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J. Tang (Wuxi Inst. Technology, China),
L. Miao (School Math. Sci., Yangzhou Univ., China),
B. Gao (School Math. and Statistics, Yili Normal Univ., China)

SOME CONDITIONS FOR CYCLIC CHIEF FACTORS OF FINITE GROUPS*
ДЕЯКІ УМОВИ НА ЦИКЛІЧНІ ГОЛОВНІ ФАКТОРИ СКІНЧЕННИХ ГРУП

A subgroup H of a finite group G is called $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for every maximal subgroup H_1 of H. The main purpose of the paper is to study the influence of $\mathcal{M}$-supplemented subgroups on the cyclic chief factors of finite groups.

Підгрупа H скінченної групи G називається $\mathcal{M}$-доповненою в G, якщо існує підгрупа B групи G така, що $G = HB$, а H_1B є власною підгрупою G для кожної максимальної підгрупи H_1 в H. Основною метою статті є вивчення впливу $\mathcal{M}$-доповнених підгруп на циклічні головні фактори скінченних груп.

1. Introduction. All groups in this paper are finite. Most of the notation is standard and can be found in [2, 7, 8]. In what follows, $\mathcal{U}$ denotes the formation of all supersoluble groups and $\mathcal{N}$ denotes the formation of all nilpotent groups. The symbol $\mathcal{A}(p-1)$ [12] stands for the formation of all Abelian groups of exponent dividing $p-1$ where p is a prime. $F^*(E)$ stands for the generalized Fitting subgroup of E, which coincides with the product of all normal quasinilpotent subgroups of E [8] (Chapter X). Following Doerk and Hawkes [2], we use $[A]B$ to denote the semidirect product of the groups A and B, where B is an operator group of A. $Z_{\mathcal{U}}(G)$ is the product of all such normal subgroups H of G whose G-chief factors are cyclic [2].

Let $\mathcal{F}$ be a class of groups. If $1 \in \mathcal{F}$, then we write $G^{\mathcal{F}}$ to denote the intersection of all normal subgroups N of a group G with $G/N \in \mathcal{F}$. The class $\mathcal{F}$ is said to be a formation if either $\mathcal{F} = \varnothing$ or $1 \in \mathcal{F}$ and every homomorphic image of $G/G^{\mathcal{F}}$ belongs to $\mathcal{F}$ for any group G. The formation $\mathcal{F}$ is said to be solubly saturated if $G \in \mathcal{F}$ whenever $G/\Phi(N) \in \mathcal{F}$ for some soluble normal subgroup N of a group G.

In this paper, as a continuation of the Theorem of [10], we mainly prove the following theorem.

Theorem 1.1. *Let $X = E$ or $X = F^*(E)$ be two normal subgroups of a group G. Suppose that every noncyclic Sylow subgroup P of X has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|D|$ is $\mathcal{M}$-supplemented in G, then each chief factor of G below E is cyclic.*

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Definition 1.1. *A subgroup H is called $\mathcal{M}$-supplemented in a group G, if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for every maximal subgroup H_1 of H.*

Recall that a subgroup H is called weakly S-permutable in a group G [10], if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{sG}$. In fact, the following example indicates that the $\mathcal{M}$-supplementation of subgroups cannot be deduced from Skiba's result.

***Example* 1.1.** Let $G = S_4$ be the symmetric group of degree 4 and $H = \langle(1234)\rangle$ be a cyclic subgroup of order 4. Then $G = HA_4$ where A_4 is the alternating group of degree 4. Obviously, H is $\mathcal{M}$-supplemented in G because the group H has an unique maximal subgroup. On the other hand, we have $H_{sG} = 1$. Otherwise, if H is S-permutable in G, then H is normal in G, a contradiction. If $H_{sG} = \langle(13)(24)\rangle$ is S-permutable in G, then $\langle(13)(24)\rangle$ is normal in G, also is a contradiction. Therefore H is not weakly S-permutable in G.

2. Proof of Theorem 1.1. In order to prove Theorem 1.1, we first list here some lemmas.

Lemma 2.1 ([9], Lemmas 2.1 and 2.2)**.** *Let G be a group. Then the following hold*:

(1) *If $H \leq M \leq G$ and H is $\mathcal{M}$-supplemented in G, then H is also $\mathcal{M}$-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H \leq G$. If H is $\mathcal{M}$-supplemented in G, then H/N is $\mathcal{M}$-supplemented in G/N.*

(3) *Let K be a normal π'-subgroup and H be a π-subgroup of G for a set π of primes. Then H is $\mathcal{M}$-supplemented in G if and only if HK/K is $\mathcal{M}$-supplemented in G/K.*

(4) *If P is a p-subgroup of G where $p \in \pi(G)$ and P is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $P \cap B = P_1 \cap B = \Phi(P) \cap B$ and $|G\colon P_1B| = p$ for every maximal subgroup P_1 of P.*

Lemma 2.2 ([3], Theorem 1.8.17)**.** *Let N be a nontrivial soluble normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.3 ([10], Lemma 1)**.** *Given a normal p-subgroup E of a group G, if $E \leq Z_{\mathcal{U}}(G)$, then $(G/C_G(E))^{\mathcal{A}(p-1)} \leq O_p(G/C_G(E))$.*

Lemma 2.4 ([10], Lemma 2)**.** *Given a normal subgroup E of a group G, if every G-chief factor of $F^*(E)$ is cyclic, then so is every G-chief factor of E.*

Lemma 2.5 ([1], Lemma 3.5)**.** *Let P be a normal p-subgroup of a group G. If each subgroup of P of order p is complemented in G, then $P \leq Z_{\mathcal{U}}(G)$.*

***Proof of Theorem* 1.1.** Suppose that this theorem is false and consider a counterexample (G, E) for which $|G||E|$ is minimal. Take a Sylow p-subgroup P of E, where p is the smallest prime divisor of the order of E, and put $C = C_G(P)$. If $X = E$, then we have following claims.

(1) E is supersoluble and $E \neq G$.

Corollary 3.3 of [9] shows that E is supersoluble and hence $E \neq G$ by the choice of G.

(2) If T is a Hall subgroup of E, then the hypotheses of Theorem 1.1 hold for (T, T). Furthermore, if T is normal in G, then the hypotheses of the theorem hold for (G, T) and $(G/T, E/T)$. The claim follows directly from Lemma 2.1.

(3) If T is a nontrivial normal Hall subgroup of E, then $T = E$.

Suppose that $1 \neq T \neq E$. Since T is a characteristic subgroup of E, it follows that T is normal in G, and by (2) the hypotheses of Theorem 1.1 hold for $(G/T, E/T)$ and (G, T). Then $E/T \leq Z_{\mathcal{U}}(G/T)$ and $T \leq Z_{\mathcal{U}}(G)$ by the choice of (G, E). So, $E \leq Z_{\mathcal{U}}(G)$. This contradiction shows that $T = E$.

(4) $E = P$.

Suppose that $E \neq P$. By (1), there exists a normal Hall p'-subgroup V of E and $1 \neq V \neq E$, which contradicts (3). Consequently, $E = P$ and P is noncyclic.

(5) $|D| > p$.

Suppose that $|D| = p$. Then every minimal subgroup of P is $\mathcal{M}$-supplemented in G. Indeed, every minimal subgroup of P is complemented in G and so by Lemma 2.5, $E \leq Z_{\mathcal{U}}(G)$, a contradiction.

(6) $|N| \leq |D|$ for any minimal normal subgroup N of G contained in E.

Assume that $|D| < |N|$. Suppose $H < N$ and H is a subgroup of N with order $|D|$. By hypotheses, there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H. Clearly, $G = HB = NB$ and $N \cap B \trianglelefteq G$. If $N \cap B = N$, then $G = B$, a contradiction. If $N \cap B = 1$, then $H = N$, also is a contradiction.

(7) If N is a minimal normal subgroup of G contained in E, then the hypotheses are still true for $(G/N, E/N)$.

If $|D| = |N|$, then N is $\mathcal{M}$-supplemented in G. There exists a subgroup B of G such that $G = NB$ and $TB < G$ for every maximal subgroup T of N. Clearly, $N \nleq TB$ and $|G\colon TB| = p$. Hence $|N| = p$, contrary to (5). So we may assume that $|N| < |D|$, then every subgroup H/N of P/N with order $|D|/|N|$ is $\mathcal{M}$-supplemented in G/N by Lemma 2.1(2). It follows that the hypotheses are still true for $(G/N, E/N)$.

(8) $P \cap \Phi(G) \neq 1$.

If $P \cap \Phi(G) = 1$, then by Lemma 2.2, $P = R_1 \times \ldots \times R_t$ with minimal normal subgroups $R_1, ..., R_t$ of G contained in P. Let L be any minimal normal subgroup of G contained in P. We get $|D| \geq |L|$ by (6). Now we suppose that $L \leq H \leq P$ with $|H| = |D|$. By hypotheses, there exists $B \leq G$ such that $G = HB$ and $H_iB < G$ for every maximal subgroup H_i of H. Since $|G\colon H_iB| = p$ by Lemma 2.1(4) and $P \cap \Phi(G) = 1$, there exists a maximal subgroup H_i of H with $L \nleq H_i$ and hence $H = LH_i$ as well as $G = HB = LH_iB$ and $L \cap H_iB \trianglelefteq G$. As L is minimal normal in G, we get $L \nleq H_iB$ and thus $|L| = |G\colon H_iB| = p$, otherwise, if $L \leq H_iB$, then $H_iB = LH_iB = HB = G$, a contradiction. By hypotheses and (7), $(G/L, E/L)$ satisfies the condition of Theorem 1.1. The minimal choice of G implies that $E/L \leq Z_{\mathcal{U}}(G/L)$ and hence $E \leq Z_{\mathcal{U}}(G)$, a contradiction.

(9) $\Phi(P) \neq 1$.

By (8), $P \cap \Phi(G) \neq 1$. Then there exists a minimal normal subgroup L of G contained in $P \cap \Phi(G)$ and L is an elementary Abelian p-group.

If $|D| = |L|$, then we may choose a subgroup $H \leq L$. By hypotheses, H is $\mathcal{M}$-supplemented in G, i.e., there exists a subgroup B of G such that $G = HB$ and $TB < G$ for every maximal subgroup T of H. Since $L \leq \Phi(G)$, we get $G = HB = LB = B$, a contradiction.

So we have $|D| > |L|$ and fix $H \leq P$ with $L < H$ where $|H| = |D|$. By hypotheses, H is $\mathcal{M}$-supplemented in G, i.e., there exists a subgroup B of G such that $G = HB$ and $TB < G$ for every maximal subgroup T of H. By Lemma 2.1(4), $|G\colon TB| = p$ and $H \cap B = T \cap B \leq \Phi(H) \leq \Phi(P)$. Since L is a minimal normal subgroup of G and TB is a maximal subgroup of G for every maximal subgroup T of H, we have $G = LTB$ or $L \leq TB$. If $G = LTB$ for some maximal subgroup T of H, we obtain $G = TB$ since L is contained in $P \cap \Phi(G)$, a contradiction. Therefore $L \leq TB$ for every maximal subgroup T of H. Moreover, if $L \nleq T_i$ for some maximal subgroup T_i of H, then $H = LT_i$ and hence $T_iB = LT_iB = HB = G$, a contradiction. Therefore we have $L \leq T$ for every maximal subgroup T of H and hence $L \leq \Phi(H) \leq \Phi(P)$, that is, $\Phi(P) \neq 1$.

(10) $C_G(P/\Phi(P))/C$ is a p-group.
Firstly, we obtain $\Phi(P) \neq 1$ by (9). And then we suppose that this claim is false. Pick a p'-element aC of $C_G(P/\Phi(P))/C$, where $a \in C_G(P/\Phi(P))\backslash C$. Put $G_0 = [P](G/C)$. Then aC is a nontrivial p'-element of G/C, that is, aC is a p'-automorphism of the p-group P and $aC \in C_{G_0}(P/\Phi(P))$, which contradicts Theorem 1.4 of [6] (Chapter 5). Hence, $C_G(P/\Phi(P))/C$ is a p-group.

(11) $P/\Phi(P) \nleq Z_{\mathcal{U}}(G/\Phi(P))$.

Suppose that $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$. Then $(G/C_G(P/\Phi(P)))^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.3. Since

$$(G/C_G(P/\Phi(P)))^{\mathcal{A}(p-1)} \cong (G/C/C_G(P/\Phi(P))/C)^{\mathcal{A}(p-1)} =$$
$$= (G/C)^{\mathcal{A}(p-1)}C_G(P/\Phi(P))/C/C_G(P/\Phi(P))/C,$$

we get that $(G/C)^{\mathcal{A}(p-1)}$ is a p-group by (10). Take an arbitrary chief factor H/K of G below $\Phi(P)$ and $C = C_G(P) \leq C_G(H/K)$. Then

$$(G/C_G(H/K))^{\mathcal{A}(p-1)} \cong (G/C/C_G(H/K)/C)^{\mathcal{A}(p-1)} =$$
$$= (G/C)^{\mathcal{A}(p-1)}C_G(H/K)/C/C_G(H/K)/C$$

and hence $(G/C_G(H/K))^{\mathcal{A}(p-1)}$ is a p-group since $(G/C)^{\mathcal{A}(p-1)}$ is a p-group. On the other hand, we have $O_p(G/C_G(H/K)) = 1$ by Lemma 3.9 of [3] (Chapter 1) and then $(G/C_G(H/K))^{\mathcal{A}(p-1)} = 1$. So $G/C_G(H/K) \in \mathcal{A}(p-1)$ and hence $|H/K| = p$ by Lemma 4.1 of [11] (Chapter 1). Therefore $P \leq Z_{\mathcal{U}}(G)$. This contradiction completes the proof of (11).

The final contradiction. It follows form (7), (8), (9) that $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$, which contradicts (11).

If $X = F^*(E)$, then $F^*(E) \leq Z_{\mathcal{U}}(G)$ by the proof in the case $X = E$, which by Lemma 2.4 implies that $E \leq Z_{\mathcal{U}}(G)$.

Theorem 1.1 is proved.

Note that if $\mathcal{F}$ is a solubly saturated formation and $G/E \in \mathcal{F}$, where every chief factor of G below E is cyclic, then $G \in \mathcal{F}$ (Lemma 3.3 in [4]). Therefore from Theorem 1.1 we get the following corollary.

***Corollary* 2.1.** *Let $\mathcal{F}$ be a solubly saturated formation containing all supersoluble groups and $X \leq E$ normal subgroups of a group G such that $G/E \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup P of X has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G. If either $X = E$ or $X = F^*(E)$, then $G \in \mathcal{F}$.*

In detail, if $\mathcal{F}$ is a saturated formation containing $\mathcal{N}$, then both $\mathcal{F}^*$ and ${\mathcal{F}_p}^*$ are solubly saturated formations, where $\mathcal{F}^*$ and ${\mathcal{F}_p}^*$ denote the class of all quasi-$\mathcal{F}$-groups and the class of all p-quasi-$\mathcal{F}$-groups, respectively (Theorem A in [5]). Hence we get the following corollary.

***Corollary* 2.2.** *Let E be a normal subgroup of a group G such that G/E is p-quasisupersoluble. Suppose that every noncyclic Sylow subgroup P of X has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ is $\mathcal{M}$-supplemented in G, where $X = E$ or $X = F^*(E)$. Then G is p-quasisupersoluble.*

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