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ALMOST EVERYWHERE CONVERGENCE OF CESÀRO MEANS OF TWO VARIABLE WALSH – FOURIER SERIES WITH VARYING PARAMETERS *

ЗБІЖНІСТЬ МАЙЖЕ СКРІЗЬ СЕРЕДНІХ ЧЕЗАРО ДЛЯ РЯДІВ УОЛША – ФУР'Є ДВОХ ЗМІННИХ ЗІ ЗМІННИМИ ПАРАМЕТРАМИ

We prove that the maximal operator of some (C, β_n) means of cubical partial sums of two variable Walsh – Fourier series of integrable functions is of weak type (L_1, L_1). Moreover, the (C, β_n)-means $\sigma_{2^n}^{\beta_n} f$ of the function $f \in L_1$ converge a.e. to f for $f \in L_1(I^2)$, where I is the Walsh group for some sequences $1 > \beta_n \searrow 0$.

Доведено, що максимальний оператор від деяких середніх (C, β_n) кубічних часткових сум рядів Уолша – Фур'є двох змінних для інтегровних функцій має слабкий тип (L_1, L_1). Більш того, (C, β_n)-середні $\sigma_{2^n}^{\beta_n} f$ для функції $f \in L_1$ збігаються майже скрізь до f для $f \in L_1(I^2)$, де I — група Уолша для деяких послідовностей $1 > \beta_n \searrow 0$.

1. Introduction and main results. In 1939, for the two-dimensional trigonometric Fourier partial sums $S_{j,j} f$ Marcinkiewicz [9] proved that for all $f \in L \log L([0, 2\pi]^2)$ the relation

$$\sigma_n^1 f = \frac{1}{n+1} \sum_{j=0}^{n} S_{j,j} f \to f$$

holds a.e. as $n \to \infty$. Zhizhiashvili [12] improved this result and showed that for $f \in L([0, 2\pi]^2)$ the (C, α)-means

$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{j=0}^{n} A_{n-j}^{\alpha-1} S_{j,j} f$$

converge to f a.e. for any $\alpha > 0$. Dyachenko [4] proved this result for dimensions greater than 2. In papers [8, 11] by Goginava and Weisz one can find that the $(C, 1)$-means $\sigma_n^1 f$ of the double Walsh – Fourier series of a function $f \in L_1([0, 1]^2)$ converges to f a.e. Recently, Gát [5] proved this result with respect to two-dimensional Vilenkin systems. The d-dimensional Walsh – Fourier case is discussed in [7].

For the one dimensional trigonometric system it can be found in Zygmund [13, p. 94] that the Cesàro means or $(C, \alpha)(\alpha > 0)$ means $\sigma_n^\alpha f$ of the Fourier series of a function $f \in L_1([-\pi, \pi])$ converge a.e. to f as $n \to \infty$. Moreover, it is known that the maximal operator of the (C, α)-means $\sigma_*^\alpha := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha|$ is of weak type (L_1, L_1), i.e.,

$$\sup_{\gamma > 0} \gamma \lambda(\sigma_*^\alpha f > \gamma) \leq C \|f\|_1, \qquad f \in L_1([-\pi, \pi]).$$

This result can be found implicitly in Zygmund [13, p. 154 – 156].

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In 2007 Akhobadze [1] (see also [2]) introduced the notion of Cesàro means of Fourier series with variable parameters for one-dimensional functions. In paper [3], we proved the almost everywhere convergence of the the Cesàro (C, α_n)-means of integrable functions $\sigma_n^{\alpha_n} f \to f$, where $\mathbb{N} \supset \mathbb{N}_{\alpha,K} \ni$ $\ni n \to \infty$ for $f \in L^1(I)$, where I is the Walsh group for every sequence $\alpha = (\alpha_n),\ 0 < \alpha_n < 1$. The main aim of this paper is to investigate to two-dimensional version of this issue.

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see, e.g., [10]) and others. Denote by $\mathbb{N} := \{0, 1, \ldots\}$, $\mathbb{P} := \mathbb{N}\backslash\{0\}$, the set of natural numbers, the set of positive integers and $I := [0, 1)$ the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set $B(B \subset I)$. Denote by $L^p(I)$ the usual Lebesgue spaces and $\|.\|_p$ the corresponding norms $(1 \le p \le \infty)$. Set

$$\mathcal{J} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right\} : p, n \right\}$$

the set of dyadic intervals and for given $x \in I$ set the definition of the nth $(n \in \mathbb{N})$ Walsh–Paley function at point $x \in I$ as

$$\omega_n(x) := \prod_{j=0}^{\infty} (-1)^{x_j n_j},$$

where $\mathbb{N} \ni n = \sum_{n=0}^{\infty} n_j 2^j$, $n_j \in \{0, 1\}$, $j \in \mathbb{N}$, and $x = \sum_{j=0}^{\infty} x_j 2^{-(j+1)}$, $x_j \in \{0, 1\}$, $j \in \mathbb{N}$. Remark that if x is a dyadic rational, that is, $x \in \{p/2^n : p, n \in \mathbb{N}\}$, then choose the expansion terminates in zeros.

For any $x,\ y \in I$ and $k,\ n \in \mathbb{N}$ define the so-called dyadic or logical addition as

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}, \qquad k \oplus n := \sum_{n=0}^{\infty} |k_i - n_i| 2^i.$$

By the definition of ω_n we have

$$\omega_{k \oplus n} = \omega_k \omega_n, \qquad |\omega_n| = 1,$$

and also the almost everywhere equality

$$\omega_n(x + y) = \omega_n(x)\omega_n(y).$$

Denote by

$$\hat{f}(n) := \int_I f \omega_n d\lambda, \qquad D_n := \sum_{k=0}^{n-1} \omega_k, \qquad K_n^1 := \frac{1}{n+1} \sum_{k=0}^{n} D_k$$

the Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels, respectively. At some places, if it does not cause confusion, we simple write K_n $\left(\text{instead of } K_n^1\right)$. It is also known that the Fejér or $(C, 1)$-means of f is

$$\sigma_n^1 f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n^1(y + x) d\lambda(x) =$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \int_I f(x) D_k(y + x) d\lambda(x), \qquad n \in \mathbb{N}, \quad y \in I.$$

Now, for the two variable case we have for $x = \left(x^1, x^2\right)$, $y = \left(y^1, y^2\right) \in I^2$, $n = (n_1, n_2) \in \mathbb{N}^2$ the two-dimensional Fourier coefficients

$$\hat{f}(n_1, n_2) := \int\limits_{I\times I} f(x^1, x^2)\omega_{n_1}(x^1)\omega_{n_2}(x^2)d\lambda(x^1, x^2),$$

the rectangular partial sums of the two-dimensional Fourier series

$$S_{n_1,n_2}f(y^1, y^2) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\omega_{k_1}(y^1)\omega_{k_2}(y^2)$$

and the rectangular Dirichlet kernels

$$D_{n_1,n_2}(z) := D_{n_1}(z^1)D_{n_2}(z^2) = \sum_{k_1=0}^{n_k-1}\sum_{k_2=0}^{n_k-1} \omega_{k_1}(z^1)\omega_{k_2}(z^2), \qquad z = (z^1, z^2) \in I^2.$$

We have the nth Marcinkiewicz mean and kernel

$$\sigma_n^1 f(y) := \frac{1}{n+1}\sum_{k=0}^{n} S_{j,j}f(y), \qquad K_n^1(z) = \frac{1}{n+1}\sum_{j=0}^{n} D_{j,j}(z)$$

and so we get

$$\sigma_n^1 f(y^1, y^2) = \int\limits_{I\times I} f(x^1, x^2)K_n^1(y^1 + x^1, y^2 + x^2)d\lambda(x^1, x^2).$$

Denote by $K_n^{\alpha_a}$ the kernel of the summability method (C, α_a)-Marcinkiewicz and call it the (C, α_a) kernel or the Cesàro – Marcinkiewicz kernel for $\alpha_a \in \mathbb{R} \setminus \{-1, -2, \ldots\}$

$$K_n^{\alpha_a}(x_1, x_2) = \frac{1}{A_n^{\alpha_a}}\sum_{k=0}^{n} A_{n-k}^{\alpha_a - 1}D_{j,j}(x_1, x_2),$$

where

$$A_k^{\alpha_a} = \frac{(\alpha_a + 1)(\alpha_a + 2)\ldots(\alpha_a + k)}{k!}.$$

The (C, α_n) Cesàro – Marcinkiewicz means of integrable function f for two variables are

$$\sigma_n^{\alpha_n} f(y^1, y^2) = \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1}S_{k,k}f(y^1, y^2) = \int\limits_{I\times I} f(x^1, x^2)K_n^{\alpha_n}(y^1 + x^1, y^2 + x^2)d\lambda(x^1, x^2),$$

$$\sigma_n^{\alpha_n} f(y^1, y^2) = \frac{1}{A_n^{\alpha_n}}\sum_{k=0}^{n}\int\limits_{I\times I} A_{n-k}^{\alpha_n - 1}f(x^1, x^2)D_k(y^1 + x^1)D_k(y^2 + x^2)d\lambda(x^1, x^2).$$

Over all of this paper we suppose that monotone decreasing sequences (α_n) and (β_n) satisfy

$$\beta_n = \alpha_{2^n}, \qquad \frac{\alpha_N}{A_N^{\alpha_N}}\log^{\delta}\left(1 + \frac{N}{n}\right) \le C\frac{\alpha_n}{A_n^{\alpha_n}}, \qquad N \ge n, \quad n, N \in \mathbb{P}, \tag{1.1}$$

for some $\delta > 1$ and for some positive constant C. We remark that from condition (1.1) it follows that sequence $\left(\frac{\alpha_n}{A_n^{\alpha_n}}\right)$ is quasimonotone decreasing. That is, for some $C > 0$ we have

$$\frac{\alpha_N}{A_N^{\alpha_N}} \le C\frac{\alpha_n}{A_n^{\alpha_n}}, \qquad N \ge n, \quad n, N \in \mathbb{P}.$$

The main aim of this paper is to prove the following theorem.

Theorem 1.1. *Suppose that monotone decreasing sequence* $1 > \beta_n > 0$ *satisfies the condition* $\dfrac{A_{2^n}^{\beta_n}}{\beta_n}\dfrac{\beta_N}{A_{2^N}^{\beta_N}}(N+1-n)^{\delta} \le C$ *for every* $\mathbb{N} \ni N \ge n \ge 1$ *and for some* $\delta > 1$. *Let* $f \in L_1(I^2)$. *Then we have the almost everywhere convergence*

$$\sigma_{2^n}^{\beta_n} f \to f.$$

***Remark* 1.1.** In the proof of Theorem 1.1 we define the sequence (α_n) in a way that $\alpha_{2^k} = \beta_k$, and, for any $2^k \le n < 2^{k+1}$, let $\alpha_n = \alpha_{2^k} = \beta_k$. Then the sequence (α_n) satisfies that it is decreasing and $\dfrac{A_n^{\alpha_n}}{\alpha_n}\dfrac{\alpha_N}{A_N^{\alpha_N}}\log^{\delta}\left(1+\dfrac{N}{n}\right) \le C$ for every $\mathbb{N} \ni N \ge n \ge 1$. That is, condition (1.1) is fulfilled.

We give two examples for sequences (β_n) like above. Example 1: $\beta_k = \alpha_{2^k} = \alpha_n = \alpha \in (0,1)$ for every natural number k, n.

Example 2: Let $\alpha_n = 1/n$. Then it is not difficult to have that $A_n^{\alpha_n} \to 1$ and it should be fulfilled for sequence (α_n) that $CN/n \ge \log^{\delta}(1+N/n)$ for some $\delta > 1$ and it trivially holds.

Introduce the following notations: for a, n, $j \in \mathbb{N}$, let $n_{(j)} := \sum_{i=0}^{j-1} n_i 2^i$, that is, $n_{(0)} = 0$, $n_{(1)} = n_0$ and, for $2^B \le n < 2^{B+1}$, let $|n| := B$, $n = n_{(B+1)}$. Moreover, introduce the following functions and operators for $n \in \mathbb{N}$ and $1 > \alpha_a > 0$, $a \in \mathbb{N}$, where (x^1, x^2), $(y^1, y^2) \in I^2$:

$$T_n^{\alpha_a}(x^1,x^2) := \frac{1}{A_n^{\alpha_a}}\sum_{j=0}^{2^B-1} A_{n-j}^{\alpha_a-1} D_{j,j}(x^1,x^2),$$

$$\bar{T}_n^{\alpha_a}(x^1,x^2) := D_{2^B}(x^1)\frac{1}{A_n^{\alpha_a}}\left|\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^2)\right|, \qquad \bar{\bar{T}}_n^{\alpha_a}(x^1,x^2) := \bar{T}_n^{\alpha_a}(x^2,x^1),$$

$$\tilde{T}_n^{\alpha_a}(x^1,x^2) := \frac{1}{A_n^{\alpha_a}} D_{2^B,2^B}(x^1,x^2)\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} +$$

$$+\frac{1-\alpha_a}{A_n^{\alpha_a}}\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1}\frac{j+1}{n_{(B)}+j+1}\left|K_j^1(x^1,x^2)\right| + A_n^{\alpha_a-1}2^B\left|K_{2^B-1}^1(x^1,x^2)\right|,$$

$$t_n^{\alpha_a} f(y^1,y^2) := \int\limits_{I\times I} f(x^1,x^2) T_n^{\alpha_a}(y^1+x^1, y^2+x^2)\, d\lambda(x^1,x^2),$$

$$\tilde{t}_n^{\alpha_a} f(y^1,y^2) := \int\limits_{I\times I} f(x^1,x^2) \tilde{T}_n^{\alpha_a}(y^1+x^1, y^2+x^2)\, d\lambda(x^1,x^2).$$

Now we need several lemmas in the next section.

2. Proofs.

Lemma 2.1. *Let* $1 > \alpha_a > 0,\ f \in L_1(I \times I)$ *such that*

$$\operatorname{supp} f \subset I_k(u^1) \times I_k(u^2), \qquad \int\limits_{I_k(u^1)\times I_k(u^2)} f d\lambda = 0$$

for some dyadic rectangle $(u^1, u^2) \in I^2$. *Then we have*

$$\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}} \sup_{n,a\in\mathbb{N}} |\tilde{t}_n^{\alpha_a} f| d\lambda \leq C\|f\|_1. \tag{2.1}$$

We also prove that

$$|T_n^{\alpha_a}(x^1, x^2)| \leq \tilde{T}_n^{\alpha_a}(x^1, x^2) + \bar{T}_n^{\alpha_a}(x^1, x^2) + \bar{\bar{T}}_n^{\alpha_a}(x^1, x^2). \tag{2.2}$$

Proof. First, we start with the proof of the inequality $|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a} + \bar{T}_n^{\alpha_a} + \bar{\bar{T}}_n^{\alpha_a}$.

Recall that $B = |n|$. Then, by equality $D_{2^B - j} = D_{2^B} - \omega_{2^B-1} D_j$ and $n_{(B)} = \sum_{j=0}^{B-1} n_j 2^j,\ n_{(B)} + 2^B = n$,

$$A_n^{\alpha_a} T_n^{\alpha_a}(x) = \sum_{j=0}^{2^B-1} A_{2^B+n_{(B)}-j}^{\alpha_a-1} D_{j,j}(x) = \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{2^B-j,2^B-j}(x) =$$

$$= D_{2^B}(x^1)D_{2^B}(x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} - \omega_{2^B-1}(x^1)D_{2^B}(x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^1) -$$

$$-\omega_{2^B-1}(x^2)D_{2^B}(x^1) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(x^2) +$$

$$+\omega_{2^B-1}(x^1)\omega_{2^B-1}(x^2) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1, x^2) =: (1) - (2) - (3) + (4).$$

So, by the help of the Abel transform, we get

$$|(4)| = \left| \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{j,j}(x^1, x^2) \right| =$$

$$= \left| \sum_{j=0}^{2^B-1} (A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1}) \sum_{i=0}^{j} D_{i,i}(x^1, x^2) + A_{n_{(B)}+2^B}^{\alpha_a-1} \sum_{i=0}^{2^B-1} D_{i,i}(x^1, x^2) \right| =$$

$$= \left| (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} K_j^1(x^1, x^2) + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1(x^1, x^2) \right| \leq$$

$$\leq (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \left| K_j^1(x^1, x^2) \right| +$$

$$+(1-\alpha_a)\sum_{j=2^k}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1}\frac{j+1}{n_{(B)}+j+1}\left|K_j^1(x^1,x^2)\right| + A_n^{\alpha_a-1}2^B\left|K_{2^B-1}^1(x^1,x^2)\right| =:$$

$$=: I + II + III.$$

By the above written, we have

$$A_n^{\alpha_a}\left|T_n^{\alpha_a}(x^1,x^2)\right| \le D_{2^B,2^B}(x^1,x^2)\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1} + D_{2^B}(x^1)\left|\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}D_j(x^2)\right| +$$

$$+D_{2^B}(x^2)\left|\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}D_j(x^1)\right| + \left|\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}D_{j,j}(x^1,x^2)\right|.$$

Thus,

$$\left|T_n^{\alpha_a}(x^1,x^2)\right| \le \tilde{T}_n^{\alpha_a}(x^1,x^2) + D_{2^B}(x^1)\frac{1}{A_n^{\alpha_a}}\left|\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}D_j(x^2)\right| +$$

$$+D_{2^B}(x^2)\frac{1}{A_n^{\alpha_a}}\left|\sum_{j=0}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}D_j(x^1)\right| = \tilde{T}_n^{\alpha_a}(x^1,x^2) + \bar{T}_n^{\alpha_a}(x^1,x^2) + \bar{\bar{T}}_n^{\alpha_a}(x^1,x^2).$$

For $n < 2^k$ and $(x^1,x^2)\in I_k(u^1)\times I_k(u^2)$, we have that $\tilde{T}_n^{\alpha_a}(y+x)$ depends (with respect to x) only on coordinates $x_0^1,\ldots,x_{k-1}^1,x_0^2,\ldots,x_{k-1}^2$, thus $\tilde{T}_n^{\alpha_a}(y+x)=\tilde{T}_n^{\alpha_a}(y+u)$ and, consequently,

$$\int\limits_{I_k(u^1)\times I_k(u^2)} f(x^1,x^2)\tilde{T}_n^{\alpha_a}(y^1+x^1,y^2+x^2)d\lambda(x^1,x^2) =$$

$$=\tilde{T}_n^{\alpha_a}(y^1+u^1,y^2+u^2)\int\limits_{I_k(u^1)\times I_k(u^2)} f(x^1,x^2)d\lambda(x^1,x^2)=0.$$

Observe that

$$\overline{I_k(u^1)\times I_k(u^2)} = \overline{I_k(u^1)}\times\overline{I_k(u^2)}\cup I_k(u^1)\times\overline{I_k(u^2)}\cup\overline{I_k(u^1)}\times I_k(u^2).$$

Since for any $j < 2^k$ we obtain that the kernel $K_j^1(y+x)$ depends (with respect to x) only on coordinates $x_0^1,\ldots,x_{k-1}^1,x_0^2,\ldots,x_{k-1}^2$, then

$$\int\limits_{I_k(u^1)\times I_k(u^2)} f(x)|K_j^1(y+x)|d\lambda(x) = |K_j^1(y+u)|\int\limits_{I_k(u^1)\times I_k(u^2)} f(x)d\lambda(x)=0$$

gives $\int_{I_k(u^1)\times I_k(u^2)} f(x)I(y+x)d\lambda(x)=0$. On the other hand,

$$II = (1-\alpha_a)\sum_{j=2^k}^{2^B-1}A_{n_{(B)}+j}^{\alpha_a-1}\frac{j+1}{n_{(B)}+j+1}|K_j^1(y^1+x^1,y^2+x^2)| \le$$

$$\leq \sup_{j\geq 2^k} |K_j^1(x^1,x^2)|(1-\alpha_a)\sum_{j=0}^{n} A_j^{\alpha_a-1} = A_n^{\alpha_a}(1-\alpha_a)\sup_{j\geq 2^k}|K_j^1(x^1,x^2)|.$$

This, by Lemma 3 in [5], gives

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} II d\lambda \leq \int\limits_{\overline{I_k\times I_k}} \sup_{j\geq 2^k} |K_j^1(x^1,x^2)| d\lambda \leq C.$$

The situation with III is similar. So, just as in the case of II we apply Lemma 3 in [5]:

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} III d\lambda \leq \int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k} |K^1_{2^{|n|}-1}| d\lambda \leq C.$$

Therefore, substituting $z^1 = (x^1+y^1), z^2 = (x^2+y^2)$, where $z \in \overline{I_k\times I_k}$ and, consequently, $D_{2^B,2^B}(z^1,z^2)=0$, we obtain

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda =$$

$$= \int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \left| \int\limits_{I_k\times I_k} f(x^1,x^2)\tilde{T}_n^{\alpha_a}(y^1+x^1,y^2+x^2)d\lambda(x^1,x^2)\right| d\lambda(y^1,y^2) \leq$$

$$\leq \int\limits_{\overline{I_k\times I_k}} \int\limits_{I_k\times I_k} |f(x^1,x^2)| \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} [II(y^1+x^1,y^2+x^2)+$$

$$+III(y^1+x^1,y^2+x^2)]d\lambda(x^1,x^2)d\lambda(y^1,y^2) =$$

$$= \int\limits_{I_k\times I_k} |f(x^1,x^2)| \int\limits_{\overline{I_k\times I_k}} \sup_{n\geq 2^k, a\in\mathbb{N}} \frac{1}{A_n^{\alpha_a}} II(z^1,z^2) + III(z^1,z^2)d\lambda(z^1,z^2)d\lambda(x^1,x^2) \leq$$

$$\leq C \int\limits_{I_k\times I_k} |f(x^1,x^2)|d\lambda(x^1,x^2).$$

This gives

$$\int\limits_{\overline{I_k\times I_k}} \sup_{n,a\in\mathbb{N}} \left|\tilde{t}_n^{\alpha_a} f\right| d\lambda \leq C\|f\|_1.$$

Lemma 2.1 is proved.

Now, we just proved the lemma which means that maximal operator $\sup_{n,a}|\tilde{t}_n^{\alpha_a}|$ is quasilocal. The following lemma shows that the one-dimensional operator which maps $f \in L_1(I)$ to

$$\sup_n \left| f * \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n} A_j^{\alpha_n-1}|K_j| \right|$$

is quasilocal. This lemma is interesting itself if one investigates Cesàro means with variable parameters and in the proof we introduce methods which will also be used later.

Lemma 2.2. *Let* (α_n) *be a monotone decreasing sequence and* $\left(\dfrac{\alpha_n}{n^{\alpha_n}}\right)$ *be a quasidecreasing sequences with* $1 > \alpha_n > 0,\ n \in \mathbb{N}$. *Then*

$$\int\limits_{\overline{I_k}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n} A_j^{\alpha_n-1} |K_j| \leq C.$$

Proof. Recall that K_n denotes the one-dimensional Fejér kernel, that is, $K_n = K_n^1$. By [6] we get

$$\int\limits_{\overline{I_k}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=2^k}^{n} A_j^{\alpha_n-1} |K_j(x)|\, dx \leq \int\limits_{\overline{I_k}} \sup_{j\geq 2^k} |K_j(x)| \sup_{n} \frac{1}{A_n^{\alpha_n}} \sum_{l=2^k}^{n} A_l^{\alpha_n-1} dx \leq$$

$$\leq \int\limits_{\overline{I_k}} \sup_{j\geq 2^k} |K_j(x)|\, dx \leq C.$$

On the other hand, if $j < 2^k$, by $\bar{I}_k = \bigcup_{a=0}^{k-1} (I_a \backslash I_{a+1})$, we have

$$\int\limits_{\overline{I_k}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{2^k-1} A_j^{\alpha_n-1} |K_j| \leq$$

$$\leq \sum_{a=0}^{k-1} \int\limits_{I_a\backslash I_{a+1}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=2^a}^{2^k-1} A_j^{\alpha_n-1} |K_j| + \sum_{a=0}^{k-1} \int\limits_{I_a\backslash I_{a+1}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{2^a-1} A_j^{\alpha_n-1} |K_j| =:$$

$$=: I + II.$$

For I we obtain

$$I \leq \sum_{a=0}^{k-1} \int\limits_{I_a\backslash I_{a+1}} \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=a}^{k-1} \sum_{j=2^b}^{2^{b+1}-1} A_j^{\alpha_n-1} |K_j| \leq$$

$$\leq \sum_{a=0}^{k-1} \sum_{b=a}^{k-1} \int\limits_{I_a\backslash I_{a+1}} \sup_{j\geq 2^b} |K_j| \sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{l=2^b}^{2^{b+1}-1} A_l^{\alpha_n-1},$$

where

$$\sup_{n\geq 2^k} \frac{1}{A_n^{\alpha_n}} \sum_{l=2^b}^{2^{b+1}-1} A_l^{\alpha_n-1} \leq \sup_{n\geq 2^k} \frac{A_{2^{b+1}-1}^{\alpha_n} - A_{2^b-1}^{\alpha_n}}{A_n^{\alpha_n}} =$$

$$= \sup_{n\geq 2^k} \frac{A_{2^b-1}^{\alpha_n}}{A_n^{\alpha_n}} \left[\frac{(2^b+\alpha_n)\dots(2^{b+1}-1+\alpha_n)}{2^b(2^b+1)\dots(2^{b+1}-1)} - 1 \right] =$$

$$= \sup_{n\geq 2^k} \frac{A_{2^b-1}^{\alpha_n}}{A_n^{\alpha_n}} \left[\left(1+\frac{\alpha_n}{2^b}\right)\left(1+\frac{\alpha_n}{2^b+1}\right)\dots\left(1+\frac{\alpha_n}{2^b+2^b-1}\right) - 1 \right] \leq$$

$$\leq \sup_{n\geq 2^k}\frac{A_{2^b}^{\alpha_n}}{A_n^{\alpha_n}}\left[\left(1+\frac{\alpha_n}{2^b}\right)^{2^b}-1\right]\leq C\sup_{n\geq 2^k}\frac{A_{2^b}^{\alpha_n}}{A_n^{\alpha_n}}\alpha_n\leq C\sup_{n\geq 2^k}\left(\frac{2^b}{n}\right)^{\alpha_n}\alpha_n\leq$$

$$\leq C\sup_{n\geq 2^k}\left(2^b\right)^{\alpha_{2^k}}\left(\frac{\alpha_n}{n^{\alpha_n}}\right)\leq C\left(2^b\right)^{\alpha_{2^k}}\left(\frac{\alpha_{2^k}}{(2^k)^{\alpha_{2^k}}}\right),$$

where the inequality $\dfrac{A_{2^b}^{\alpha_n}}{A_n^{\alpha_n}}\leq C\left(\dfrac{2^b}{n}\right)^{\alpha_n}$ is given from [3] (Lemma 2.4). Besides, since (α_n) is a monotone decreasing sequences, then $(2^b)^{\alpha_n}\leq (2^b)^{\alpha_{2^k}}$. Sequence $\left(\dfrac{\alpha_n}{n^{\alpha_n}}\right)$ is quasidecreasing. Moreover, $\left(1+\dfrac{\alpha_n}{2^b}\right)^{2^b}-1\leq C\alpha_n$ for any $0<\alpha_n<1,\ b\in\mathbb{N}$.

Thus, by (2.3) [8],

$$I\leq C\sum_{a=0}^{k-1}\sum_{b=a}^{k-1}\frac{2^a}{2^b}(b-a)\,\alpha_{2^k}\left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}}=C\sum_{b=0}^{k-1}\sum_{a=0}^{b}\frac{2^a}{2^b}(b-a)\,\alpha_{2^k}\left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}}\leq$$

$$\leq C\sum_{b=0}^{k-1}\alpha_{2^k}\left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}}\leq C\alpha_{2^k}\sum_{l=0}^{\infty}\frac{1}{2^{l\alpha_k}}\leq C\alpha_{2^k}\frac{1}{1-2^{\alpha_{2^k}}}\leq C.$$

We have to discuss II in the case when $j<2^a$ and, thus, $|K_j(x)|\leq j$. Besides, $A_j^{\alpha_n-1}j=\alpha_nA_{j-1}^{\alpha_n}$ and we get

$$\sum_{j=0}^{2^a-1}A_j^{\alpha_n-1}|K_j(x)|\leq\alpha_n\sum_{j=0}^{2^a-1}A_j^{\alpha_n}\leq\alpha_nA_{2^a}^{\alpha_n+1}=\alpha_nA_{2^a+1}^{\alpha_n}\left(\frac{2^a+1}{\alpha_n+1}\right).$$

Besides, by [3] (Lemma 2.4) and by the fact that the sequence (α_n/n^{α_n}) is quasidecreasing, we have

$$\sup_{n\geq 2^k}\frac{\alpha_nA_{2^a+1}^{\alpha_n}}{A_n^{\alpha_n}}\frac{2^a+1}{\alpha_n+1}\leq C2^a\sup_{n\geq 2^k}\alpha_n\left(\frac{2^a+1}{n}\right)^{\alpha_n}\leq C2^a\alpha_{2^k}\left(\frac{2^a}{2^k}\right)^{\alpha_{2^k}}.$$

Then

$$II\leq C\sum_{a=0}^{k-1}\frac{1}{2^a}2^a\alpha_{2^k}\left(\frac{2^a}{2^k}\right)^{\alpha_{2^k}}\leq C\sup_k\alpha_{2^k}\sum_{l=0}^{\infty}\frac{1}{2^{l\alpha_{2^k}}}\leq C.$$

Lemma 2.2 is proved.

Next we prove the following lemma.

Lemma 2.3. *Suppose that for the monotone decreasing sequence* (α_n) *the condition* (1.1) *is fulfilled. Let* $a\colon I\setminus\{0\}\longmapsto\mathbb{N}$ *be defined as* $a(x)=a$ *for* $x\in(I_a\backslash I_{a+1})$. *Then the inequality*

$$\int\limits_{I_k\times\overline{I_k}}\sup_{n\geq 2^k}\frac{1}{A_n^{\alpha_n}}\sum_{s=k}^{|n|}\sum_{j=0}^{2^{a(x^2)}}A_j^{\alpha_n-1}\left|K_j(x^2)\right|D_{2^s}(x^1)d(x^1,x^2)\leq C$$

holds.

Proof. Since $\int_{I_k\times\overline{I_k}} = \sum_{a=0}^{k-1}\int_{I_k\times(I_a\setminus I_{a+1})}$, we have to check the values of the integrand on $I_k\times(I_a\setminus I_{a+1})$. That is, $x^2\in I_a\setminus I_{a+1}$. Thus, $\left|K_j(x^2)\right|\le Cj$ gives

$$A_j^{\alpha_n-1}.j=\frac{\alpha_n\ldots(\alpha_n+j-1)}{j!}j=\alpha_n\frac{(1+\alpha_n)\ldots(j-1+\alpha_n)}{(j-1)!}=\alpha_nA_{j-1}^{\alpha_n}.$$

Hence it follows that

$$\sum_{j=0}^{2^a}A_j^{\alpha_n-1}\left|K_j(x^2)\right|\le C\sum_{j=1}^{2^a}\alpha_nA_{j-1}^{\alpha_n}=C\alpha_nA_{2^a-1}^{\alpha_n+1}=$$

$$=C\alpha_n\frac{(2+\alpha_n)\ldots(2^a+\alpha_n)}{(2^a-1)!}=C\alpha_n\left(\frac{2^a}{1+\alpha_n}\right)A_{2^a}^{\alpha_n}\le C\alpha_n2^aA_{2^a}^{\alpha_n},$$

that is, we have to investigate

$$\sum_{a=0}^{k-1}\int\limits_{I_k}\sup_{n\ge2^k}\frac{\alpha_n}{A_n^{\alpha_n}}A_{2^a}^{\alpha_n}\sum_{s=k}^{|n|}D_{2^s}(x^1)d(x^1).$$

Recall that $\int\limits_{I_a\setminus I_{a+1}}2^a\le1,\ A_{2^a}^{\alpha_n}\le A_{2^a}^{\alpha_{2^k}}$, since $\alpha_n\searrow$ and $n\ge2^k$. Also recall that

$$\frac{\alpha_n}{A_n^{\alpha_n}}\le\frac{C}{\log^\delta\left(1+\dfrac{n}{2^k}\right)}\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}},$$

which gives

$$\frac{\alpha_n}{A_n^{\alpha_n}}A_{2^a}^{\alpha_n}\le C\alpha_{2^k}\frac{A_{2^a}^{\alpha_{2^k}}}{A_{2^k}^{\alpha_{2^k}}}\frac{1}{\log^\delta\left(1+\dfrac{n}{2^k}\right)}.$$

That is, we have to investigate

$$\sum_{a=0}^{k-1}\alpha_{2^k}\frac{A_{2^a}^{\alpha_{2^k}}}{A_{2^k}^{\alpha_{2^k}}}\int\limits_{I_k}\sup_{n\ge2^k}\frac{1}{\log^\delta\left(1+\dfrac{n}{2^k}\right)}\sum_{s=k}^{|n|}D_{2^s}(x^1)d(x^1).$$

Check the integral above $\int_{I_k}=\sum_{t=k}^{\infty}\int_{I_t\setminus I_{t+1}}$ and the integral on $I_t\setminus I_{t+1}$ can be estimated by

$$\int\limits_{I_t\setminus I_{t+1}}\sup_{n\ge2^k}\frac{C}{(1+|n|-k)^\delta}\sum_{s=k}^{\min(t,|n|)}2^sd(x^1)\le\frac{C}{(t+1-k)^\delta}$$

and henceforth, by $\delta>1$, $\sum_{t=k}^{\infty}\dfrac{1}{(1+t-k)^\delta}\le C$. We have, by Lemma 2.4 in [3], that

$$\sum_{a=0}^{k-1}\alpha_{2^k}\frac{A_{2^a}^{\alpha_{2^k}}}{A_{2^k}^{\alpha_{2^k}}}\le2\sum_{a=0}^{k-1}\alpha_{2^k}\left(\frac{2^a+1}{2^k}\right)^{\alpha_{2^k}}\le C\sum_{a=0}^{k-1}\alpha_{2^k}\left(\frac{2^a}{2^k}\right)^{\alpha_{2^k}}\le$$

$$\leq C\alpha_{2^k}\sum_{j=0}^{\infty}\left(\frac{1}{2^{\alpha_{2^k}}}\right)^j=\frac{C\alpha_{2^k}}{1-\left(\frac{1}{2}\right)^{\alpha_{2^k}}}\leq C.$$

Lemma 2.3 is proved.

Let (α_n) be a monotone decreasing sequences such that $0<\alpha_n<1$ with property (1.1). That is, for some $\delta>1, C>0$,

$$\frac{A_n^{\alpha_n}}{\alpha_n}\frac{\alpha_N}{A_N^{\alpha_N}}\log^{\delta}\left(1+\frac{N}{n}\right)\leq C$$

for every $\mathbb{N}\ni N\geq n\geq 1$.

We prove the following lemma.

Lemma 2.4.

$$\sum_{a=0}^{k-1}\int\limits_{I_k\times(I_a\setminus I_{a+1})}\sup_{n>2^k}\frac{1}{A_n^{\alpha_n}}\sum_{s=k}^{|n|}\sum_{b=a}^{k-1}\sum_{j=2^b+1}^{2^{b+1}}A_j^{\alpha_n-1}\left|K_j(x^2)\right|D_{2^s}(x^1)d(x^1,x^2)\leq C.$$

Proof. By the result of Goginava [8], that is, by

$$\int\limits_{I_a\setminus I_{a+1}}\sup_{n\geq 2^b}\left|K_j(x^2)\right|d(x^2)\leq C\left(\frac{b-a}{2^{b-a}}\right), \tag{2.3}$$

we have to investigate

$$\mathbf{B_1}:=\sum_{a<k}\int\limits_{I_k}\sup_{n>2^k}\frac{1}{A_n^{\alpha_n}}\sum_{s=k}^{|n|}\sum_{b=a}^{k-1}\frac{b-a}{2^{b-a}}\sum_{j=2^b+1}^{2^{b+1}}A_j^{\alpha_n-1}D_{2^s}(x^1)d(x^1).$$

So, we have

$$\sum_{j=2^b+1}^{2^{b+1}}A_j^{\alpha_n-1}=A_{2^{b+1}}^{\alpha_n}-A_{2^b}^{\alpha_n}=A_{2^b}^{\alpha_n}\left[\frac{(2^b+1+\alpha_n)\ldots(2^{b+1}+\alpha_n)}{(2^b+1)\ldots(2^{b+1})}-1\right]=$$

$$=A_{2^b}^{\alpha_n}\left[\left(1+\frac{\alpha_n}{2^b+1}\right)\ldots\left(1+\frac{\alpha_n}{2^{b+1}}\right)-1\right]\leq A_{2^b}^{\alpha_n}\left[0\left(1+\frac{\alpha_n}{2^b}\right)^{2^b}-1\right]\leq C\alpha_nA_{2^b}^{\alpha_n}.$$

On the other hand, by $\displaystyle\int_{I_k}=\sum\nolimits_{t=k}^{\infty}\int_{I_t\setminus I_{t+1}}$ it follows that

$$\int\limits_{I_k}\sup_{n>2^k}\frac{1}{(|n|+1-k)^{\delta}}\sum_{s=k}^{|n|}D_{2^s}(x^1)d(x^1)=\sum_{t=k}^{\infty}\int\limits_{I_t\setminus I_{t+1}}\sup_{n>2^k}\frac{1}{(|n|+1-k)^{\delta}}\sum_{s=k+1}^{\min(t,|n|)}2^s\leq$$

$$\leq\sum_{t=k}^{\infty}\left(\int\limits_{I_t\setminus I_{t+1}}\sup_{t\geq|n|>k}\frac{1}{(|n|+1-k)^{\delta}}2^{|n|}+\int\limits_{I_t\setminus I_{t+1}}\sup_{|n|>t}\frac{1}{(|n|+1-k)^{\delta}}2^t\right)=:$$

$$=:\sum_{t=k}^{\infty}\left(\mathbf{B_{2,1}}+\mathbf{B_{2,2}}\right).$$

Now we have

$$\sum_{t=k}^{\infty}(\mathbf{B_{2,2}}) \leq \sum_{t=k}^{\infty}\frac{1}{(t+1-k)^{\delta}} \leq C,$$

$$\sum_{t=k}^{\infty}(\mathbf{B_{2,1}}) \leq \sum_{t=k}^{\infty}\sup_{t\geq|n|>k}\frac{2^{|n|+1-t}}{(|n|-k)^{\delta}} \leq \sum_{t=k+1}^{\infty}\frac{1}{(t-k)^{\delta}} \leq C.$$

That is, for $\mathbf{B_1}$ we get

$$\mathbf{B_1} \leq C\sum_{a<k}\sup_{n>2^k}\frac{1}{A_n^{\alpha_n}}\sum_{b=a}^{k-1}\alpha_n A_{2^b}^{\alpha_n}\frac{b-a}{2^{b-a}}\log^{\delta}\left(1+\frac{n}{2^k}\right)\times$$

$$\times\sum_{t=k}^{\infty}\int\limits_{I_t\setminus I_{t+1}}\sup_{n>2^k}\frac{1}{(|n|+1-k)^{\delta}}\sum_{s=k}^{\min(t,|n|)}D_{2^s}(x^1)dx^1 \leq$$

$$\leq C\sum_{a<k}\sup_{n>2^k}\frac{\alpha_n}{A_n^{\alpha_n}}\log^{\delta}\left(1+\frac{n}{2^k}\right)\sum_{b=a}^{k-1}A_{2^b}^{\alpha_n}\frac{b-a}{2^{b-a}} \leq$$

$$\leq C\sum_{a<k}\sum_{b=a}^{k-1}A_{2^b}^{\alpha_{2^k}}\frac{b-a}{2^{b-a}}\sup_{n>2^k}\frac{\alpha_n}{A_n^{\alpha_n}}\log^{\delta}\left(1+\frac{n}{2^k}\right) =:$$

$$=: \mathbf{B_3}.$$

Recall that $A_{2^b}^{\alpha_n} \leq A_{2^b}^{\alpha_{2^k}}$. Since $n > 2^k$ and (α_n) is a monotone decreasing sequence, by the properties of (α_n) we have $\dfrac{\alpha_n}{A_n^{\alpha_n}}\log^{\delta}\left(1+\dfrac{n}{2^k}\right) \leq C\dfrac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}$, and then by Lemma 2.4 for the Cesàro numbers in [3]

$$\mathbf{B_3} \leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}\sum_{a<k}\sum_{b=a}^{k-1}A_{2^b}^{\alpha_{2^k}}\frac{b-a}{2^{b-a}} = C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}\sum_{b=0}^{k-1}A_{2^b}^{\alpha_{2^k}}\sum_{a=0}^{b}\frac{b-a}{2^{b-a}} \leq$$

$$\leq C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}\sum_{b=0}^{k-1}A_{2^b}^{\alpha_{2^k}} \leq C\sum_{b=0}^{k-1}\alpha_{2^k}\left(\frac{2^b+1}{2^k}\right)^{\alpha_{2^k}} \leq C\sum_{b=0}^{k-1}\alpha_{2^k}\left(\frac{2^b}{2^k}\right)^{\alpha_{2^k}} \leq C,$$

again just as at the end of the proof of Lemma 2.3.

Lemma 2.4 is proved.

***Corollary* 2.1.** *Let* $1 > \alpha_a > 0$ *fulfill property* (1.1). *Then by Lemmas* 2.3 *and* 2.4, *as a direct consequence, we have*

$$\int\limits_{I_k\times\overline{I_k}}\sup_{n>2^k}\frac{1}{A_n^{\alpha_n}}\sum_{s=k}^{|n|}\sum_{j=0}^{2^k}A_j^{\alpha_n-1}\left|K_j(x^2)\right|D_{2^s}(x^1)d(x^1,x^2) \leq C.$$

Moreover, we prove the following lemma.

Lemma 2.5.

$$\int\limits_{I_k\times\overline{I_k}} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{j=2^k+1}^{2^{|n|}} A_j^{\alpha_n-1} \left|K_j(x^2)\right| D_{2^s}(x^1) d(x^1,x^2) \le C,$$

where $1>\alpha_a>0$ *is a decreasing sequence with property* (1.1).

Proof. By the result of Goginava [8] (see (2.3)) we have $\int_{I\setminus I_k} \sup_{j\ge 2^b} \left|K_j(x^1)\right| d(x^1) \le$ $\le C\dfrac{b-k+1}{2^{b-k}}$ for any $b\ge k$. That is the integral in Lemma 2.5 is bounded by

$$C\int\limits_{I_k} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{s=k}^{|n|} \sum_{b=k}^{|n|-1} \frac{b-k+1}{2^{b-k}} \sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n-1} D_{2^s}(x^1) d(x^1) =: \mathbf{B_4}.$$

As in the proof of Lemma 2.4 we have $\sum_{j=2^b+1}^{2^{b+1}} A_j^{\alpha_n-1} \le C\alpha_n A_{2^b}^{\alpha_n}$. In the proof of Lemma 2.4 we can find inequality

$$\int\limits_{I_k} \sup_{n>2^k} \frac{1}{(|n|+1-k)^\delta} \sum_{s=k}^{|n|} D_{2^s}(x^1) d(x^1) \le C$$

and henceforth

$$\mathbf{B_4} \le \int\limits_{I_k} \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=k}^{|n|-1} \frac{b-k+1}{2^{b-k}} \alpha_n A_{2^b}^{\alpha_n} (|n|+1-k)^\delta \, \frac{1}{(|n|+1-k)^\delta} \sum_{s=k}^{|n|} D_{2^s}(x^1) d(x^1) \le$$

$$\le C \sup_{n>2^k} \frac{1}{A_n^{\alpha_n}} \sum_{b=k}^{|n|} \frac{b-k+1}{2^{b-k}} \alpha_n A_{2^b}^{\alpha_n} \log^\delta\left(1+\frac{n}{2^k}\right) \int\limits_{I_k} \sup_{n>2^k} \frac{1}{(|n|+1-k)^\delta} \sum_{s=k}^{|n|} D_{2^s}(x^1) d(x^1) \le$$

$$\le C \sup_{n>2^k} \frac{\alpha_n}{A_n^{\alpha_n}} \sum_{b=k}^{|n|} \frac{b-k+1}{2^{b-k}} A_{2^b}^{\alpha_n} \log^\delta\left(1+\frac{n}{2^k}\right) =: \mathbf{B_5}.$$

So, by $\dfrac{\alpha_n}{A_n^{\alpha_n}} \log^\delta\left(1+\dfrac{n}{2^k}\right) \le C\dfrac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}}$, we have

$$\mathbf{B_5} \le C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \sup_{n>2^k} \sum_{b=k}^{|n|} \frac{b-k+1}{2^{b-k}} A_{2^b}^{\alpha_n}.$$

Since (α_n) is a monotone decreasing, then $A_{2^b}^{\alpha_n} \le A_{2^b}^{\alpha_{2^k}}$. Thus, by [3] (Lemma 2.4) (second inequality below)

$$\mathbf{B_5} \le C\frac{\alpha_{2^k}}{A_{2^k}^{\alpha_{2^k}}} \left[A_{2^k}^{\alpha_{2^k}} + \frac{2}{2}A_{2^{k+1}}^{\alpha_{2^k}} + \frac{3}{2^2}A_{2^{k+2}}^{\alpha_{2^k}} + \frac{4}{2^3}A_{2^{k+3}}^{\alpha_{2^k}} + \ldots\right] \le$$

$$\leq C\alpha_{2^k}\sum_{j=0}^{\infty}\left(\frac{2^{k+j}+1}{2^k}\right)^{\alpha_{2^k}}\frac{j}{2^j}\leq C\alpha_{2^k}\sum_{j=0}^{\infty}\frac{j}{2^{j(1-\alpha_{2^k})}}\leq C$$

as it holds $0<\alpha_{2^k}\leq 1-\alpha_2<1$.

Lemma 2.5 is proved.

Corollary 2.1 and Lemma 2.5 give the following corollary.

***Corollary* 2.2.** *Let* $0<\alpha_n<1$ *be a monotone decreasing sequence and*

$$\frac{\alpha_N}{A_N^{\alpha_N}}\frac{A_n^{\alpha_n}}{\alpha_n}\log^{\delta}\left(1+\frac{N}{n}\right)\leq C$$

for every $N\geq n\geq 1$. *Then*

$$\int\limits_{I_k\times\overline{I_k}}\sup_{n>2^k}\frac{1}{A_n^{\alpha_n}}\sum_{s=k+1}^{|n|}\sum_{j=0}^{2^{|n|}}A_j^{\alpha_n-1}\left|K_j(x^2)\right|D_{2^s}(x^1)d(x^1,x^2)\leq C.$$

By the help of Corollary 2.2 and Lemma 2.1 we prove that operator

$$t^*f(y):=\sup_n|t_n^{*,\alpha_n}f(y)|:=\sup_n\left|\int\limits_{I\times I}f(x)|T_n^{\alpha_n}(x+y)|d\lambda(x)\right|$$

is quasilocal.

Lemma 2.6. *Suppose that sequence* (α_n) *fulfills the conditions of Corollary* 2.2. *Let* $f\in L_1(I\times\times I)$ *such that* $\operatorname{supp} f\subset I_k(u^1)\times I_k(u^2)$, $\int_{I_k(u^1)\times I_k(u^2)}f\,d\lambda=0$ *for some dyadic rectangle. Then we have*

$$\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}}t^*f\,d\lambda\leq C\|f\|_1.$$

Besides, operator t^* *is of strong type* (L_∞,L_∞).

Proof. Recall that for any $m,\ n\leq 2^k$ we have $\hat f(m,n)=0$, and then

$$t^*f(y):=\sup_{n>2^k}|t_n^{*,\alpha_n}f(y)|.$$

The proof this lemma is based on Lemma 2.1. More precisely, on inequalities (2.1) and (2.2), that is,

$$\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}}t^*f\,d\lambda\ \leq\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}}\sup_{n>2^k}|\tilde t_n^{\alpha_n}f|d\lambda+$$

$$+\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}}\sup_{n>2^k}|\bar t_n^{\alpha_n}f|d\lambda+\int\limits_{\overline{I_k(u^1)\times I_k(u^2)}}\sup_{n>2^k}|\bar{\bar t}_n^{\alpha_n}f|d\lambda=:A_1+A_2+A_3.$$

Lemma 2.1 means that $A_1\leq C\|f\|_1$. Since the difference between terms A_2 and A_3 is only the interchange of variables therefore it is enough to discuss A_2 only. By the theorem of Fubini and the shift invariance of the Lebesgue measure, we have

$$A_2 \leq \int\limits_{I_k(u^1)\times I_k(u^2)} |f(x^1,x^2)| \int\limits_{\overline{I_k\times I_k}} \sup_{n>2^k} \bar{T}_n^{\alpha_n}(z^1,z^2)d\lambda(z)d\lambda(x).$$

Therefore, if we could prove the inequality $\int_{\overline{I_k\times I_k}} \sup_{n>2^k} \bar{T}_n^{\alpha_n}(z^1,z^2)d\lambda(z) \leq C$, then the proof of Lemma 2.6 would be complete.

By the help of the Abel transform, we get

$$A_n^{\alpha_n}\bar{T}_n^{\alpha_n}(z^1,z^2) = D_{2^B}(z^1)\left|\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j(z^2)\right| =$$

$$= D_{2^B}(z^1)\left|\sum_{j=0}^{2^B-1}(A_{n_{(B)}+j}^{\alpha_a-1} - A_{n_{(B)}+j+1}^{\alpha_a-1})\sum_{i=0}^{j} D_i + A_{n_{(B)}+2^B}^{\alpha_n-1}\sum_{i=0}^{2^B-1} D_i(z^2)\right| =$$

$$= D_{2^B}(z^1)\left|(1-\alpha_n)\sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1}\frac{j+1}{n_{(B)}+j+1}K_j^1(z^2) + A_n^{\alpha_n-1}2^B K_{2^B-1}^1(z^2)\right| \leq$$

$$\leq D_{2^B}(z^1)\sum_{j=0}^{2^B-1} A_j^{\alpha_n-1}\left|K_j^1(z^2)\right| + D_{2^B}(z^1)A_n^{\alpha_n-1}2^B\left|K_{2^B-1}^1(z^2)\right|. \tag{2.4}$$

Use the facts that $\overline{I_k\times I_k} = \bar{I}_k \times I_k \cup \bar{I}_k \times \bar{I}_k \cup I_k \times \bar{I}_k$ and $D_{2^B}(z^1) = 0$ for $n > 2^k$, that is, $B = |n| \geq k$ in the case of $z^1 \in \bar{I}_k$. Moreover, $2^B A_n^{\alpha_n-1}/A_n^{\alpha_n} \leq 1$, then by Corollary 2.2 the proof of the sublinearity of operator t^*f is complete. On the other hand,

$$\|t^*f\|_\infty \leq \sup_n \left|\int\limits_{I\times I} \|f\|_\infty |T_n^{\alpha_n}(x+y)|d\lambda(x)\right| \leq C\|f\|_\infty$$

as it comes from (2.4) and the fact that the L_1-norms of the Fejér kernels and also the Dirichlet kernels with indices of the form 2^m are uniformly bounded.

Lemma 2.6 is proved.

Now, we can prove the main tool in order to have Theorem 1.1 for operators

$$\sigma_*^\beta f := \sup_{n\in\mathbb{N}}\left|\sigma_{2^n}^{\beta_n} f\right| = \sup_{n\in\mathbb{N}}\left|f * K_{2^n}^{\beta_n}\right|$$

and

$$\tilde{\sigma}_*^\beta f := \sup_{n\in\mathbb{N}}\left|\tilde{\sigma}_{2^n}^{\beta_n} f\right| = \sup_{n\in\mathbb{N}}\left|f * |K_{2^n}^{\beta_n}|\right|.$$

Lemma 2.7. *The operators $\tilde{\sigma}_*^\beta$ and σ_*^β are of weak type (L_1, L_1).*

Proof. First, we prove Lemma 2.7 for operator $\tilde{\sigma}_*^\beta$. We apply the Calderon – Zygmund decomposition lemma [10]. That is, let $f \in L_1(I^2)$ and $\|f\|_1 < \eta$. Then there is a decomposition

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that $\|f_0\|_\infty \le C\eta$, $\|f_0\|_1 \le C\|f\|_1$ and $I^j \times I^j = I_{k_j}(u^{j,1}) \times I_{k_j}(u^{j,2})$ are disjoint dyadic rectangles for which

$$\operatorname{supp} f_j \subset I^j \times I^j, \quad \int\limits_{I^j\times I^j} f_j d\lambda = 0, \quad \lambda(F) \le \frac{C\|f_1\|}{\eta}, \quad (u^{j,1}, u^{j,2}) \in I \times I, \quad k_j \in \mathbb{N}, \quad j \in \mathbb{P},$$

where $F = \cup_{j=1}^{\infty} I^j \times I^j$. By the σ-sublinearity of the maximal operator with an appropriate constant C, we have

$$\lambda(\tilde{\sigma}_*^\beta f > 2C\eta) \le \lambda(\tilde{\sigma}_*^\beta f_0 > C\eta) + \lambda\left(\tilde{\sigma}_*^\beta \left(\sum_{i=1}^{\infty} f_i\right) > C_\eta\right) := I + II.$$

Notice that

$$K_{2^n}^{\beta_n}(x) = T_{2^n}^{\alpha_{2^n}}(x) + \frac{D_{2^n}(x^1)D_{2^n}(x^2)}{A_{2^n}^{\alpha_{2^n}}}$$

and keep in mind that operator $\sup_n |f * (D_{2^n} \times D_{2^n})|$ is quasilocal and it is of weak type (L_1, L_1) and it is also of type (L_p, L_p) for each $1 < p \le \infty$ [10]. Since by Lemma 2.6 $\|\tilde{\sigma}_*^\alpha f_0\|_\infty \le$ $\le C\|f_0\|_\infty \le C\eta$, then we have $I = 0$. So,

$$\lambda\left(\tilde{\sigma}_*^\beta \left(\sum_{i=1}^{\infty} f_i\right) > C\eta\right) \le \lambda(F) + \lambda\left(\bar{F} \cap \left\{\tilde{\sigma}_*^\beta \left(\sum_{i=1}^{\infty} f_i\right) > C\eta\right\}\right) \le$$

$$\le \frac{C\|f\|_1}{\eta} + \frac{C}{\eta}\sum_{i=1}^{\infty} \int\limits_{I^j\times I^j} \tilde{\sigma}_*^\beta f_j d\lambda =: \frac{C\|f\|_1}{\eta} + \frac{C}{\eta}\sum_{i=1}^{\infty} III_j,$$

where

$$III_j := \int\limits_{I^j\times I^j} \tilde{\sigma}_*^\beta f_j d\lambda =$$

$$= \int\limits_{I_{k_j}(u^j)\times I_{k_j}(u^j)} \sup_{n\in\mathbb{N}} \left| \int\limits_{I_{k_j}(u^j)\times I_{k_j}(u^j)} f_j(x) \left|K_{2^n}^{\beta_n}(y+x)\right| d\lambda(x^1, x^2) \right| d\lambda(y^1, y^2).$$

The forthcoming estimation of III_j is given by the help Lemma 2.6:

$$III_j \le C\|f_j\|_1.$$

That is, operator $\tilde{\sigma}_*^\beta$ is of weak type (L_1, L_1) and same holds for operator σ_*^β.

Lemma 2.7 is proved.

***Proof of Theorem* 1.1.** Let $P \in \mathbf{P}$ be a two-dimensional Walsh polynomial, that is, $P(x) =$ $= \sum_{i,j=0}^{2^k-1} c_{i,j}\omega_i(x^1)\omega_j(x^2)$. Then for all natural number $m \ge 2^k$ we have that $S_{m,m}P \equiv P$. Consequently, the statement $\sigma_{2^n}^{\beta_n} P \to P$ holds everywhere. This follows from the fact that for any fixed j it holds $\dfrac{A_{2^n-j}^{\beta_n-1}}{A_{2^n}^{\beta_n}} \to 0$ since, for instance, for $j = 1$ we have $\dfrac{A_{2^n-1}^{\beta_n-1}}{A_{2^n}^{\beta_n}} = \dfrac{\beta_n 2^n}{(2^n - 1 + \beta_n)(2^n + \beta_n)} \to 0.$

Now, let $\eta,\ \epsilon > 0,\ f \in L_1(I^2)$. Let $P \in \mathbf{P}$ be a two-dimensional Walsh polynomial such that $\|f - P\|_1 < \eta$. Then

$$\lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n} f - f\right| > \epsilon\right) \leq$$

$$\leq \lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n}(f - P)\right| > \frac{\epsilon}{3}\right) + \lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n} P - P\right| > \frac{\epsilon}{3}\right) + \lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n} P - f\right| > \frac{\epsilon}{3}\right) \leq$$

$$\leq \lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n}(f - P)\right| > \frac{\epsilon}{3}\right) + 0 + \frac{3}{\epsilon}\|P - f\|_1 \leq C\|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C}{\epsilon}\eta$$

because σ_*^β is of weak type (L_1, L_1). This holds for all $\eta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$\lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n} f - f\right| > \epsilon\right) = 0$$

and, consequently,

$$\lambda\left(\overline{\lim_{n\in\mathbb{N}}}\left|\sigma_{2^n}^{\beta_n} f - f\right| > 0\right) = 0.$$

This finally gives $\sigma_{2^n}^{\beta_n} f \longrightarrow f$ a.e.

Theorem 1.1 is proved.

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