dragomir S. S., Cerone P., Roumeliotis J., Wang S.* A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis // Bull. math. Soc. sci. math. roum. – 1999. – **42(90)**, № 4. – P. 301 – 314.
27. *Dragomir S. S., Rassias Th. M. (Eds).* Ostrowski type inequalities and applications in numerical integration. – Kluwer Acad. Publ., 2002.

28. *Dragomir S. S., Wang S.* A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules // Tamkang J. Math. – 1997. – **28**. – P. 239 – 244.
29. *Dragomir S. S., Wang S.* Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules // Appl. Math. Lett. – 1998. – **11**. – P. 105 – 109.
30. *Dragomir S. S., Wang S.* A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules // Indian J. Math. – 1998. – **40**, № 3. – P. 245 – 304.
31. *Douglas R.* Banach algebra techniques in operator theory. – Acad. Press, 1972.
32. *Farforovskaya Yu. B.* Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich – Rubinshtein metric (in Russian) // Vestn. Leningrad. Gos. Univ. Ser. Mat., Mekh., Astron. – 1967. – **4**. – P. 155 – 156.
33. *Farforovskaya Yu. B.* An estimate of the $\mathrm{norm}\, \|f(B) - f(A)\|$ for self-adjoint operators A and B (in Russian) // Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. – 1976. – **56**. – P. 143 – 162.
34. *Farforovskaya Yu. B., Nikolskaya L.* Modulus of continuity of operator functions // Algebra i Analiz. – 2008. – **20**, № 3. – P. 224 – 242 (transl. in St. Petersburg Math. J. – 2009. – **20**, № 3. – P. 493 – 506).
35. *Feng Y., Zhao W.* Refinement of Hermite – Hadamard inequality // Far East J. Math. Sci. – 2012. – **68**, № 2. – P. 245 – 250.
36. *Fink A. M.* Bounds on the deviation of a function from its averages // Czechoslovak Math. J. – 1992. – **42(117)**, № 2. – P. 298 – 310.
37. *Gao X.* A note on the Hermite – Hadamard inequality // J. Math. Inequal. – 2010. – **4**, № 4. – P. 587 – 591.
38. *Hwang S.-R., Tseng K.-L., Hsu K.-C.* Hermite – Hadamard type and Fejér type inequalities for general weights (I) // J. Inequal. Appl. – 2013. – **2013**, № 170.
39. *Kato T.* Continuity of the map $S \to |S|$ for linear operators // Proc. Jap. Acad. – 1973. – **49**. – P. 143 – 162.
40. *Kırmacı U. S., Dikici R.* On some Hermite – Hadamard type inequalities for twice differentiable mappings and applications // Tamkang J. Math. – 2013. – **44**, № 1. – P. 41 – 51.
41. *Mikusiński J.* The Bochner Integral. – Birkhäuser-Verlag, 1978.
42. *Muddassar M., Bhatti M. I., Iqbal M.* Some new s-Hermite – Hadamard type inequalities for differentiable functions and their applications // Proc. Pakistan Acad. Sci. – 2012. – **49**, № 1. – P. 9 – 17.
43. *Matić M., Pečarić J.* Note on inequalities of Hadamard's type for Lipschitzian mappings // Tamkang J. Math. – 2001. – **32**, № 2. – P. 127 – 130.
44. *Ostrowski A.* Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert // Comment. math. helv. – 1938. – **10**. – P. 226 – 227.
45. *Rudin W.* Functional analysis. – McGraw Hill, 1973.
46. *Sarikaya M. Z.* On new Hermite – Hadamard Fejér type integral inequalities // Stud. Univ. Babeş-Bolyai Math. – 2012. – **57**, № 3. – P. 377 – 386.
47. *Wąsowicz S., Witkowski A.* On some inequality of Hermite – Hadamard type // Opusc. Math. – 2012. – **32**, № 3. – P. 591 – 600.
48. *Xi B.-Y., Qi F.* Some integral inequalities of Hermite – Hadamard type for convex functions with applications to means // J. Funct. Spaces Appl. – 2012. – Art. ID 980438. – 14 p.
49. *Zabandan G., Bodaghi A., Kılıçman A.* The Hermite – Hadamard inequality for r-convex functions // J. Inequal. Appl. – 2012. – **2012**, № 215. – 8 p.
50. *Zhao C. -J., Cheung W. -S., Li X.-Y.* On the Hermite – Hadamard type inequalities // J. Inequal. Appl. – 2013. – **2013**, № 228.

Received 06.10.15