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A CONSTRUCTION OF REGULAR SEMIGROUPS WITH QUASIIDEAL REGULAR $*$-TRANSVERSALS*

ПОБУДОВА РЕГУЛЯРНИХ НАПІВГРУП ІЗ КВАЗІІДЕАЛЬНИМИ РЕГУЛЯРНИМИ $*$-ТРАНВЕРСАЛЯМИ

Let S be a semigroup and let "$*$" be a unary operation on S satisfying the following identities:

$$xx^*x = x, \qquad x^*xx^* = x^*, \qquad x^{***} = x^*, \qquad (xy^*)^* = y^{**}x^*, \qquad (x^*y)^* = y^*x^{**}.$$

Then $S^* = \{x^* \mid x \in S\}$ is called a *regular $*$-transversal* of S in the literatures. We propose a method for the construction of regular semigroups with quasiideal regular $*$-transversals based on the use of fundamental regular semigroups and regular $*$-semigroups.

Нехай S — напівгрупа, а "$*$" — унарна операція на S, що задовольняє такі тотожності:

$$xx^*x = x, \qquad x^*xx^* = x^*, \qquad x^{***} = x^*, \qquad (xy^*)^* = y^{**}x^*, \qquad (x^*y)^* = y^*x^{**}.$$

Тоді $S^* = \{x^* \mid x \in S\}$ має в літературі назву *регулярної $*$-трансверсалі* S. Запропоновано новий метод побудови регулярних напівгруп з квазіідеальними регулярними $*$-трансверсалями з використанням фундаментальних регулярних напівгруп та регулярних $*$-напівгруп.

1. Introduction. Let S be a semigroup. We denote the set of all idempotents of S by $E(S)$ and the set of all inverses of $x \in S$ by $V(x)$. Recall that

$$V(x) = \{a \in S \mid xax = x, \ axa = a\}$$

for any $x \in S$. A semigroup S is called *regular* if $V(x) \neq \varnothing$ for any $x \in S$, and a regular semigroup S is called *inverse* if $E(S)$ is a commutative subsemigroup of S, or equivalently, the cardinality of $V(x)$ is equal to 1 for any x in S.

Recall from Petrich and Reilly [11] that *a unary semigroup* is a (2,1)-algebra $(S, \cdot, ^*)$ where $(S, \cdot)$, is a semigroup and the mapping $a \mapsto a^*$ is a unary operation on S. For brevity, we denote $(S, \cdot, ^*)$ by $(S, \ ^*)$. It is well known that a regular semigroup S is inverse if and only if there exists a unary operation "$*$" on S satisfying the following identities:

$$xx^*x = x, \qquad (x^*)^* = x, \qquad (xy)^* = y^*x^*, \qquad xx^*yy^* = yy^*xx^*. \tag{1.1}$$

Thus, inverse semigroups can be regarded as a class of unary semigroups.

Inspired by the above identity (1.1), *regular $*$-semigroups* were introduced in [10]. Recall that a unary semigroup $(S, \ ^*)$ is called *a regular $*$-semigroup* if the following identities are satisfied:

$$xx^*x = x, \qquad (x^*)^* = x, \qquad (xy)^* = y^*x^*. \tag{1.2}$$

Obviously, the class of regular $*$-semigroups forms a class of unary semigroups and contains the class of inverse semigroups as a subclass. Regular $*$-semigroups are investigated in many papers (see, for example, [5, 6, 10, 18, 19]).

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On the other hand, Blyth and McFadden [1] introduced the concept of inverse transversals for regular semigroups. A subsemigroup S° of a semigroup S is called an *inverse transversal* of S if $V(x) \cap S^\circ$ contains one element exactly for all $x \in S$. Clearly, in this case, S° is an inverse subsemigroup of S. From the remarks following Theorem 2 in Tang [12] and Theorem 4.8 in Tang [13], we can deduce easily that a regular semigroup S contains an inverse transversal if and only if there exists a unary operation "$*$" on S satisfying the following identities:

$$\begin{gathered} xx^*x = x, \qquad x^*xx^* = x^*, \qquad x^{***} = x^*, \qquad (x^*y)^* = y^*x^{**}, \\ (xy^*)^* = y^{**}x^*, \qquad x^*x^{**}y^*y^{**} = y^*y^{**}x^*x^{**}. \end{gathered} \tag{1.3}$$

In this case, $S^\circ = \{x^* \mid x \in S\}$ is an inverse transversal of S. Therefore, the class of regular semigroups with inverse transversals is a class of unary semigroups which also contains the class of inverse semigroups as a subclass. Inverse transversals of regular semigroups are studied extensively (see, for example, [1 – 3, 12, 13]).

Now, let $(S, {}^*)$ be a unary semigroup and the unary operation "$*$" satisfy the following identities:

$$xx^*x = x, \qquad x^*xx^* = x^*, \qquad x^{***} = x^*, \qquad (xy^*)^* = y^{**}x^*, \qquad (x^*y)^* = y^*x^{**}. \tag{1.4}$$

Then $S^* = \{x^* \mid x \in S\}$ is called a *regular $*$-transversal* of S from Li [8]. Clearly, $(S^*, {}^*)$ is a regular $*$-semigroup in this case. Moreover, combining the facts (1.2) and (1.3), we can see that regular semigroups having regular $*$-transversals are generalizations of regular $*$-semigroups and regular semigroups with inverse transversals.

Regular $*$-transversals have received serious attention in the literatures (see, e.g., [7 – 9, 14 – 16]). Recently, the author initiated the investigations of regular semigroups with regular $*$-transversals by *fundamental approaches* in Wang [17] in which fundamental regular semigroups with a quasiideal regular $*$-transversal are constructed and a fundamental representation of any regular semigroup with quasiideal regular $*$-transversals is obtained. Recall from Howie [4] that a semigroup is *fundamental* if its maximum idempotent-separating congruence is the identity congruence.

In this paper, we shall continue to study regular semigroups with regular $*$-transversals by fundamental approaches. After giving some necessary preliminaries, we give a construction method of regular semigroups with quasiideal regular $*$-transversals by using regular $*$-semigroups and fundamental regular semigroups constructed in Wang [17].

2. Preliminaries. Let $(S, {}^*)$ be a regular $*$-semigroup. Then we write $(S, {}^*) \in \mathbf{r}$ and $F_S = \{e \in E(S) \mid e^* = e\}$, and call F_S the set of projections of $(S, {}^*)$. It is easy to see that $F_S = \{xx^* \mid x \in S\} = \{x^*x \mid x \in S\}$. On regular $*$-semigroups, we have the following basic results.

Lemma 2.1 [10, 18]. *Let $(S, {}^*) \in \mathbf{r}$. Then*

(1) $(\forall e, f \in F_S)\ ef \in F_S \Longrightarrow ef = fe \in F_S$;

(2) $(\forall x \in S)\ x \in E(S) \Longleftrightarrow x^* \in E(S)$;

(3) $(F_S)^2 \subseteq E(S)$ *and* $xF_Sx^*, x^*F_Sx \subseteq F_S$ *for all* $x \in S$.

Now, let $(S, {}^*)$ be a unary semigroup and S^* be a regular $*$-transversal of S. Then we write $(S, {}^*) \in \mathbf{rt}$. Thus, $(S^*, {}^*) \in \mathbf{r}$ if $(S, {}^*) \in \mathbf{rt}$. A *quasiideal* of a semigroup S is a subsemigroup T of S which satisfies that $TST \subseteq T$. If $(S, {}^*) \in \mathbf{rt}$ and S^* is a quasiideal of S, then we write $(S, {}^*) \in \mathbf{qit}$. In this case, we denote $I_S = \{aa^* \mid a \in S\}$ and $\Lambda_S = \{a^*a \mid a \in S\}$.

Lemma 2.2 (Lemmas 4.1 and 4.2 in [8], Corollary 2.5 in [17]). *Let* $(S,\ ^*) \in \mathbf{qit}$. *Then*
(1) $I_S = \{e \in E(S) \mid e\mathcal{L}e^*\}$, $\Lambda_S = \{f \in E(S) \mid f\mathcal{R}f^*\}$ *and* $F_{S^*} = I_S \cap \Lambda_S$;
(2) $g^{**} = g^* \in F_{S^*}$ *for all* $g \in I_S \cup \Lambda_S$;
(3) $fg \in S^*$ *and so* $fg = (fg)^{**}$ *for all* $f \in \Lambda_S$ *and* $g \in I_S$;
(4) $(xy)^{**} = x^{**}x^*xyy^*y^{**}$ *for all* $x, y \in S$.
Now, let $(S,\ ^*) \in \mathbf{qit}$ and $e \in I_S, f \in \Lambda_S$. Denote

$$\langle e\rangle = eI_Se = \{eie \mid i \in I_S\}, \langle f\rangle = f\Lambda_Sf = \{f\lambda f \mid \lambda \in \Lambda_S\}.$$

Lemma 2.3 (Lemma 2.10 in [17]). *Let* $(S,\ ^*) \in \mathbf{qit}$, $a \in S$, $e \in I_S$, $f \in \Lambda_S$ *and* $p \in F_{S^*}$. *Then* $\langle p\rangle \subseteq F_{S^*}$ *and*
(1) $\langle e\rangle = eF_{S^*}e^* = \{x \in I_S \mid exe = x\}$ *and* $\langle f\rangle = f^*F_{S^*}f = \{x \in \Lambda_S \mid fxf = x\}$;
(2) $xyx \in \langle e\rangle$ *for all* $x, y \in \langle e\rangle$ *and* $xyx \in \langle f\rangle$ *for all* $x, y \in \langle f\rangle$;
(3) $a^*xa \in \langle a^*a\rangle$ *for all* $x \in \langle aa^*\rangle$ *and* $aya^* \in \langle aa^*\rangle$ *for all* $y \in \langle a^*a\rangle$.

Recall that a *semiband* is a semigroup which is generated by its idempotents. Let C be a regular semiband, $(C,\ ^*) \in \mathbf{qit}$ and use I and Λ to denote I_C and Λ_C, respectively. In view of Lemma 2.3, we have $xyx \in \langle e\rangle$ for all $x, y \in \langle e\rangle$ and $e \in I \cup \Lambda$.

Now, let $e, f \in I \cup \Lambda$. A bijection α from $\langle e\rangle$ onto $\langle f\rangle$ is called a *pre-isomorphism* if

$$\big(\forall x, y \in \langle e\rangle\big) \qquad (xyx)\alpha = (x\alpha)(y\alpha)(x\alpha). \tag{2.1}$$

Clearly, $e\alpha = f$ in the case. Moreover, we say that $\langle e\rangle$ and $\langle f\rangle$ are *pre-isomorphic* if there exists a pre-isomorphism from $\langle e\rangle$ onto $\langle f\rangle$. In this case, we write $\langle e\rangle \simeq \langle f\rangle$ and denote the set of all pre-isomorphisms from $\langle e\rangle$ onto $\langle f\rangle$ by $T_{e,f}$. The following result shows that pre-isomorphisms exist indeed. As usual, we use ι_M to denote the identity transformation on the nonempty set M.

Lemma 2.4 (Proposition 3.1 in [17]). *Let* C *be a regular semiband and* $(C,\ ^*) \in \mathbf{qit}$. *Define*

$$\pi_a : \langle aa^*\rangle \to \langle a^*a\rangle, \quad x \mapsto a^*xa.$$

Then $\pi_a \in T_{aa^*,a^*a}$. *Moreover, the inverse mapping of* π_a *is*

$$\pi_a^{-1} : \langle a^*a\rangle \to \langle aa^*\rangle, \quad y \mapsto aya^*$$

and $\pi_a^{-1} \in T_{a^*a,aa^*}$. *In particular, we have* $\pi_p = \iota_{\langle p\rangle} = \pi_p^{-1}$ *for any* $p \in F_{C^*}$.

On pre-isomorphisms in general, we have the following results.

Lemma 2.5 (Lemma 3.2 in [17]). *Let* $e \in I$, $f \in \Lambda$, $x \in \langle e\rangle$, $y \in \langle f\rangle$ *and* $\alpha \in T_{e,f}$. *Then*
(1) $\alpha^{-1} \in T_{f,e}$;
(2) $\langle x\rangle\alpha = \langle x\alpha\rangle$, $\langle y\rangle\alpha^{-1} = \langle y\alpha^{-1}\rangle$;
(3) $(x\alpha)^* = (x\alpha)f^*, x\alpha = (x\alpha)^*f$;
(4) $(y\alpha^{-1})^* = e^*(y\alpha^{-1}), y\alpha^{-1} = e(y\alpha^{-1})^*$.

Denote $\mathcal{U} = \{(e, f) \in I \times \Lambda | \langle e\rangle \simeq \langle f\rangle\}$ and define a multiplication "$\circ$" on the set

$$T_C = \bigcup_{(e,f)\in\mathcal{U}} T_{e,f}$$

as follows: for $\alpha \in T_{e,f}$ and $\beta \in T_{g,h}$,

$$\alpha \circ \beta = \alpha\pi^{-1}_{g(fg)^*f}\beta, \qquad \alpha^* = \pi_f\alpha^{-1}\pi_e,$$

where the composition is the that in the symmetric inverse semigroup on the set $I \cup \Lambda$.

Lemma 2.6 (Lemma 3.3, Theorem 3.5, Corollary 3.9 and Theorem 3.10 in [17]). *With the above notations, we have the following results*:

(1) $(T_C, \ ^*) \in \mathbf{qit}$, T_C *is fundamental and* $T_C^* = \{\alpha \in T_C | \alpha \in T_{p,q}, p, q \in F_{C^*}\}$;

(2) *if* $\alpha, \beta \in T_C$ *and* $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$, *then* $\alpha \circ \beta \in T_{j,k}$, *where* $j = (fg(fg)^*f)\alpha^{-1}$, $k = (g(fg)^*fg)\beta$;

(3) $\alpha^* \in T_{f^*,e^*}$, $\alpha^{**} \in T_{e^*,f^*}$, $\alpha \circ \alpha^* = \pi_e$, $\alpha^* \circ \alpha = \pi_f$.

In the rest of this section, we let $(S, \ ^*) \in \mathbf{qit}$ and C be the semiband generated by $E(S)$. Then we have the following lemma.

Lemma 2.7 (Lemma 4.1 in [17]). *Let* $(S, \ ^*) \in \mathbf{qit}$. *Then* $(C, \ ^*) \in \mathbf{qit}$. *In this case*, $C^* = C \cap S^*$, $I_S = I_C$, $\Lambda_S = \Lambda_C$ *and* $F_{S^*} = F_{C^*}$.

By Lemmas 2.6 and 2.7, we can construct a fundamental regular semigroup T_C with a quasiideal regular $*$-transversal T_C^*. For $a \in S$, by Lemma 2.3 (3), we can define

$$\rho_a \colon \langle aa^* \rangle \to \langle a^*a \rangle, \quad x \mapsto a^*xa.$$

Then the inverse mapping ρ_a^{-1} of ρ_a is

$$\rho_a^{-1} \colon \langle a^*a \rangle \to \langle aa^* \rangle, \quad y \mapsto aya^*.$$

Observe that $\rho_a = \pi_a$ for every $a \in C$ where π_a is defined as in Lemma 2.4. Moreover, we also need the following result.

Lemma 2.8 (Lemma 4.2 and Theorem 4.3 in [17]). *Let* $a, b \in S$. *Then*

(1) $\rho_a \in T_{aa^*,a^*a}$ *and* $\rho_a^{-1} \in T_{a^*a,aa^*}$;

(2) $\rho_a \circ \rho_b = \rho_{ab}$ *and* $(\rho_a)^* = \rho_{a^*}$ *in* T_C.

3. Main result. In this section, a structure theorem of regular semigroups with a quasiideal regular $*$-transversal is obtained by using a fundamental regular semigroup and a regular $*$-semigroup.

Let C be a semiband, $(C, \ ^*) \in \mathbf{qit}$ and $(R, \ ^*) \in \mathbf{r}$. Assume that $(C^*, \ ^*)$ is a common (2,1)-subalgebra of $(R, \ ^*)$ and $(C, \ ^*)$ such that $R \cap C = C^*$ and $F_R = F_{C^*}$. By Lemma 2.6 (1), we have $(T_C, \ ^*) \in \mathbf{qit}$ and T_C is fundamental.

Now, let $a \in R$. Then $a^*xa \in a^*aF_Ra^*a$ for every x in $aa^*F_Raa^*$ by Lemma 2.1 (3). Therefore, we can define a mapping as follows:

$$\lambda_a \colon aa^*F_Raa^* \to a^*aF_Ra^*a, \quad x \mapsto a^*xa.$$

It can be proved easily that $xyx \in aa^*F_Raa^*$ and

$$(xyx)\lambda_a = (x\lambda_a)(y\lambda_a)(x\lambda_a) \tag{3.1}$$

for all $x, y \in aa^*F_Raa^*$.

Lemma 3.1. *With the above notations, the following statements hold for all* $a, b \in R$:

(1) $\lambda_a \in T_{aa^*,a^*a} \subseteq T_C^*$. *In particular, if* $a \in C^*$, *then* $\lambda_a = \pi_a$ *where* π_a *is defined as in Lemma* 2.4;

(2) $(\lambda_a)^* = \lambda_{a^*}$ *and* $\lambda_a \circ \lambda_b = \lambda_{ab}$ *in* T_C^* *where ab is taken in* R.

Proof. (1) Since $F_R = F_{C^*}$ and $aa^*, a^*a \in F_R$, we have $aa^*, a^*a \in F_{C^*} = I \cap \Lambda$ and $(aa^*)^* = aa^*$, $(a^*a)^* = a^*a$ in C. By Lemma 2.3 (1), we obtain

$$\langle aa^* \rangle = aa^* F_{C^*} aa^* = aa^* F_R aa^*, \qquad \langle a^*a \rangle = a^*a F_{C^*} a^*a = a^*a F_R a^*a$$

in C. This implies that $\operatorname{dom} \lambda_a = \langle aa^* \rangle$ and $\operatorname{ran} \lambda_a = \langle a^*a \rangle$. Moreover, it is easy to see that λ_a is bijective. In fact, the inverse mapping λ_a^{-1} of λ_a is

$$\lambda_{a^*} : \langle a^*a \rangle \to \langle aa^* \rangle, \quad y \mapsto aya^*.$$

Combining identities (2.1) and (3.1), we can obtain that $\lambda_a \in T_{aa^*, a^*a}$. If $a \in C^*$, then the product a^*xa can be taken both in C and in R for all $x \in \langle aa^* \rangle$ and thus we have $\lambda_a = \pi_a$ where π_a is defined as in Lemma 2.4.

(2) Since $aa^*, a^*a \in F_R = F_{C^*}$, we have $\pi_{a^*a} = \iota_{\langle a^*a \rangle}$ and $\pi_{aa^*} = \iota_{\langle aa^* \rangle}$ by Lemma 2.4. This implies that

$$(\lambda_a)^* = \pi_{a^*a} \lambda_a^{-1} \pi_{aa^*} = \iota_{\langle a^*a \rangle} \lambda_a^{-1} \iota_{\langle aa^* \rangle} = \lambda_a^{-1} = \lambda_{a^*}$$

in T_C. On the other hand, since a^*a, $bb^* \in F_R = F_{C^*}$, we have

$$\lambda_a \circ \lambda_b = \lambda_a \pi^{-1}_{bb^*(a^*abb^*)^*a^*a} \lambda_b = \lambda_a \pi^{-1}_{bb^*bb^*a^*aa^*a} \lambda_b = \lambda_a \pi^{-1}_{bb^*a^*a} \lambda_b = \lambda_a \lambda^{-1}_{bb^*a^*a} \lambda_b$$

by item (1) and the fact that $(bb^*)(a^*a) \in C^*$, and

$$(a^*abb^*a^*a)\lambda_a^{-1} = a(a^*abb^*a^*a)a^* = ab(ab)^*, (bb^*a^*abb^*)\lambda_b = b^*(bb^*a^*abb^*)b = (ab)^*ab$$

whence $\lambda_a \circ \lambda_b \in T_{ab(ab)^*, (ab)^*ab}$ by Lemma 2.6 (2). Moreover, it follows that

$$x(\lambda_a \circ \lambda_b) = x(\lambda_a \lambda^{-1}_{bb^*a^*a} \lambda_b) = b^*(bb^*a^*a(a^*xa)(bb^*a^*a)^*)b = b^*a^*xab = (ab)^*xab = x\lambda_{ab}$$

for all $x \in \operatorname{dom}(\lambda_a \circ \lambda_b) = \operatorname{dom} \lambda_{ab}$. This implies that $\lambda_a \circ \lambda_b = \lambda_{ab}$.

Lemma 3.1 is proved.

Now, let

$$W = \{(\alpha, a) \in T_C \times R \mid \alpha^{**} = \lambda_a\}$$

and define a binary operation and a unary operation "$*$" as follows: for $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$ and $(\alpha, a), (\beta, b) \in W$,

$$(\alpha, a)(\beta, b) = \big(\alpha \circ \beta, a(fg)b\big), \qquad (\alpha, a)^* = (\alpha^*, a^*),$$

where $fg \in C^* \subseteq R$ by Lemma 2.2 (3) and the product $a(fg)b$ is taken in R.

Theorem 3.1. *With the above notations*, $(W, {}^*) \in \mathbf{qit}$. *Conversely, any* $(S, {}^*) \in \mathbf{qit}$ *can be constructed in this way.*

Proof. The binary operation and the unary operation are well-defined. In fact, let $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$ and $(\alpha, a), (\beta, b) \in W$. Then $\alpha^{**} = \lambda_a$ and $\beta^{**} = \lambda_b$. In view of Lemma 2.2 (4), we have

$$(\alpha \circ \beta)^{**} = \alpha^{**} \circ (\alpha^* \circ \alpha \circ \beta \circ \beta^*) \circ \beta^{**} = \lambda_a \circ (\pi_f \circ \pi_g) \circ \lambda_b = \lambda_a \circ \pi_{fg} \circ \lambda_b$$

by Lemmas 2.6 (3) and 2.8. Since $fg \in C \cap R = C^*$ by Lemma 2.2 (3), we have $\pi_{fg} = \lambda_{fg}$ by Lemma 3.1 (1), this shows that

$$(\alpha \circ \beta)^{**} = \lambda_a \circ \lambda_{fg} \circ \lambda_b = \lambda_{a(fg)b}$$

by Lemma 3.1 (2) and so

$$(\alpha, a)(\beta, b) = \big(\alpha \circ \beta, a(fg)b\big) \in W.$$

On the other hand, observe that $(\alpha^*)^{**} = (\alpha^{**})^* = (\lambda_a)^* = \lambda_{a^*}$ by Lemma 3.1 (2) again, it follows that $(\alpha, a)^* = (\alpha^*, a^*) \in W$.

The above binary operation is associative. In fact, let

$$\alpha \in T_{e,f}, \quad \beta \in T_{g,h}, \quad \gamma \in T_{s,t}, \quad \alpha \circ \beta \in T_{j,k}, \quad \beta \circ \gamma \in T_{p,q}, \quad (\alpha, a), (\beta, b), (\gamma, c) \in W,$$

where $k = (g(fg)^* fg)\beta$ by Lemma 2.6 (2). By Lemma 2.5 (3), we have $k = k^* h$ and so $ks = k^*(hs)$. Since $(\alpha \circ \beta, a(fg)b) \in W$ and $\alpha \circ \beta \in T_{j,k}$, it follows that

$$T_{j^*,k^*} \ni (\alpha \circ \beta)^{**} = \lambda_{a(fg)b} \in T_{a(fg)b(a(fg)b)^*,(a(fg)b)^* a(fg)b}$$

by Lemma 2.6 (3), whence $k^* = (a(fg)b)^* a(fg)b$. Thus, we get

$$(a(fg)b)(ks)c = (a(fg)b)k^*(hs)c = \big(a(fg)b\big) \cdot \big(a(fg)b\big)^* a(fg)b) \cdot (hs)c = a(fg)b(hs)c.$$

Dually, we can prove that $a(fp)(b(hs)c) = a(fg)b(hs)c$. Thus,

$$\big[(\alpha, a)(\beta, b)\big](\gamma, c) = \big((\alpha \circ \beta) \circ \gamma, (a(fg)b)(ks)c\big) =$$
$$= \big(\alpha \circ (\beta \circ \gamma), a(fp)(b(hs)c)\big) = (\alpha, a)\big[(\beta, b)(\gamma, c)\big].$$

Let $\alpha \in T_{e,f}, \beta \in T_{g,h}$ and $(\alpha, a), (\beta, b) \in W$. Then

$$\alpha^* \in T_{f^*,e^*}, \quad \beta^* \in T_{h^*,g^*} \quad T_{e^*,f^*} \ni \alpha^{**} = \lambda_a \in T_{aa^*,a^*a}, \quad T_{g^*,h^*} \ni \beta^{**} = \lambda_b \in T_{bb^*,b^*b} \quad (3.2)$$

by Lemmas 2.6 (3) and 3.1 (1). This implies that $e^* = aa^*$ and $f^* = a^*a$. Thus, we have

$$(\alpha, a)(\alpha, a)^*(\alpha, a) = (\alpha, a)(\alpha^*, a^*)(\alpha, a) = (\alpha \circ \alpha^* \circ \alpha, a(ff^*)a^*(e^*e)a) = (\alpha, a).$$

Similarly, we can see that $(\alpha, a)^*(\alpha, a)(\alpha, a)^* = (\alpha, a)^*$. On the other hand, observe that $(fh^*)^* = = h^{**}f^* = h^*f^*$ by (1.4), it follows that

$$\big[(\alpha, a)(\beta, b)^*\big]^* = \big[(\alpha, a)(\beta^*, b^*)\big]^* = \big((\alpha \circ \beta^*)^*, b^{**}(fh^*)^* a^*\big) =$$
$$= \big(\beta^{**} \circ \alpha^*, b^{**}(h^* f^*)a^*\big) = (\beta^{**}, b^{**})(\alpha^*, a^*) = (\beta, b)^{**}(\alpha, a)^*.$$

Similarly, we can see that $\big[(\alpha, a)^*(\beta, b)\big]^* = (\beta, b)^*(\alpha, a)^{**}$.

Recall that $T_C^* = \{\alpha \in T_C \mid \alpha \in T_{p,q}, p, q \in F_{C^*}\}$ by Lemma 2.6 (1). We assert that $W^* = = \{(\alpha, a) \in W | \alpha \in T_C^*\}$. Obviously, $W^* \subseteq \{(\alpha, a) \in W \mid \alpha \in T_C^*\}$. On the other hand, if $(\alpha, a) \in W$ and $\alpha \in T_C^*$, then $\alpha = \alpha^{**}, a^{**} = a$ and $(\alpha^*, a^*) \in W$. This implies that $(\alpha, a) = = (\alpha^*, a^*)^* \in W^*$. Thus, $\{(\alpha, a) \in W \mid \alpha \in T_C^*\} \subseteq W^*$. Now, let $(\alpha, a), (\gamma, c) \in W^*$ and $(\beta, b) \in W$. Since $(T_C, {}^*) \in \mathbf{qit}$, $\alpha \circ \beta \circ \gamma \in T_C^*$. This implies that $(\alpha, a)(\beta, b)(\gamma, c) \in W^*$ and so $(W, {}^*) \in \mathbf{qit}$.

Conversely, let $(S, {}^*) \in \mathbf{qit}$ and C be the semiband generated by $E(S)$. Then $(C, {}^*) \in \mathbf{qit}$, $(S^*, {}^*)$ is a regular $*$-semigroup, and $(C^*, {}^*)$ is a (2,1)-subalgebra of $(S^*, {}^*)$ and $(C, {}^*)$ such that $S^* \cap C = C^*$ and $F_{S^*} = F_{C^*}$ by Lemma 2.7. By the direct part, we have a semigroup

$$W = \{(\alpha, a) \in T_C \times S^* \mid \alpha^{**} = \lambda_a\}$$

and $(W, \ ^*) \in \mathbf{qit}$ with respect to

$$(\alpha, a)(\beta, b) = \big(\alpha \circ \beta, a(fg)b\big), \qquad (\alpha, a)^* = (\alpha^*, a^*),$$

where $(\alpha, a), (\beta, b) \in W$, $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$ and $fg \in C^*$ by Lemma 2.2 (3). Observe that $\lambda_a = \rho_a$ where ρ_a is defined as in the statements before Lemma 2.8 for all $a \in S^*$.

In what follows, we prove that

$$\psi \colon S \to W, \quad x \mapsto (\rho_x, x^{**})$$

is a unary isomorphism through the following steps where ρ_x is defined as in the statements before Lemma 2.8.

(1) By Lemma 2.8 (2), we have $(\rho_x)^* = \rho_{x^*}$ and so $(\rho_x)^{**} = \rho_{x^{**}} = \lambda_{x^{**}}$ and

$$(x\psi)^* = (\rho_x, x^{**})^* = ((\rho_x)^*, x^{***}) = (\rho_{x^*}, x^{***}) = x^*\psi.$$

This implies that ψ is well-defined and preserves the unary operation "$*$."

(2) Since $\rho_x \in T_{xx^*,x^*x}, \rho_y \in T_{yy^*,y^*y}$ by Lemma 2.8 (1), we have

$$(xy)\psi = (\rho_{xy}, (xy)^{**}) = (\rho_x \circ \rho_y, x^{**}(x^*xyy^*)y^{**}) = (\rho_x, x^{**})(\rho_y, y^{**})$$

by Lemma 2.2 (4) and Lemma 2.8 (2).

(3) If $x, y \in S$ and $(\rho_x, x^{**}) = (\rho_y, y^{**})$, then $\rho_x = \rho_y$, $x^{**} = y^{**}$. This implies that

$$T_{xx^*,x^*x} \ni \rho_x = \rho_y \in T_{yy^*,y^*y}$$

by Lemma 2.8 (1) and so

$$xx^* = yy^*, \qquad x^*x = y^*y, \qquad x^{**} = y^{**}.$$

Thus, $x = xx^*x^{**}x^*x = yy^*y^{**}y^*y = y$.

(4) If $(\alpha, a) \in W$ and $\alpha \in T_{e,f}$, then $\alpha^{**} = \lambda_a = \rho_a$ since $a \in S^*$. By Lemma 2.6 (3), Lemma 2.8 and the fact that $e, f \in C$, we obtain

$$\alpha = \alpha \circ \alpha^* \circ \alpha^{**} \circ \alpha^* \circ \alpha = \pi_e \circ \rho_a \circ \pi_f = \rho_e \circ \rho_a \circ \rho_f = \rho_{eaf}.$$

This shows that $(\alpha, a) = (\rho_{eaf}, a)$. Since $(\alpha, a) \in W$ and $\alpha \in T_{e,f}$, we have $a \in S^*$, $e^* = aa^*$ and $f^* = a^*a$ by (3.2). This implies that

$$(eaf)^{**} = (ea)^{**}((ea)^*eaff^*)f^{**} = e^*aa^*e^*eaff^* = a$$

by Lemma 2.2 (4) and item (1.4). Therefore $(eaf)\psi = (\rho_{eaf}, (eaf)^{**}) = (\alpha, a)$.

Theorem 3.1 is proved.

We end our paper by giving the following example.

***Example* 3.1.** Let S be a completely simple semigroup and H be an $\mathcal{H}$-class of S. Then H is a group and contains exactly one inverse of a for any $a \in S$. Denote the identity of H by e° and the unique inverse of a in H by a° for $a \in S$. Then H is an inverse transversal of S such that $HSH \subseteq H$ and so $(S, {}^*) \in \mathbf{qit}$ with the operation "$*$" defined by $a^* = a^\circ$ for any $a \in S$. Obviously, $H = S^*$ and $F_{S^*} = \{e^\circ\}$ in the case. Consider $C = \langle E(S) \rangle$. Then $(C, {}^*) \in \mathbf{qit}$, $C^* = H \cap C$ and $F_{C^*} = \{e^\circ\}$. Moreover,

$$I = I_C = I_S = \{e \in E(S) \mid e\mathcal{L}e^\circ\}, \qquad \Lambda = \Lambda_C = \Lambda_S = \{f \in E(S) \mid f\mathcal{R}e^\circ\}$$

by Lemmas 2.7 and 2.2 (1). For any $a \in S^* = H$, the mapping

$$\lambda_a \colon \langle aa^* \rangle = \{e^\circ\} \to \langle a^*a \rangle = \{e^\circ\}, \quad x \mapsto a^*xa$$

is always $\iota_{\{e^\circ\}}$. On the other hand, for any $e \in I$ and $f \in \Lambda$, we have $\langle e \rangle = \{e\}$ and $\langle f \rangle = \{f\}$. Denote

$$\sigma_{e,f} \colon \langle e \rangle \to \langle f \rangle, \qquad e \mapsto f, \quad e \in I, \quad f \in \Lambda.$$

Then $T_{e,f} = \{\sigma_{e,f}\}$ for all $e \in I$ and $f \in \Lambda$ and so $T_C = \{\sigma_{e,f} \mid e \in I,\ f \in \Lambda\}$. By Lemma 2.6 (2),

$$\sigma_{e,f} \circ \sigma_{g,h} \in T_{(fg(fg)^*f)\sigma_{e,f}^{-1},(g(fg)^*fg)\sigma_{g,h}} = T_{f\sigma_{e,f}^{-1},g\sigma_{g,h}} = T_{e,h}$$

for all $\sigma_{e,f}, \sigma_{g,h} \in T_C$. This implies that

$$\sigma_{e,f} \circ \sigma_{g,h} = \sigma_{e,h}, \qquad \sigma_{e,f}^* = \iota_{\{e^\circ\}}$$

for all $e, g \in I$ and $f, h \in \Lambda$. Thus, we can form the following semigroup:

$$W = \{(\alpha, a) \in T_C \times H \mid \alpha^{**} = \lambda_a\}$$

with the operation

$$(\sigma_{e,f}, a)(\sigma_{g,h}, b) = \big(\sigma_{e,h}, a(fg)b\big),$$

where $fg \in S^* = H$ by Lemma 2.2 (3). Observe that $\lambda_a = \iota_{\{e^\circ\}} = \sigma_{e,f}^{**}$ for all $a \in H$ and $e \in I$, $f \in \Lambda$, it follows that $W = T_C \times H$. It is routine to check that W is isomorphic to the semigroup $M = I \times H \times \Lambda$ with respect to the following binary operation:

$$(e, a, f)(g, b, h) = (e, a(fg)b, h).$$

By Theorem 3.1, S is isomorphic to M. However, M is just a Rees matrix semigroup over the group H. Thus, we obtain the well-known Rees constructions of completely simple semigroups by applying Theorem 3.1.

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