DOI: 10.37863/umzh.v73i3.195

UDC 517.9

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ON SOLVABILITY OF ONE CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS*

ПРО РОЗВ'ЯЗНІСТЬ ОДНОГО КЛАСУ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ТРЕТЬОГО ПОРЯДКУ

One-dimensional mixed problem for one class of third order partial differential equation with nonlinear right-hand side is considered. The concept of generalized solution for this problem is introduced. By the Fourier method, the problem of existence and uniqueness of generalized solution for this problem is reduced to the problem of solvability of the countable system of nonlinear integro-differential equations. Using Bellman's inequality, the uniqueness of generalized solution is proved. Under some conditions on initial functions and the right-hand side of the equation, the existence theorem for the generalized solution is proved using the method of successive approximations.

Розглянуто одновимірну мішану задачу для одного класу диференціальних рівнянь третього порядку з частинними похідними з нелінійною правою частиною. Введено поняття узагальненого розв'язку для цієї задачі. За допомогою методу Фур'є задачу існування та єдиності узагальненого розв'язку зведено до задачі розв'язності зліченної системи нелінійних інтегро-диференціальних рівнянь. З використанням нерівності Беллмана доведено єдиність узагальненого розв'язку. При деяких умовах на початкові функції та праву частину рівняння на основі методу послідовних ітерацій доведено теорему про існування узагальненого розв'язку.

1. Introduction. In the last century, there has been considerable interest in local and nonlocal boundary-value problems for partial differential equations with time and spatial variables. The theory and applications of local and nonlocal boundary-value problems for third order PDEs have been studied by many mathematicians (see [1 – 12]).

Many problems of elasticity theory such as the problem of longitudinal oscillations of non uniform viscoelastic rod, the problem of longitudinal impact of perfectly rigid body on non uniform finite-length viscoelastic rod with variable cross section, the problem of wave propagation in a viscoelastic body, etc. are reduced to the solution in the domain $D = (0,T) \times (0,\pi)$ (T is any positive number) to the mixed problem for the equation

$$u_{tt}(t,x) - \alpha u_{txx}(t,x) = F(u)(t,x), \tag{1.1}$$

with initial and boundary conditions

$$u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x), \quad 0 \le x \le \pi, \tag{1.2}$$

$$u(t,0) = 0, \qquad u(t,\pi) = 0, \quad 0 \le t \le T, \tag{1.3}$$

where $0 < \alpha$ is a fixed number, F is in general a given nonlinear operator, φ, ψ are the given functions from certain space of functions. Definition of the solution of problem (1.1) – (1.3) is given

* This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan (grant No. EIF-ETL-2020-2(36)-16/04/1-M04) and by the Scientific and Technological Research Council of Turkey (TUBITAK) with Azerbaijan National Academy of Sciences (project No. P19042020).

on the next section. Earlier in [13 – 15], the existence and uniqueness of the classical solution of the problem (1.1) – (1.3) for $t > 0$ and $0 < x < \pi$ as well as its behavior for $t \to +\infty$ have been considered. In [16 – 18], existence and uniqueness theorems for classical solution of this problem have been proved under some conditions on problem data and the properties of the solution have been explored. A priori estimates which allow obtaining the conditions for the existence, uniqueness and asymptotic stability of the solution have been treated in [19, 20]. Similar matters have been studied in [21 – 24]. The existence and uniqueness of classical, generalized and almost everywhere solutions of (1.1) – (1.3) have been considered in [25 – 27]. Using the method of successive approximations and the principles of Krasnoselski, Schauder and Leray – Schauder, in [27] proved local and global existence and uniqueness theorems for classical, generalized and almost everywhere solutions of the problem (1.1) – (1.3).

Fourier's well-known method of separation of variables is applicable also for solving the problem (1.1) – (1.3). Among the works devoted to justification of the Fourier method for solving such problems we can mention the works [27 – 31]. Note that in [31] introduced the Banach spaces $B^{2,1}_{2,2,T}$ of the functions $u(t,x)$ of the form

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t) \sin nx, \tag{1.4}$$

considered in the set $\overline{D}$, with $u_n(t) \in C^{(2)}\big([0,T]\big)$, equipped with the finite norm

$$\|u\|_{B^{2,1}_{2,2,T}} = \left(\sum_{n=1}^{\infty} \left(n^2 \max_{0 \le t \le T} \big|u_n(t)\big| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \left(n \max_{0 \le t \le T} \big|u_n'(t)\big| \right)^2 \right)^{\frac{1}{2}}.$$

The generalized solutions of (1.1) – (1.3) are considered in the space $B^{2,1}_{2,2,T}$. Using Fourier method, they reduced the problem (1.1) – (1.3) to the countable system of nonlinear integro-differential equations, which, in turn, was reduced to finding a fixed point of some nonlinear operator in corresponding space. In [32, 33], the continuous dependence (in some sense) of solution of the problem (1.1) – (1.3) on F, φ, ψ has been considered. In [34, 35], the estimates for the classical, generalized and almost everywhere solutions of (1.1) – (1.3) have been treated. In works [1, 12, 14, 16 – 19] using the operator approach, it is established stability estimates for the solution of the boundary-value problem for third order partial differential equations.

In this paper, we consider the problem (1.1) – (1.3) in the Banach space $B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ of functions of the form (1.4) with the coefficients $u_n(t) \in C^{(2)}\big([0,T]\big)$, equipped with the norm

$$\|u\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} = \left(\left(\sum_{n=1}^{\infty} \left(n^{1+\frac{2}{q}} \max_{0 \le t \le T} \big|u_n(t)\big| \right)^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} \left(n^{\frac{2}{q}} \max_{0 \le t \le T} \big|u_n'(t)\big| \right)^p \right)^{\frac{1}{p}} \right),$$

where $p > 2$ and p, q are the numbers conjugate of each other. We prove the existence and uniqueness theorems for the generalized solution of (1.1) – (1.3). To justify our method, we use the analog of Littlewood – Paley theorem for vector-valued coefficients of decomposition of a function with respect to the system $\{\sin nx\}$ in the sense of b-basis [36 – 38]. The matter of existence of the solution is reduced to finding the coefficients of the sought solution in concrete Banach space of sequences of functions.

2. Some basic notations and auxiliary facts. In this section, we introduce some that we need notations and the definition of the generalized solution of the problem (1.1)–(1.3). By $L_{p,p-2}(0,\pi)$, $p \geq 2$, we denote the Banach space of functions $f(x) \in L_p(0,\pi)$, with

$$\{f_n\}_{n\in N} \in l_{p,p-2}, \qquad f_n = \frac{2}{\pi}\int\limits_0^\pi f(x)\sin nx dx,$$

equipped with the norm

$$\|f\|_{L_{p,p-2}(0,\pi)} = \left(\sum_{n=1}^\infty n^{p-2}\,|f_n|^p\right)^{\frac{1}{p}},$$

where $l_{p,p-2}$ the Banach space of sequences of scalars $\lambda = \{\lambda_n\}_{n\in N}$, with the norm $\|\lambda\|_{l_{p,p-2}} = \left(\sum_{n=1}^\infty n^{p-2}|\lambda_n|^p\right)^{\frac{1}{p}}$, $L_{p,p-2}\big([0,T],L_p(0,\pi)\big)$, $p \geq 2$, will denote the set of vector-functions $u(t)\colon [0,T] \to L_p(0,\pi)$, such that

$$\|u\|^p_{L_{p,p-2}\left([0,T],L_p(0,\pi)\right)} = \sum_{n=1}^\infty n^{p-2}\int\limits_0^T \big|u_n(t)\big|^p dt < +\infty,$$

where $u_n(t) = \dfrac{2}{\pi}\displaystyle\int_0^\pi u(t,x)\sin nx dx$. The space $L_{p,p-2}\big([0,T],L_p(0,\pi)\big)$ is a Banach space with respect to the norm $\|u\|_{L_{p,p-2}\left([0,T],L_p(0,\pi)\right)}$.

Let X be some Banach space. Denote by the $W_p^{(1)}([a,b],X)$ set of vector-functions $u\colon [a,b] \to X$ such that for all $t \in [0,T]$ there exist in X a strong limit $\lim_{\Delta t\to 0}\dfrac{u(t+\Delta t)-u(t)}{\Delta t} = u'(t)$ and it holds

$$\|u\|^p_{W_p^{(1)}([a,b],X)} = \int\limits_a^b \|u(t)\|^p_X dt + \int\limits_a^b \big\|u'(t)\big\|^p_X dt < +\infty.$$

The space $W_p^{(1)}([a,b],X)$ is a Banach space with respect to the norm $\|u\|_{W_p^{(1)}([a,b],X)}$.

In the sequel, the elements of the space $W_1^{(1)}\big([0,T],L_q(0,\pi)\big)$ will be denoted in the form $u = u(t,x)$, where $t \in (0,T)$, $x \in (0,\pi)$.

Before giving the definition for the generalized solution of the problem (1.1)–(1.3), we make some remarks.

***Remark* 2.1.** If $p > 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then the space $B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ is continuously embedded in the space $B^{2,1}_{2,2,T}$ and the following relation is true:

$$\|u\|_{B^{2,1}_{2,2,T}} \leq \left(\frac{\pi^2}{6}\right)^{\frac{2-q}{2q}} \|u\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}}.$$

In fact, for $u(t,x)\in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$, we have

$$\|u\|_{B^{2,1}_{2,2,T}}=\left(\sum_{n=1}^{\infty}\left(n^2\max_{0\le t\le T}\left|u_n(t)\right|\right)^2\right)^{\frac12}+\left(\sum_{n=1}^{\infty}\left(n\max_{0\le t\le T}\left|u_n'(t)\right|\right)^2\right)^{\frac12}=$$

$$=\left(\sum_{n=1}^{\infty}\left(n^{1+\frac{2}{q}}\max_{0\le t\le T}\left|u_n(t)\right|\right)^2 n^{\frac{2(q-2)}{q}}\right)^{\frac12}+\left(\sum_{n=1}^{\infty}\left(n^{\frac{2}{q}}\max_{0\le t\le T}\left|u_n'(t)\right|\right)^2 n^{\frac{2(q-2)}{q}}\right)^{\frac12}.$$

Applying Hölder's inequality with index $\dfrac{p}{2}$ $\left(\text{whose conjugate is } \dfrac{q}{2-q}\right)$ to every sum in the last equality, we obtain

$$\|u\|_{B^{2,1}_{2,2,T}}\le\left(\sum_{n=1}^{\infty}\left(n^{1+\frac{2}{q}}\max_{0\le t\le T}\left|u_n(t)\right|\right)^p\right)^{\frac1p}\left(\sum_{n=1}^{\infty}n^{-2}\right)^{\frac{2-q}{2q}}+$$

$$+\left(\sum_{n=1}^{\infty}\left(n^{\frac{2}{q}}\max_{0\le t\le T}\left|u_n'(t)\right|\right)^p\right)^{\frac1p}\left(\sum_{n=1}^{\infty}n^{-2}\right)^{\frac{2-q}{2q}}=$$

$$=\left(\sum_{n=1}^{\infty}n^{-2}\right)^{\frac{2-q}{2q}}\left(\left(\sum_{n=1}^{\infty}\left(n^{1+\frac{2}{q}}\max_{0\le t\le T}\left|u_n(t)\right|\right)^p\right)^{\frac1p}+\left(\sum_{n=1}^{\infty}\left(n^{\frac{2}{q}}\max_{0\le t\le T}\left|u_n'(t)\right|\right)^p\right)^{\frac1p}\right)=$$

$$=\left(\frac{\pi^2}{6}\right)^{\frac{2-q}{2q}}\|u\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}}.$$

Obviously, if $u(t,x)\in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$, then the convergence of the series $\sum_{n=1}^{\infty}n\|u_n\|_{C[0,T]}$ and $\sum_{n=1}^{\infty}\|u_n'\|_{C[0,T]}$ implies $u(t,x),\ u_t(t,x),u_x(t,x)\in C(\overline{D})$.

***Remark* 2.2.** For $u(t,x)\in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ there exist partial derivatives $u_{xx},\ u_{tx}$ and

$$u_{xx}(t,x)=-\sum_{n=1}^{\infty}n^2u_n(t)\sin nx,\qquad u_{tx}(t,x)=\sum_{n=1}^{\infty}nu_n'(t)\cos nx.$$

Besides $u_{xx},u_{tx}\in C\big([0,T];L_{p,p-2}(0,\pi)\big)$. In fact, we have

$$\sum_{n=1}^{\infty}\big(n^2|u_n(t)|\big)^p n^{p-2}\le\sum_{n=1}^{\infty}\left(n^{1+\frac{2}{q}}\|u_n\|_{C[0,T]}\right)^p<+\infty\quad\forall t\in[0,T].$$

Then, by Paley theorem (see [39, p. 182]), there exists $f(t,x)\in L_p(0,\pi)$ such that $f(t,x)= =-\sum_{n=1}^{\infty}n^2u_n(t)\sin nx$ and

$$\int\limits_0^{\pi}\left|f(t,x)\right|^pdx\le A_p\sum_{n=1}^{\infty}n^{p-2}\big(n^2|u_n(t)|\big)^p.$$

Hence we obtain $f(t,x) \in C\big([0,T]; L_{p,p-2}(0,\pi)\big)$. Moreover, $f(t,x) = u_{xx}(t,x)$. In fact,

$$f(t,x) = \frac{\partial}{\partial x}\int\limits_0^x f(t,y)dy = \frac{\partial}{\partial x}\left(-\sum_{n=1}^{\infty} n^2 u_n(t)\int\limits_0^x \sin nydy\right) =$$

$$= \frac{\partial^2}{\partial x^2}\int\limits_0^x \left(\sum_{n=1}^{\infty} nu_n(t)\cos ny\right)dy = \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} nu_n(t)\int\limits_0^x \cos nydy\right) =$$

$$= \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} u_n(t)\sin nx\right) = u_{xx}(t,x).$$

The derivative u_{tx} is treated similarly.

We introduce the following definition the generalized solution of (1.1)–(1.3).

Definition 2.1. *The function* $u(t,x) \in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ *satisfying condition* (1.2), *is called a generalized solution of* (1.1)–(1.3) *if for any function* $v(t,x) \in W_1^{(1)}\big([0,T], L_q(0,\pi)\big)$ *such that* $v(T,x) = 0$ *on a.e.* $[0,\pi]$ *the integral identity*

$$\int\limits_0^T\int\limits_0^{\pi} \{u_t(t,x)v_t(t,x) - \alpha u_{xx}(t,x)v_t(t,x) + F(u)(t,x)v(t,x)\}\,dxdt -$$

$$-\alpha\int\limits_0^{\pi} \varphi''(x)v(0,x)dx + \int\limits_0^{\pi} \psi(x)v(0,x)dx = 0 \tag{2.1}$$

is fulfilled.

When obtaining the main results we will need the following lemma.

Lemma 2.1 (Gronwall–Bellman)**.** *Let the functions* $u(t)$, $f(t)$ *be continuous and nonnegative for* $t \geq t_0$ *and it hold*

$$u(t) \leq a + \int\limits_{t_0}^{t} f(\tau)u(\tau)d\tau, \quad t \geq t_0,$$

where a *is some positive constant. Then*

$$u(t) \leq ae^{\int_{t_0}^t f(\tau)d\tau}, \quad t \geq t_0.$$

3. Existence and uniqueness of generalized solution. In this section, we prove the existence and uniqueness theorem for the generalized solution of the problem (1.1)–(1.3).

Lemma below can be proved similarly to the one proved in [31] for $p = 2$.

Lemma 3.1. *Let* $u(t,x)$ *be a generalized solution of the problem* (1.1)–(1.3) *and* $F(u) \in$ $\in L_{p,p-2}\big([0,T], L_p(0,\pi)\big)$. *Then the coefficients* $u_n(t)$ *are the solutions of the following countable of nonlinear integro-differential equations*:

$$u_n(t) = \varphi_n + \frac{1}{\alpha n^2}\big(1 - e^{-\alpha n^2 t}\big)\psi_n + \frac{1}{\alpha n^2}\int\limits_0^t F_n(u,\tau)\big(1 - e^{-\alpha n^2(t-\tau)}\big)d\tau, \tag{3.1}$$

where φ_n, ψ_n *and* $F_n(u,t)$ *are the Fourier coefficients by system* $\{\sin nx\}$ *of the functions* $\varphi(x), \psi(x)$ *and* $F(u)(t,x)$, *respectively.*

Proof. Substituting the function of the form

$$v_{\tau,n}(t,x) = \begin{cases} \dfrac{2}{\pi}(t-\tau)\sin nx, & 0 \le t \le \tau, \quad 0 \le x \le \pi, \\ 0, & \tau < t \le T, \quad 0 \le x \le \pi, \end{cases}$$

into (2.1) with fixed $n \in N$ and $\tau \in [0,T]$, we obtain

$$\int\limits_0^\tau \big\{u_n'(t) + \alpha n^2 u_n(t) + (t-\tau)F_n(u,t)\big\}dt - \alpha n^2\varphi_n\tau - \psi_n\tau = 0.$$

On differentiating the last equality twice in τ, we have

$$u_n''(\tau) + \alpha n^2 u_n'(\tau) - F_n(u,\tau) = 0, \quad \tau \in [0,T].$$

Taking into account the conditions $u_n(0) = \varphi_n$ and $u_n'(0) = \psi_n$, we obtain (3.1).

Lemma 3.1 is proved.

When obtaining the main result we need the following lemma.

Lemma 3.2. *Let the operator* P *be defined in the space* $L_{p,p-2}\big([0,T], L_p(0,\pi)\big)$ *by the formula*

$$P(f)(t,x) = \sum_{n=1}^{\infty} \frac{1}{\alpha n^2}\int\limits_0^t f_n(\tau)\big(1 - e^{-\alpha n^2(t-\tau)}\big)d\tau \sin nx, \quad f \in L_{p,p-2}\big([0,T], L_p(0,\pi)\big),$$

where $f_n(t) = \dfrac{2}{\pi}\displaystyle\int_0^\pi f(t,x)\sin nx dx$. *Then* $P\colon L_{p,p-2}\big([0,T], L_p(0,\pi)\big) \to B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ *and*

$$\big\|P(f)\big\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} \le L\|f\|_{L_{p,p-2}\left([0,T],L_p(0,\pi)\right)}, \tag{3.2}$$

where $L = \dfrac{q^{\frac{1}{q}}T^{\frac{1}{q}} + \alpha^{\frac{1}{p}}}{\alpha q^{\frac{1}{q}}}$.

Proof. For every $f \in L_{p,p-2}\big([0,T], L_p(0,\pi)\big)$, we have

$$\big\|P(f)\big\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} = \left(\sum_{n=1}^{\infty}\left\{n^{1+\frac{2}{q}}\max_{[0,T]}\left|\frac{1}{\alpha n^2}\int\limits_0^t f_n(\tau)(1 - e^{-\alpha n^2(t-\tau)})d\tau\right|\right\}^p\right)^{\frac{1}{p}} +$$

$$+\left(\sum_{n=1}^{\infty}\left\{n^{\frac{2}{q}}\max_{[0,T]}\left|\int\limits_0^t f_n(\tau)e^{-\alpha n^2(t-\tau)}d\tau\right|\right\}^p\right)^{\frac{1}{p}}.$$

Applying the Hölder inequality, we obtain

$$\|P(f)\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} \le \frac{1}{\alpha}\left(\sum_{n=1}^{\infty} n^{p-2}\int_0^T |f_n(\tau)|^p d\tau \max_{[0,T]}\left(\int_0^t (1-e^{-\alpha n^2(t-\tau)})^q d\tau\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} +$$

$$+\left(\sum_{n=1}^{\infty} n^{\frac{2p}{q}}\int_0^T |f_n(\tau)|^p dt \max_{[0,T]}\left(\int_0^t e^{-\alpha n^2(t-\tau)} d\tau\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \le$$

$$\le \frac{T^{\frac{1}{q}}}{\alpha}\left(\sum_{n=1}^{\infty} n^{p-2}\int_0^T |f_n(\tau)|^p d\tau\right)^{\frac{1}{p}} +$$

$$+\left(\sum_{n=1}^{\infty} n^{\frac{2p}{q}}\int_0^T |f_n(\tau)|^p dt \left(\frac{1}{\alpha n^2 q}\right)^{\frac{p}{q}} \max_{[0,T]}\left(1-e^{-\alpha n^2 t}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}.$$

Taking into account that

$$\max_{[0,T]}\int_0^t (1-e^{-\alpha n^2(t-\tau)})^q d\tau \le T \qquad \text{and} \qquad \max_{[0,T]}\int_0^t e^{-\alpha n^2(t-\tau)} d\tau \le \frac{1}{\alpha n^2 q},$$

we have

$$\|P(f)\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} \le \frac{T^{\frac{1}{q}}}{\alpha}\left(\sum_{n=1}^{\infty} n^{p-2}\int_0^T |f_n(\tau)|^p d\tau\right)^{\frac{1}{p}} + \frac{1}{\alpha^{\frac{1}{q}} q^{\frac{1}{q}}}\left(\sum_{n=1}^{\infty}\int_0^T |f_n(\tau)|^p dt\right)^{\frac{1}{p}} \le$$

$$\le \frac{q^{\frac{1}{q}}T^{\frac{1}{q}}+\alpha^{\frac{1}{p}}}{\alpha q^{\frac{1}{q}}}\left(\sum_{n=1}^{\infty} n^{p-2}\int_0^T |f_n(\tau)|^p d\tau\right)^{\frac{1}{p}} = L\left(\int_0^T\left(\sum_{n=1}^{\infty} n^{p-2}|f_n(\tau)|^p\right)d\tau\right)^{\frac{1}{p}}.$$

Lemma 3.2 is proved.

3.1. Uniqueness of solution. Now let us state the main uniqueness result for the generalized solution of the problem (1.1)–(1.3).

Theorem 3.1. *Let the following conditions be satisfied*:

1) $F: B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T} \to L_{p,p-2}\big([0,T], L_{p,p-2}(0,\pi)\big)$;

2) $\forall u(t,x), v(t,x) \in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ *and* $t\in[0,T]$:

$$\big\|F(u)(t,\cdot)-F(v)(t,\cdot)\big\|_{L_{p,p-2}(0,\pi)} \le c(t)\|u-v\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}, \tag{3.3}$$

where $c(t)\in L_p(0,T)$ *is some positive function.*

Then the problem (1.1)–(1.3) *can not have more than one generalized solution.*

Proof. Assume the contrary, i.e., assume that the problem (1.1) – (1.3) has at least two different generalized solutions $u(t,x)$ and $v(t,x)$. Let $\{u_n(t)\}_{n\in N}$ and $\{v_n(t)\}_{n\in N}$ be the sequences of coefficients of the functions $u(t,x)$ and $v(t,x)$, respectively. From Lemma 3.1, we obtain

$$u(t,x)-v(t,x)=\sum_{n=1}^{\infty}\left(u_n(t)-v_n(t)\right)\sin nx=$$

$$=\sum_{n=1}^{\infty}\frac{1}{\alpha n^2}\int\limits_0^t\big(F_n(u,\tau)-F_n(v,\tau)\big)\big(1-e^{-\alpha n^2(t-\tau)}\big)d\tau\sin nx=$$

$$=P\big(F(u)(t,x)-F(v)(t,x)\big).$$

Then, for all $t\in[0,T]$ by virtue of (3.2), using (3.3), we have

$$\left\|u-v\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}=\left\|P(F(u)-F(v))\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}\le L^p\int\limits_0^t c^p(\tau)\left\|u-v\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,\tau}}d\tau.$$

Hence, by Lemma 2.1, we obtain $\|u-v\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}=0$. Consequently, $u(t,x)=v(t,x)$.

Theorem 3.1 is proved.

Lemma 3.3. *Assume* $\varphi(x)\in W_p^{(2)}(0,\pi),\{n^2\varphi_n\}_{n\in N}\in l_{p,p-2},\varphi(0)=\varphi(\pi)=0,\ \psi(x)\in$ $\in W_p^{(1)}(0,\pi),\ \{n\psi_n\}_{n\in N}\in l_{p,p-2},\ \psi(0)=\psi(\pi)=0$ *and let* $w_n(t)=\varphi_n+\dfrac{1}{\alpha n^2}\big(1-e^{-\alpha n^2 t}\big)\psi_n$. *Then the function* $w(t,x)=\sum_{n=1}^{\infty}w_n(t)\sin nx$ *belongs to the space* $B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$.

Proof. It is clear that the series $\sum_{n=1}^{\infty}w_n(t)\sin nx$ is convergent. Let us show that $w(t,x)\in$ $\in B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$, i.e., let us show the convergence of the series $\sum_{n=1}^{\infty}\Big(n^{1+\frac{2}{q}}\|w_n\|_{C[0,T]}\Big)^p$ and $\sum_{n=1}^{\infty}\Big(n^{\frac{2}{q}}\|w_n'\|_{C[0,T]}\Big)^p$. We have

$$\|w_n\|_{C[0,T]}\le|\varphi_n|+\frac{1}{\alpha n^2}|\psi_n|\|w_n'\|_{C[0,T]}=|\psi_n|.$$

Then, taking into account that $\{n^2\varphi_n\}_{n\in N},\ \{n\psi_n\}_{n\in N}\in l_{p,p-2}$, we obtain

$$\left(\sum_{n=1}^{\infty}\Big(n^{1+\frac{2}{q}}\|w_n\|_{C[0,T]}\Big)^p\right)^{\frac{1}{p}}\le\left(\sum_{n=1}^{\infty}\left(n^{1+\frac{2}{q}}|\varphi_n|+\frac{n^{\frac{2}{q}}}{\alpha}|\psi_n|\right)^p\right)^{\frac{1}{p}}\le$$

$$\le\left(\sum_{n=1}^{\infty}\Big(n^{1+\frac{2}{q}}|\varphi_n|\Big)^p\right)^{\frac{1}{p}}+\frac{1}{\alpha}\left(\sum_{n=1}^{\infty}\Big(n^{\frac{2}{q}}|\psi_n|\Big)^p\right)^{\frac{1}{p}}=$$

$$=\left(\sum_{n=1}^{\infty}\big(n^2|\varphi_n|\big)^p n^{p-2}\right)^{\frac{1}{p}}+\frac{1}{\alpha}\left(\sum_{n=1}^{\infty}\big(n|\psi_n|\big)^p n^{p-2}\right)^{\frac{1}{p}}<+\infty.$$

Also we get

$$\sum_{n=1}^{\infty}\left(n^{\frac{2}{q}}\|w_n'\|_{C[0,T]}\right)^p=\sum_{n=1}^{\infty}\left(n^{\frac{2}{q}}|\psi_n|\right)^p=\sum_{n=1}^{\infty}\left(n|\psi_n|\right)^p n^{p-2}<+\infty.$$

Consequently, $w(t,x)\in B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$.

Lemma 3.3 is proved.

3.2. Existence of solution. Now let us consider solvability of the generalized solution of the problem.

Theorem 3.2. *Let the following conditions be satisfied*:

1) $\varphi(x)\in W_p^{(2)}(0,\pi)$, $\{n^2\varphi_n\}_{n\in N}\in l_{p,p-2}$, $\varphi(0)=\varphi(\pi)=0$, $\psi(x)\in W_p^{(1)}(0,\pi)$, $\{n\psi_n\}_{n\in N}\in l_{p,p-2}$, $\psi(0)=\psi(\pi)=0$;

2) $F: B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}\to L_{p,p-2}\big([0,T],L_{p,p-2}(0,\pi)\big)$ $\forall u\in B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$, $t\in[0,T]$:

$$\big\|F(u)(t,\cdot)\big\|_{L_{p,p-2}(0,\pi)}\le a(t)+b(t)\|\mathrm{u}\|_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}, \tag{3.4}$$

where $a(t),b(t)\in L_p(0,T)$ *are some positive functions*;

3) $\forall u(t,x),v(t,x)\in K\Big(\|\mathrm{u}\|_{B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}}\le R\Big)$ $t\in[0,T]$:

$$\big\|F(u)(t,\cdot)-F(v)(t,\cdot)\big\|_{L_{p,p-2}(0,\pi)}\le c(t)\,\|u-v\|_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}, \tag{3.5}$$

where $c(t)\in L_p(0,T)$, $R^p=A\exp\int_0^T B^p(t)dt$, $A=2^{p-1}\|w\|^p_{B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}}+L_0^p\|a\|^p_{L_p(0,T)}$, $B(t)=$ $=L_0b(t)$, $L_0=2^{\frac{2}{q}}L$.

Then the problem (1.1) – (1.3) *has a unique generalized solution.*

Proof. Consider the operator Q in the space $B_{p,p,T}^{1+\frac{2}{q},\frac{2}{q}}$ defined by the formula

$$Q(u)(t,x)=w(t,x)+P(F(u)(t,x)).$$

By using (3.2), (3.3), we obtain

$$\big\|Q(u)\big\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}\le 2^{p-1}\left(\|w\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}+\|P(F(u))\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}\right)\le$$

$$\le 2^{p-1}\left(\|w\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}+L^p\int_0^t\|F(u)\|^p_{L_{p,p-2}(0,\pi)}\,dt\right)\le$$

$$\le 2^{p-1}\left(\|w\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}+2^{p-1}L^p\int_0^t(a^p(\tau)+b^p(\tau)\|u\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}})d\tau\right)=$$

$$=2^{p-1}\,\|w\|^p_{B_{p,p,t}}+L_0^p\int_0^t a^p(\tau)d\tau+L_0^p\int_0^t b^p(\tau)\|u\|^p_{B_{p,p,t}^{1+\frac{2}{q},\frac{2}{q}}}d\tau\le$$

$$\le A + \int\limits_0^t B^p(\tau)\|u\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} d\tau. \tag{3.6}$$

Let us build the sequence $\{u_k(t,x)\}_{k=0}^{\infty} \subset B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ as follows:

$$u_0(t,x) = 0, \qquad u_k(t,x) = Q(u_{k-1})(t,x), \quad k = 1, 2, \ldots .$$

According to (3.6), for every $t \in [0,T]$, we obtain

$$\|u_1\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} = \|Q(u_0)\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} \le A \le A + A\int\limits_0^t B^p(\tau)d\tau,$$

$$\|u_2\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} = \|Q(u_1)\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} \le A + \int\limits_0^t B^p(\tau)\,\|u_1\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} d\tau \le$$

$$\le A + \int\limits_0^t B^p(\tau)(A + A\int\limits_0^\tau B^p(s)ds)d\tau =$$

$$= A + A\int\limits_0^t B^p(\tau)d\tau + A\int\limits_0^t B^p(\tau)\int\limits_0^\tau B^p(s)dsd\tau =$$

$$= A\left(1 + \int\limits_0^t B^p(\tau)d\tau + \int\limits_0^t \frac{\dfrac{d}{dt}\left(\int_0^\tau B^p(s)ds\right)^2}{2}d\tau\right) =$$

$$= A\left(1 + \int\limits_0^t B^p(\tau)d\tau + \frac{\left(\int_0^t B^p(\tau)d\tau\right)^2}{2}\right).$$

Continuing this process in a similar way, we get

$$\|u_k\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}} \le A\left(1 + \int\limits_0^t B^p(\tau)d\tau + \ldots + \frac{\left(\int_0^t B^p(\tau)d\tau\right)^k}{k!}\right), \quad k = 0, 1, \ldots .$$

Hence it follows

$$\|u_k\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} \le A\exp\int\limits_0^T B^p(t)dt = R^p, \qquad u_k(t,x) \in K, \quad k = 0, 1, \ldots .$$

Let us estimate $\left\|u_{n+k}-u_k\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}}$ for any $n,k=1,2,\ldots$. Taking into account (3.2), (3.4) and (3.5), we have

$$\|u_{n+k}-u_k\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}=\left\|Q(u_{n+k-1})-Q(u_{k-1})\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}=$$

$$=\left\|P\big(F(u_{n+k-1})-F(u_{k-1})\big)\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}\le$$

$$\le L^p\int\limits_0^t\left\|F(u_{n+k-1})-F(u_{k-1})\right\|^p_{L_{p,p-2}(0,\pi)}dt\le$$

$$\le L^p\int\limits_0^t c^p(\tau)\left\|u_{n+k-1}-u_{k-1}\right\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,\tau}}d\tau.$$

Then

$$\|u_{n+k}-u_k\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}\le L^p\int\limits_0^t c^p(\tau)\,\|u_{n+k-1}-u_{k-1}\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,\tau}}d\tau\le$$

$$\le L^p\int\limits_0^t c^p(\tau)\left(L^p\int\limits_0^\tau c^p(s)\,\|u_{n+k-2}-u_{k-2}\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,s}}\,ds\right)d\tau=$$

$$=L^{2p}\int\limits_0^t c^p(\tau)\int\limits_0^\tau c^p(s)ds\,\|u_{n+k-2}-u_{k-2}\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,\tau}}d\tau\le\ldots$$

$$\ldots\le L^{2p}\int\limits_0^t\frac{\dfrac{d}{d\tau}\left(\displaystyle\int_0^\tau c^p(s)ds\right)^2}{2}\,\|u_{n+k-2}-u_{k-2}\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,\tau}}d\tau\le$$

$$\le L^{pk}\int\limits_0^t\frac{\dfrac{d}{d\tau}\left(\displaystyle\int_0^\tau c^p(s)ds\right)^k}{k!}\,\|u_n-u_0\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,\tau}}d\tau\le$$

$$\le L^{pk}\|u_n\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}\int\limits_0^t\frac{\dfrac{d}{d\tau}\left(\displaystyle\int_0^\tau c^p(s)ds\right)^k}{k!}d\tau=$$

$$=L^{pk}\,\|u_n\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,t}}\,\frac{\left(\displaystyle\int_0^t c^p(\tau)d\tau\right)^k}{k!}.$$

Thus, the following inequality is true:

$$\|u_{n+k} - u_k\|^p_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} \le \frac{L^{pk} R^p \, \|c\|^{pk}_{L_p(0,T)}}{k!}.$$

Consequently, the sequence $\{u_k(t,x)\}_{k=1}^{\infty}$ is fundamental in $B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$, and therefore it converges to some $u(t,x) \in K$. Further, we have

$$\big\|Q(u_k) - Q(u)\big\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} = \big\|P(F(u_k) - F(u))\big\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}} \le$$

$$\le L\,\|F(u_k) - F(u)\|_{L_p([0,T],L_{p,p-2}(0,\pi))} \le L\,\|c\|_{L_p(0,T)}\,\|u_k - u\|_{B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}},$$

and, therefore, $Q(u_k)$ converges in $B^{1+\frac{2}{q},\frac{2}{q}}_{p,p,T}$ to $Q(u)$ as $k \to \infty$. Then

$$u(t,x) = \lim_{k\to\infty} u_k(t,x) = \lim_{k\to\infty} Q(u_{k-1})(t,x) = Q(u)(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin nx,$$

where

$$u_n(t) = \varphi_n + \frac{1}{\alpha n^2}\big(1 - e^{-\alpha n^2 t}\big)\psi_n + \frac{1}{\alpha n^2}\int_0^t F_n(u,\tau)\big(1 - e^{-\alpha n^2 (t-\tau)}\big)d\tau.$$

Let us show that $u(t,x)$ is a generalized solution of the problem (1.1)–(1.3). Obviously,

$$u(0,x) = \sum_{n=1}^{\infty} u_n(0)\sin nx = \sum_{n=1}^{\infty} \varphi_n \sin nx = \varphi(x),$$

$$u_t(0,x) = \sum_{n=1}^{\infty} u_n'(0)\sin nx = \sum_{n=1}^{\infty} \psi_n \sin nx = \psi(x).$$

It remains to show the validity of the identity (2.1). Assume that

$$u_m(t,x) = \sum_{n=1}^{m} u_n(t)\sin nx,$$

$$J_m = \int_0^T\int_0^\pi \Big\{u_{m,t}(t,x)v_t(t,x) - \alpha u_{m,xx}(t,x)v_t(t,x) + F(u)(t,x)v(t,x)\Big\}dxdt -$$

$$-\alpha\int_0^\pi \varphi''(x)v(0,x)dx + \int_0^\pi \psi(x)v(0,x)dx. \tag{3.7}$$

We have

$$\int_0^T \int_0^\pi u_{m,t}(t,x) v_t(t,x) dx dt =$$

$$= \int_0^T \int_0^\pi \sum_{n=1}^m u_n'(t) \sin nx v_t(x,t) dx dt = \sum_{n=1}^m \int_0^\pi \left(\int_0^T u_n'(t) v_t(t,x) dt \right) \sin nx dx =$$

$$= \sum_{n=1}^m \int_0^\pi \left(u_n'(t) v(t,x)|_0^T - \int_0^T u_n''(t) v(t,x) dt \right) \sin nx dx =$$

$$= - \int_0^\pi \sum_{n=1}^m \psi_n \sin nx v(0,x) dx - \int_0^T \int_0^\pi \sum_{n=1}^m u_n''(t) \sin nx v(x,t) dx dt,$$

$$\int_0^T \int_0^\pi u_{m,xx}(t,x) v_t(t,x) dx dt = - \int_0^T \int_0^\pi \sum_{n=1}^m n^2 u_n(t) v_t(t,x) \sin nx dx dt =$$

$$= - \sum_{n=1}^m n^2 \int_0^\pi \left(u_n(t) v(t,x)|_0^T - \int_0^T u_n'(t) v(t,x) dt \right) \sin nx dx =$$

$$= \int_0^\pi \sum_{n=1}^m n^2 \varphi_n \sin nx v(0,x) dx + \int_0^T \int_0^\pi \sum_{n=1}^m n^2 u_n'(t) v(t,x) \sin nx dx dt.$$

Consequently, taking into account $u_n''(t) + \alpha n^2 u_n'(t) = F_n(u,t)$, we obtain

$$J_m = \int_0^T \int_0^\pi \big(F(u)(t,x) - \sum_{n=1}^m F_n(u,t) \sin nx \big) v(t,x) dx dt +$$

$$+ \int_0^\pi \left(\psi(x) - \sum_{n=1}^m \psi_n \sin nx \right) v(0,x) dx +$$

$$+ \alpha \int_0^\pi \left(\varphi''(x) + \sum_{n=1}^m n^2 \varphi_n \sin nx \right) v(0,x) dx.$$

As a result, we have

$$|J_m| \le \left\| F(u)(t,x) - \sum_{n=1}^m F_n(u,t) \sin \frac{n\pi}{\ell} x \right\|_{L_p(D)} \big\| v(t,x) \big\|_{L_q(D)} +$$

$$+ \left\| \psi(x) - \sum_{n=1}^m \psi_n \sin nx \right\|_{L_p(0,\pi)} \big\| v(0,x) \big\|_{L_q(0,\pi)} +$$

$$+\alpha \left\| \varphi''(x) + \sum_{n=1}^{m} n^2 \varphi_n \sin nx \right\|_{L_p(0,\pi)} \left\| v(0,x) \right\|_{L_q(0,\pi)} \to 0$$

as $m \to \infty$. Passing to the limit in (3.7) as $m \to \infty$, we obtain the validity of the integral identity (2.1). The uniqueness of generalized solution follows from Theorem 3.1.

Theorem 3.2 is proved.

4. Conclusion. For the initial boundary-value problem (1.1) – (1.3) with nonlinear right-hand side and zero boundary conditions, the concept of generalized solution belonging to Banach space is introduced. Under some conditions, the existence and uniqueness of generalized solution is proved. In particular, for we obtain the previously known results in this field. Note that, using the known technique, we can obtain similar results for the same problem with nonzero boundary data. Moreover, the same technique is also applicable to the multidimensional analog of this problem. Of course, this problem can be treated by other methods too, for example, by operator method. To do so, you need to define the corresponding mapping operators and use the methods of this theory.

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Received 20.07.18,
after revision — 14.02.20