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# APPROXIMATION OF GENERAL $\alpha$-CUBIC FUNCTIONAL EQUATIONS IN 2-BANACH SPACES

# НАБЛИЖЕННЯ ЗАГАЛЬНИХ $\alpha$-КУБІЧНИХ ФУНКЦІОНАЛЬНИХ РІВНЯНЬ У 2-БАНАХОВИХ ПРОСТОРАХ

We introduce a new $\alpha$-cubic functional equation and investigate the generalized Hyers – Ulam stability of this functional equation in 2-Banach spaces.

Введено нове $\alpha$-кубічне функціональне рівняння та вивчено узагальнену стійкість Хайєрса – Улама цього функціонального рівняння в 2-банахових просторах.

**1. Introduction and preliminaries.** Speaking of the stability of a functional equation, we follow the question raised in 1940 by S. M. Ulam [29]:

*When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation*?

The first partial answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by D. H. Hyers (see [11]). This result was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [24] for linear mappings by considering an *unbounded Cauchy difference*. In 1994, a further generalization was obtained by P. Găvruta [10]. J. M. Rassias (see [19 – 23]) solved the Ulam problem for different mappings. In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [28].

During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers – Ulam stability to a number of functional equations and mappings (see [8, 9, 15, 16, 25 – 27]). We also refer the readers to the books: P. Czerwik [4] and D. H. Hyers, G. Isac and Th. M. Rassias [12].

Jun and Kim [13] introduced the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{1.1}$$

and they established the general solution and the generalized Hyers – Ulam stability problem for functional equation (1.1). It is easy to see that the function $f(x)=cx^3$ is a solution of (1.1). Thus, it is natural that (1.1) is called a cubic functional equation and every solution of (1.1) is said to be a cubic mapping. Jun et al. [14] introduced the Euler – Lagrange type cubic functional equation

$$f(ax+y)+f(ax-y)=af(x+y)+af(x-y)+2a(a^2-1)f(x) \tag{1.2}$$

for a fixed integer $a$ with $a\neq 0,\pm 1$, and they showed that functional equation (1.1) is equivalent to functional equation (1.2).

In the 1960s, S. Gahler [6, 7] introduced the concept of linear 2-normed spaces.

**Definition 1.1.** *Let $X$ be a linear space over $\mathbb{R}$ with $\dim X > 1$ and $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties*:

(a) $\|x, y\| = 0$ *if and only if $x$ and $y$ are linearly dependent,*

(b) $\|x, y\| = \|y, x\|$,

(c) $\|\lambda x, y\| = |\lambda| \|x, y\|$,

(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

*for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$.*

*Then the function $\|\cdot,\cdot\|$ is called a 2-norm on $X$ and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space. A standard example of a 2-normed space is $\mathbb{R}^2$ equipped with the 2-norm defined as $\|x, y\| =$ the area of the triangle having vertices 0, $x$ and $y$.*

It follows from (d), that $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ and $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$. Hence the functions $x \longrightarrow \|x, y\|$ are continuous functions of $X$ into $\mathbb{R}$ for each fixed $y \in X$.

**Definition 1.2.** *A sequence $\{x_n\}$ in a linear 2-normed space $X$ is called a Cauchy sequence if there are two points $y, z \in X$ such that $y$ and $z$ are linearly independent*,

$$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0,$$

*and*

$$\lim_{m,n\to\infty} \|x_m - x_n, z\| = 0.$$

**Definition 1.3.** *A sequence $\{x_n\}$ in a linear 2-normed space $X$ is called a convergent sequence if there is an $x \in X$ such that*

$$\lim_{m,n\to\infty} \|x_n - x, y\| = 0$$

*for all $y \in X$. If $\{x_n\}$ converges to $x$, write $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and call $x$ the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.*

**Definition 1.4.** *A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.*

**Lemma 1.1** [18]. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*

**Lemma 1.2** [18]. *For a convergent sequence $\{x_n\}$ in a linear 2-normed space $X$,*

$$\lim_{n\to\infty} \|x_n, y\| = \left\|\lim_{n\to\infty} x_n, y\right\|$$

*for all $y \in X$.*

In [18] W. G Park investigated approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces. The superstability of the Cauchy functional inequality and the Cauchy – Jensen functional inequality in 2-Banach spaces under some conditions were investigated by C. Park in [17].

In this paper, we deal with the next general cubic functional equation

$$f(\alpha x + y) + f(\alpha x - y) + f(x + \alpha y) - f(x - \alpha y) =$$
$$= 2\alpha f(x + y) + 2\alpha(\alpha^2 - 1)[f(x) + f(y)] \tag{1.3}$$

with $\alpha \in \mathbb{N},\ \alpha \neq 1$.

It is easy to see that the function $f(x) = ax^3$ is a solution of functional equation (1.3). We will prove the generalized Hyers – Ulam stability of equation (1.3) in 2-Banach spaces.

Let $X$ and $Y$ be two linear spaces. For convenience, we use the following abbreviation for a given function $f\colon X \to Y$:

$$D_\alpha f(x,y) := f(\alpha x + y) + f(\alpha x - y) + f(x + \alpha y) -$$
$$- f(x - \alpha y) - 2\alpha f(x + y) - 2\alpha(\alpha^2 - 1)[f(x) + f(y)]$$

for all $x, y \in X$. We need the following two lemmas.

**Lemma 1.3** [5]. *Let $X$ and $Y$ be two linear spaces. If a mapping $f\colon X \to Y$ satisfies the functional equation*

$$f(x + \alpha y) - f(x - \alpha y) = \alpha[f(x + y) - f(x - y)] + 2\alpha(\alpha^2 - 1)f(y)$$

*for all $x, y \in X$, then $f$ is cubic.*

**Lemma 1.4.** *Let $X$ and $Y$ be two linear spaces. If a mapping $f\colon X \to Y$ satisfies* (1.3) *for all $x, y \in X$, then $f$ is cubic.*

***Proof.*** Replacing $(x,y)$ with $(0,0)$ in (1.3), we get $f(0) = 0$. Replacing $(x,y)$ with $(x,0)$ in (1.3), we have

$$f(\alpha x) = \alpha^3 f(x) \tag{1.4}$$

for all $x \in X$. By setting $x = 0$ and using (1.4), we obtain $f(-y) = -f(y)$ for all $y \in X$, that is $f$ is odd. Replacing $(x,y)$ with $(x,-y)$ in (1.3) and using oddness of $f$, we get

$$f(\alpha x - y) + f(\alpha x + y) + f(x - \alpha y) - f(x + \alpha y) = 2\alpha f(x - y) + 2\alpha(\alpha^2 - 1)[f(x) - f(y)] \tag{1.5}$$

for all $x, y \in X$. It follows from (1.3) and (1.5) that

$$f(x + \alpha y) - f(x - \alpha y) = \alpha[f(x + y) - f(x - y)] + 2\alpha(\alpha^2 - 1)f(y)$$

for all $x, y \in X$. It follows from Lemma 1.3 that $f$ is cubic.

**2. Approximate cubic mappings.** Throughout this section, let $X$ be a normed linear space, $Y$ be a 2-Banach space and $\alpha \in \mathbb{N},\ \alpha \neq 1$.

**Theorem 2.1.** *Let $\varphi\colon X \times X \times X \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{\alpha^{3n}} \varphi(\alpha^n x, \alpha^n y, z) = 0 \tag{2.1}$$

*for all $x, y, z \in X$. Suppose that $f\colon X \longrightarrow Y$ is mapping with $f(0) = 0$,*

$$\|D_\alpha f(x,y), z\| \leq \varphi(x,y,z), \tag{2.2}$$

*and*

$$\widetilde{\varphi}(x,z) =: \sum_{n=0}^{\infty} \frac{1}{\alpha^{3i}} \varphi(\alpha^i x, 0, z) < \infty \tag{2.3}$$

*exists for all $x, y, z \in X$. Then there exists a unique cubic mapping $C\colon X \longrightarrow Y$ such that*

$$\|f(x) - C(x), z\| \leq \frac{1}{2\alpha^3}\widetilde{\varphi}(x, z) \tag{2.4}$$

*for all $x, z \in X$.*

***Proof.*** Setting $y = 0$ in (2.2), we have

$$\|f(\alpha x) - \alpha^3 f(x), z\| \leq \frac{1}{2}\varphi(x, 0, z) \tag{2.5}$$

for all $x, z \in X$. Replacing $x$ with $\alpha^n x$ in (2.5) and dividing both sides of (2.5) by $\alpha^{3n+3}$, we obtain

$$\left\|\frac{1}{\alpha^{3n+3}}f(\alpha^{n+1}x) - \frac{1}{\alpha^{3n}}f(\alpha^n x), z\right\| \leq \frac{1}{2\alpha^{3n+3}}\varphi(\alpha^n x, 0, z) \tag{2.6}$$

for all $x, z \in X$ and all nonnegative integers $n$. Hence,

$$\left\|\frac{1}{\alpha^{3n+3}}f(\alpha^{n+1}x) - \frac{1}{\alpha^{3m}}f(\alpha^m x), z\right\| \leq \sum_{i=m}^{n}\left\|\frac{1}{\alpha^{3i+3}}f(\alpha^{i+1}x) - \frac{1}{\alpha^{3i}}f(\alpha^i x), z\right\| \leq$$

$$\leq \frac{1}{2\alpha^3}\sum_{i=m}^{n}\frac{1}{\alpha^{3i}}\varphi(\alpha^i x, 0, z) \tag{2.7}$$

for all $x, z \in X$ and all nonnegative integers $m$ and $n$ with $n \geq m$. Therefore, we conclude from (2.3) and (2.7) that the sequence $\left\{\frac{1}{\alpha^{3n}}f(\alpha^n x)\right\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete the sequence $\left\{\frac{1}{\alpha^{3n}}f(\alpha^n x)\right\}$ converges in $Y$ for all $x \in X$. So one can define the mapping $C\colon X \to Y$ by

$$C(x) := \lim_{n\to\infty}\frac{1}{\alpha^{3n}}f(\alpha^n x) \tag{2.8}$$

for all $x \in X$. That is

$$\lim_{n\to\infty}\left\|\frac{1}{\alpha^{3n}}f(\alpha^n x) - C(x), y\right\| = 0$$

for all $x, y \in X$. Letting $m = 0$ and passing to the limit $n \to \infty$ in (2.7), we get (2.4). Now, we show that $C\colon X \to Y$ is a cubic mapping. It follows from (2.1), (2.2), (2.8) and Lemma 1.2 that

$$\left\|D_\alpha C(x, y), z\right\| = \lim_{n\to\infty}\frac{1}{\alpha^{3n}}\|D_\alpha f(\alpha^{3n}x, \alpha^{3n}y), z\| \leq$$

$$\leq \lim_{n\to\infty}\frac{1}{\alpha^{3n}}\varphi(\alpha^n x, \alpha^n y, z) = 0$$

for all $x, y, z \in X$. By Lemma 1.1, $D_\alpha C(x, y) = 0$ for all $x, y \in X$. So by Lemma 1.4 the mapping $C\colon X \to Y$ is cubic.

To prove the uniqueness of $C$, let $C\colon X \to Y$ be another cubic mapping satisfying (2.4). Then

$$\|C(x) - C'(x), z\| = \lim_{n\to\infty}\frac{1}{\alpha^{3n}}\left\|f(\alpha^n x) - C'(\alpha^n x), z\right\| \leq$$

$$\leq \frac{1}{2\alpha^3}\lim_{n\to\infty}\frac{1}{\alpha^{3n}}\widetilde{\varphi}(\alpha^n x, z) = 0$$

for all $x, z \in X$. By Lemma 1.1, $C(x) - C'(x) = 0$ for all $x \in X$. So $C = C'$.

***Remark* 2.1.** We can formulate a similar theorem to Theorem 2.1 in which we can define the sequence $C(x) := \lim_{n\to\infty} \alpha^{3n} f\left(\frac{x}{\alpha^n}\right)$ under suitable assumption on the function $\varphi$.

***Corollary* 2.1.** *Let $\psi\colon [0,\infty) \to [0,\infty)$ be a function such that $\psi(0) = 0$ and*

(i) $\psi(ts) \le \psi(t)\psi(s)$,

(ii) $\psi(t) < t$ *for all* $t > 1$.

*Suppose that $f\colon X \longrightarrow Y$ is a mapping with $f(0) = 0$ and*

$$\|D_\alpha f(x,y), z\| \le \psi(\|x\|) + \psi(\|y\|) + \psi(\|z\|) \tag{2.9}$$

*for all $x, y, z \in X$. Then there exists a unique cubic mapping $C\colon X \to Y$ satisfying*

$$\|f(x) - C(x), z\| \le \frac{1}{2}\frac{\psi(\|x\|)}{\alpha^3 - \psi(\alpha)} + \frac{1}{2}\frac{\psi(\|z\|)}{\alpha^3 - 1} \tag{2.10}$$

*for all $x, z \in X$.*

***Proof.*** Let

$$\varphi(x,y,z) = \psi(\|x\|) + \psi(\|y\|) + \psi(\|z\|)$$

for all $x, y, z \in X$. It follows from (i) that $\psi(\alpha^n) \le \big(\psi(\alpha)\big)^n$ and

$$\varphi(\alpha^n x, \alpha^n y, z) \le \big(\psi(\alpha)\big)^n \big(\psi(\|x\|) + \psi(\|y\|)\big) + \psi(\|z\|).$$

By using Theorem 2.1, we obtain (2.10).

***Corollary* 2.2.** *Let $q$ be a nonnegative real number such that $q < 3$ and $H\colon [0,\infty) \times [0,\infty) \to \to [0,\infty)$ be a homogeneous function of degree $q$. Suppose that $f\colon X \longrightarrow Y$ is a mapping with $f(0) = 0$ and*

$$\|D_\alpha f(x,y), z\| \le H(\|x\|, \|y\|) + \|z\|$$

*for all $x, y, z \in X$. Then there exists a unique cubic mapping $C\colon X \to Y$ such that*

$$\|f(x) - C(x), z\| \le \frac{1}{2}\frac{H(\|x\|, 0) + \|z\|}{\alpha^3 - q^3} \tag{2.11}$$

*for all $x \in X$.*

***Proof.*** Let

$$\varphi(x,y,z) = H(\|x\|, \|y\|) + \|z\|$$

for all $x, y, z \in X$. By using Theorem 2.1, we obtain (2.11).

***Corollary* 2.3.** *Let $q$ be a nonnegative real number such that $q < 3$ and $H\colon [0,\infty) \times [0,\infty) \to \to [0,\infty)$ be a homogeneous function of degree $q$. Suppose that $f\colon X \longrightarrow Y$ is a mapping with $f(0) = 0$ and*

$$\|D_\alpha f(x,y), z\| \le H(\|x\|, \|y\|)\|z\|$$

*for all $x, y, z \in X$. Then there exists a unique cubic mapping $C\colon X \to Y$ such that*

$$\|f(x) - C(x), z\| \le \frac{1}{2}\frac{H(\|x\|, 0)\|z\|}{\alpha^3 - q^3} \tag{2.12}$$

*for all $x, z \in X$.*

***Proof.*** Let

$$\varphi(x, y, z) = H(\|x\|, \|y\|)\|z\|$$

for all $x, y, z \in X$. By using Theorem 2.1, we obtain (2.12).

***Corollary* 2.4.** *Let $p$ be a nonnegative real number such that $p < 3$. Suppose that $f : X \longrightarrow Y$ is a mapping with $f(0) = 0$ and*

$$\|D_\alpha f(x, y), z\| \leq \|x\|^p + \|y\|^p + \|z\|$$

*for all $x, y, z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(x) - C(x), z\| \leq \frac{1}{2} \frac{\|x\|^p + \|z\|}{\alpha^3 - q^3}$$

*for all $x, z \in X$.*

***Corollary* 2.5.** *Let $r$, $s$ be nonnegative real numbers such that $r + s < 3$. Suppose that $f : X \longrightarrow Y$ is a mapping with $f(0) = 0$ and*

$$\|D_\alpha f(x, y), z\| \leq \|x\|^r \|y\|^r \|z\|^p$$

*for all $x, y, z \in X$. Then $f$ is cubic.*

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