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ON FOUR DIMENSIONAL PARACOMPLEX STRUCTURES WITH NORDEN METRICS

ПРО ЧОТИРИВИМІРНІ ПАРАКОМПЛЕКСНІ СТРУКТУРИ З МЕТРИКАМИ НОРДЕНА

We study the almost paracomplex structures with Norden metric on Walker 4-manifolds and try to find general solutions for the integrability of these structures on suitable local coordinates. We also discuss para-Kähler (paraholomorphic) conditions for these structures.

Вивчаються майже паракомплексні структури з метрикою Нордена на 4-многовидах Уолкера. Встановлено загальні розв'язки щодо інтегровності таких структур у відповідних локальних координатах. Також обговорюються паракелерові (параголоморфні) умови для таких структур.

1. Introduction. Let M_{2n} be a Riemannian manifold with a neutral metric, i.e., with a pseudo-Riemannian metric g of signature (n,n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p,q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

An almost paracomplex manifold is an almost product manifold (M_{2n},φ), $\varphi^2=id$, such that the two eigenbundles T^+M_{2n} and T^-M_{2n} associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the following set of affinors on $M_{2n}\colon\ \{id,\varphi\}$, $\varphi^2=id$, which form a bases of a representation of the algebra of order 2 over the field of real numbers R, which is called the algebra of paracomplex (or double) numbers and is denoted by $R(j)=\left\{a_0+a_1j : j^2=1; a_0,a_1\in R\right\}$. Obviously, it is associative, commutative and unitial, i.e., it admits principal unit 1. The canonical bases of this algebra has the form $\{1,j\}$.

Let (M_{2n},φ) be an almost paracomplex manifold with almost paracomplex structure φ. For almost paracomplex structure the integrability is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+[X,Y].$$

This structure is said to be integrable if the matrix $\varphi=(\varphi_j^i)$ is reduced to the constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x\in M_{2n}$. On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection such that $\nabla\varphi=0$. A paracomplex manifold is an almost paracomplex manifold (M_{2n},φ) such that the G-structure defined by the affinor field φ is integrable. We can give another-equivalent-definition of paracomplex manifold in terms of local homeomorphisms in the space $R^n(j)=\left\{(X^1,\ldots,X^n)\colon X^i\in R(j),\ i=1,\ldots,n\right\}$ and paraholomorphic changes of charts in a way similar to [2] (see also [6]), i.e., a manifold M_{2n} with an integrable paracomplex structure φ is a real realization of the paraholomorphic manifold $M_n(R(j))$ over the algebra $R(j)$.

1.1. Norden metrics. A metric g is a Norden metric [15] if

$$g(\varphi X, \varphi Y) = g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2n})$. Metrics of this kind have been also studied under the names: pure, anti-Hermitian and B-metric (see [5, 7, 12, 17, 23, 25]). If (M_{2n}, φ) is an almost paracomplex manifold with Norden metric g, we say that (M_{2n}, φ, g) is an almost para-Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a para-Norden manifold.

1.2. Paraholomorphic (almost paraholomorphic) tensor fields. Let $\overset{*}{t}$ be a paracomplex tensor field on $M_n(R(j))$. The real model of such a tensor field is a tensor field on M_{2n} of the same order that is independent of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [12, 18, 19, 23–25, 27]). In particular, being applied to a $(0, q)$-tensor field ω, the purity means that for any $X_1, \ldots, X_q \in \Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \ldots = \omega(X_1, X_2, \ldots, \varphi X_q).$$

We define an operator

$$\Phi_\varphi : \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to the pure tensor field ω by (see [27])

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = (\varphi X)(\omega(Y_1, Y_2, \ldots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \ldots, Y_q)) +$$

$$+\omega((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q) + \ldots + \omega(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X),$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a paracomplex structure on M_{2n} and the tensor field $\Phi_\varphi \omega$ vanishes, the paracomplex tensor field $\overset{*}{\omega}$ on $M_n(R(j))$ is said to be paraholomorphic (see [12, 23, 27]). Thus a paraholomorphic tensor field $\overset{*}{\omega}$ on $M_n(R(j))$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = 0$$

for any $X, Y_1, \ldots, Y_q \in \Im_0^1(M_{2n})$. Therefore such a tensor field ω on M_{2n} is also called paraholomorphic tensor field. When φ is an almost paracomplex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi \omega = 0$ is said to be almost paraholomorphic.

1.3. Paraholomorphic Norden (para-Kähler–Norden) metrics. In a para-Norden manifold a para-Norden metric g is called a *paraholomorphic* if

$$(\Phi_\varphi g)(X, Y, Z) = 0 \tag{1}$$

for any $X, Y, Z \in \Im_0^1(M_{2n})$.

By setting $X = \partial_k,\ Y = \partial_i,\ Z = \partial_j$ in the equation (1), we see that the components $(\Phi_\varphi g)_{kij}$ of $\Phi_\varphi g$ with respect to a local coordinate system $x^1, \ldots, x^n$ may be expressed as follows:

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}\left(\partial_i \varphi_k^m - \partial_k \varphi_i^m\right) + g_{im} \partial_j \varphi_k^m.$$

If (M_{2n}, φ, g) is a para-Norden manifold with paraholomorphic Norden metric g, we say that (M_{2n}, φ, g) is a *paraholomorphic Norden manifold*.

In some aspects, paraholomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi – Civita connection.

Theorem 1 [21] (for complex version see [10])**.** *For an almost paracomplex manifold with para-Norden metric g, the condition $\Phi_\varphi g = 0$ is equivalent to $\nabla \varphi = 0$, where ∇ is the Levi – Civita connection of g.*

A para-Kähler – Norden manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost paracomplex structure φ and a pseudo-Riemannian metric g such that $\nabla \varphi = 0$, where ∇ is the Levi – Civita connection of g and the metric g is assumed to be para-Nordenian. Therefore, there exist a one-to-one correspondence between para-Kähler – Norden manifolds and para-Norden manifolds with a paraholomorphic metric. Recall that in such a manifold, the Riemannian curvature tensor is pure and paraholomorphic, also the curvature scalar is locally paraholomorphic function (see [10, 17]).

***Remark* 1.** We know that the integrability of the almost paracomplex structure φ is equivalent to the existing a torsion-free affine connection with respect to which the equation $\nabla \varphi = 0$ holds. Since the Levi – Civita connection ∇ of g is a torsion-free affine connection, we have: if $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost para-Norden manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i.e., almost paraholomorphic Norden manifolds (analogues of the almost para-Kähler manifolds with closed para-Kähler form) does not exist.

2. Walker metrics in dimension four. A neutral metric g on a 4-manifold M_4 is said to be Walker metric if there exists a 2-dimensional null distribution D on M_4, which is parallel with respect to g. For such metrics a canonical form has been obtained by Walker [26], showing the existence of suitable coordinates (x^1, x^2, x^3, x^4) around any point of M_4 where the metric expresses as

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix},$$

for some functions $a,\ b$ and c depending on the coordinates (x^1, x^2, x^3, x^4). Note that $D = \mathrm{span}\,\{\partial_1, \partial_2\} \left(\partial_i = \dfrac{\partial}{\partial x^i}\right)$. For an application of such a 4-dimensional Walker metric (see [9]). Since the observation of the existence of almost paracomplex structures on Walker 4-manifolds in a paper [20], the Walker 4-manifolds have been intensively studied, e.g., [1, 3, 4, 8, 13, 14, 16, 20, 22].

As in a resent paper [15], we shall study throughout this paper the following Walker metrics of restricted type $(c = 0)$:

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{pmatrix}. \tag{2}$$

3. Almost paracomplex structure φ in the case of $c = 0$. A natural way to construct of an almost paracomplex structure φ on a neutral 4-manifold is as follows: choose a local orthonormal basis $\{e_i\}$, $i = 1, \ldots, 4$, so that with respect to the basis the neutral metric becomes the standard form

$$g = (g(e_i, e_j)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and then define φ by

$$\varphi e_1 = e_2, \quad \varphi e_2 = e_1, \quad \varphi e_3 = e_4, \quad \varphi e_4 = e_3. \tag{3}$$

We consider the Walker metrics with $c = 0$ as follows:

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{pmatrix}, \tag{4}$$

where a and b are functions of suitable coordinates (x^1, x^2, x^3, x^4) around any point of M_4. In this case, we find a local orthonormal basis $\{e_1, e_2, e_3, e_4\}$ [14] ((14)), as follows:

$$\begin{aligned} e_1 &= \frac{1}{\sqrt[4]{a^2+4}} \left\{ \frac{1}{2}(\sqrt{a^2+4} - a)\partial_1 + \partial_3 \right\}, \\ e_2 &= \frac{1}{\sqrt[4]{b^2+4}} \left\{ \frac{1}{2}(\sqrt{b^2+4} - b)\partial_2 + \partial_4 \right\}, \\ e_3 &= \frac{1}{\sqrt[4]{a^2+4}} \left\{ -\frac{1}{2}(\sqrt{a^2+4} + a)\partial_1 + \partial_3 \right\}, \\ e_4 &= \frac{1}{\sqrt[4]{b^2+4}} \left\{ -\frac{1}{2}(\sqrt{b^2+4} + b)\partial_2 + \partial_4 \right\}. \end{aligned} \tag{5}$$

For the Walker metric (2) with $c = 0$, the dual basis $\{e^1, e^2, e^3, e^4\}$ of 1-forms to the basis (5) of vectors is given by [14] ((19))

$$e^1 = \frac{1}{\sqrt[4]{a^2+4}} \left\{ dx^1 + \frac{1}{2}(\sqrt{a^2+4} + a)dx^3 \right\},$$

$$e^2 = \frac{1}{\sqrt[4]{b^2+4}} \left\{ dx^2 + \frac{1}{2}(\sqrt{b^2+4}+b)dx^4 \right\},$$

$$e^3 = -\frac{1}{\sqrt[4]{a^2+4}} \left\{ dx^1 - \frac{1}{2}(\sqrt{a^2+4}-a)dx^3 \right\},$$

$$e^4 = -\frac{1}{\sqrt[4]{b^2+4}} \left\{ dx^2 - \frac{1}{2}(\sqrt{b^2+4}-b)dx^4 \right\}.$$

We now put $K = \sqrt[4]{(b^2+4)/(a^2+4)}$. The almost paracomplex structures defined by (3) is written explicitly as follows:

$$\varphi = e_1 \otimes e^2 + e_2 \otimes e^1 + e_3 \otimes e^4 + e_4 \otimes e^3 =$$

$$= \begin{pmatrix} 0 & \frac{1}{K} & 0 & \frac{1}{2}\left(\frac{b}{K} - aK\right) \\ K & 0 & \frac{1}{2}\left(aK - \frac{b}{K}\right) & 0 \\ 0 & 0 & 0 & K \\ 0 & 0 & \frac{1}{K} & 0 \end{pmatrix}, \tag{6}$$

where these matrices are written with respect to the coordinate basis. In this case, the triple (M_4, φ, g) is called almost para-Norden – Walker manifold.

4. φ-Integrability (para-Norden structures). If we write as $\varphi\partial_i = \sum_{j=1}^4 \varphi_i^j \partial_j$, then from (6) we can read off the nonzero components φ_i^j as follows:

$$\begin{gathered} \varphi_1^2 = K, \qquad \varphi_2^1 = \frac{1}{K}, \qquad \varphi_3^2 = \frac{1}{2}\left(aK - \frac{b}{K}\right), \\ \varphi_3^4 = \frac{1}{K}, \qquad \varphi_4^1 = \frac{1}{2}\left(\frac{b}{K} - aK\right), \qquad \varphi_4^3 = K. \end{gathered} \tag{7}$$

The almost paracomplex structure φ is integrable if and only if the torsion of φ (Nijenhuis tensor) vanishes, or equivalently the following components:

$$(N_\varphi)_{jk}^i = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m$$

all vanish (cf. [12, p. 124]), where φ_i^j are given by (7). By explicit calculation, we find the φ-integrability condition as follows.

Theorem 2. *The almost paracomplex structure φ on almost para-Norden – Walker manifolds is integrable if and only if the following PDE's hold:*

$$K_1 = 0, \quad K_2 = 0, \quad K^2 a_1 - b_1 - 2KK_3 = 0, \quad K^2 a_2 - b_2 - \frac{2}{K} K_4 = 0. \tag{8}$$

In this note, we will try to find the general solutions according to suitable local coordinates for the PDE's in above theorem.

Theorem 3. *The almost paracomplex structure φ on almost para-Norden – Walker manifolds is integrable if and only if a and b satisfy one of the following*:

type φ_A : $K = \left\{(b^2+4)/(a^2+4)\right\}^{1/4} (= C)$ *is constant, and*

$$a = a\left(x^3, x^4\right), \qquad b = b\left(x^3, x^4\right),$$

type φ_B : $a\left(x^1, x^2, x^3, x^4\right) = b\left(x^1, x^2, x^3, x^4\right)$ (*necessarily* $K = 1$),

$$\textit{and either} \quad a_1 \neq 0 \quad \textit{or} \quad a_2 \neq 0,$$

type φ_C : $K = \left\{(b^2+4)/(a^2+4)\right\}^{1/4}$*is not constant, and*

$$\begin{aligned} a\left(x^1, x^2, x^3, x^4\right) &= \psi^{-2}\left(\psi L + \phi - \frac{\psi^4 - 1}{\psi L + \phi}\right), \\ b\left(x^1, x^2, x^3, x^4\right) &= -\psi L - \phi - \frac{\psi^4 - 1}{\psi L + \phi}, \end{aligned} \tag{9}$$

with

$$L = L\left(x^1, x^2, x^3, x^4\right) = \psi_3 x^1 + \psi^{-2}\psi_4 x^2, \tag{10}$$

for a and b functions defined according to suitable coordinates values of (x^1, x^2, x^3, x^4), $\psi = \psi(x^3, x^4)$ and $\phi = \phi(x^3, x^4)$ are smooth functions of x^3 and x^4 such that $\psi(x^3, x^4) \neq 0$ and $\phi(x^3, x^4) \neq 0$ for suitable points (x^3, x^4), and $x^1 \neq -\dfrac{\psi_4}{\psi^2\psi_3}x^2 - \dfrac{\phi}{\psi\psi_3}$ must be satisfied for suitable points (x^1, x^2). Moreover, the function K depends only on x^3, x^4, and coincides with ψ.

We shall prove this theorem in three steps.

Proof. *The first step*: From the former two equations in (8), we must note that $K = \left\{(b^2+4)/(a^2+4)\right\}^{1/4}$ does not depend on x^1, x^2, and further that the latter two can be written as follows:

$$(K^2a - b)_1 = 2KK_3, \quad (K^2a - b)_2 = \frac{2}{K}K_4. \tag{11}$$

Integrating these equations with respect to x^1 and x^2, respectively, we have

$$\begin{aligned} K^2a - b &= 2KK_3x^1 + p^A(x^2, x^3, x^4), \\ K^2a - b &= \frac{2}{K}K_4x^2 + p^B(x^1, x^3, x^4), \end{aligned} \tag{12}$$

where p^A and p^B are arbitrary functions of x^2, x^3, x^4 and of x^1, x^3, x^4, respectively. Differentiating the former equation by x^2, then we have $(K^2a - b)_2 = p^A_2(x^2, x^3, x^4) = \dfrac{2}{K}K_4$, and hence $p^A(x^2, x^3, x^4) = \dfrac{2}{K}K_4x^2 + q^A(x^3, x^4)$. For p^B, similarly we can write $p^B(x^1, x^3, x^4) = 2KK_3x^1 + q^B(x^3, x^4)$. Using these p^A and p^B in (12), we see that q^A and q^B coincide with each other, and denote them by $2f(x^3, x^4)$. In fact, we obtain

$$K^2a - b = 2KK_3x^1 + \frac{2}{K}K_4x^2 + 2f(x^3, x^4). \tag{13}$$

From the relation $K = \left\{(b^2+4)/(a^2+4)\right\}^{1/4}$, we get

$$(K^4)_1 = 0 \Rightarrow bb_1(a^2+4) = aa_1(b^2+4),$$

$$(K^4)_2 = 0 \Rightarrow bb_2(a^2+4) = aa_2(b^2+4),$$

and hence

$$K^4 = \frac{b^2+4}{a^2+4} = \frac{bb_1}{aa_1} = \frac{bb_2}{aa_2}. \tag{14}$$

(End of the first step.)

In the subsequent steps of the proof, we divide the situation into two cases as follows:

Case I: K is constant.

Case II: K depends only on x^3 and x^4.

The second step: We consider here the first Case I: K is constant, denoted by $K = C$, i.e., $C^4(a^2+4) = b^2+4$. In this case, the equations (11) reduce to

$$(C^2a - b)_i = (C^2a \pm \sqrt{C^4(a^2+4)-4})_i = 0, \quad i = 1, 2.$$

There are two types of solutions to these equations as follows:

i) $C^2a - b = 0$, where a and b can be functions of x^1, x^2, x^3 and x^4, or

ii) $a_1 = a_2 = b_1 = b_2 = 0$.

For i), in fact, the relation $C^2a - b = 0$ together with $C^4(a^2+4) = b^2+4$ implies that $K = C = 1$, and that $a\left(x^1, x^2, x^3, x^4\right) = b\left(x^1, x^2, x^3, x^4\right)$ which is of type φ_B.

It is easy to see that if $C^2a - b \neq 0$, then there is another possibility of the second case ii) $a_1 = a_2 = b_1 = b_2 = 0$. Therefore, if K is constant $(K = C)$ (including $K = 1$), then $a = a\left(x^3, x^4\right)$ and $b = b\left(x^3, x^4\right)$ are solutions to (8). Such solutions are of type φ_A. Here, we must note that such a and b are subject to a relation $C^4(a^2+4) = b^2+4$. (End of the second step.)

The third step: In this final step, we consider the Case II: K is independent of x^1 and x^2. From (14), we have $K^2aa_1 = \dfrac{1}{K^2}bb_1$, and add $-ba_1$ both sides of it. Then, we have

$$(K^2a - b)a_1 = -\frac{b}{K^2}(K^2a - b)_1.$$

From the former equation in (11), we obtain

$$(K^2a - b)a_1 = -\frac{2K_3}{K}b = \frac{2K_3}{K}\left\{-K^2a + (K^2a - b)\right\}.$$

Using (13), we get

$$\left(KK_3x^1 + \frac{1}{K}K_4x^2 + f\right)a_1 + KK_3a = \frac{2K_3}{K}\left(KK_3x^1 + \frac{1}{K}K_4x^2 + f\right). \tag{15}$$

From a similar calculation, we have an analogous equation for x^2 as follows:

$$\left(KK_3x^1 + \frac{1}{K}K_4x^2 + f\right)a_2 + \frac{1}{K}K_4a = \frac{2K_4}{K^3}\left(KK_3x^1 + \frac{1}{K}K_4x^2 + f\right). \tag{16}$$

At this stage, we recall that for a function $y(t)$ of single argument t, an ODE of the form

$$(\alpha t + \beta)\frac{dy(t)}{dt} + \alpha y(t) = \gamma t + \delta \quad (\alpha,\ \beta, \gamma,\ \delta \text{ — constants})$$

has a solution $y(t) = \dfrac{\frac{1}{2}\gamma t^2 + \delta t + \alpha C}{\alpha t + \beta}$ (C — constant). If we regard the equation (15) as such an ODE with respect to x^1, with x^2, x^3, x^4 as parameters, then we have its solution as follows:

$$a = \frac{K_3^2\left(x^1\right)^2 + \dfrac{2K_3K_4}{K^2}x^1x^2 + \dfrac{2}{K}K_3fx^1 + KK_3h^A(x^2, x^3, x^4)}{KK_3x^1 + \dfrac{1}{K}K_4x^2 + f}.$$

In a similar way, we can obtain a solution to (16) as follows:

$$a = \frac{\dfrac{1}{K^4}K_4^2\left(x^2\right)^2 + \dfrac{2K_3K_4}{K^2}x^1x^2 + \dfrac{2}{K^3}K_4fx^2 + \dfrac{K_4}{K}h^B(x^1, x^3, x^4)}{KK_3x^1 + \dfrac{1}{K}K_4x^2 + f}.$$

In the above two equations, h^A and h^B are arbitrary functions of x^2, x^3, x^4 and of x^1, x^3, x^4, respectively. Comparing the above two solutions for a, we have

$$KK_3h^A(x^2, x^3, x^4) = \frac{1}{K^4}K_4^2\left(x^2\right)^2 + \frac{2}{K^3}K_4fx^2 + h(x^3, x^4),$$

$$\frac{K_4}{K}h^B(x^1, x^3, x^4) = \frac{2}{K}K_3fx^1 + K_3^2\left(x^1\right)^2 + h(x^3, x^4),$$

where $h(x^3, x^4)$ is an arbitrary function of x^3, x^4. Therefore, we see that a is written as

$$a = \frac{K_3^2\left(x^1\right)^2 + \dfrac{2K_3K_4}{K^2}x^1x^2 + \dfrac{1}{K^4}K_4^2\left(x^2\right)^2 + \dfrac{2}{K}K_3fx^1 + \dfrac{2}{K^3}K_4fx^2 + h}{KK_3x^1 + \dfrac{1}{K}K_4x^2 + f} =$$

$$= \frac{1}{K^2}\left(KK_3x^1 + \frac{1}{K}K_4x^2 + f - \frac{f^2 - K^2h}{KK_3x^1 + \dfrac{1}{K}K_4x^2 + f}\right).$$

From this expression for a, we can obtain, with (13), the explicit form of the function b as well as a:

$$b = K^2a - 2KK_3x^1 - \frac{2}{K}K_4x^2 - 2f(x^3, x^4) =$$

$$= -KK_3x^1 - \frac{1}{K}K_4x^2 - f - \frac{f^2 - K^2h}{KK_3x^1 + \dfrac{1}{K}K_4x^2 + f}.$$

These expressions for a and b contain two arbitrary functions f and h of x^3 and x^4. Taking into account of $K = \left\{(b^2+4)/(a^2+4)\right\}^{1/4}$ for the above solutions a and b, we can see that there is a relation among f, h and K as follows:

$$f^2 - K^2h = K^4 - 1.$$

At the final stage of the proof, we will arrange the expressions for a, b so that they look simple. Keeping the last expression in mind, we can regard K as one of arbitrary functions with arguments x^3, x^4, instead of h. Then, we denote $K(x^3, x^4)$ by a new symbol $\psi = \psi(x^3, x^4)$, and also put $\phi = \phi(x^3, x^4) = f(x^3, x^4)$. If we write $L = K_3x^1 + \dfrac{1}{K^2}K_4x^2 = \psi_3x^1 + \psi^{-2}\psi_4x^2$ as in (10), we have arrived at the desired expressions as in (9). Also, for a and b functions defined according to suitable coordinates values of (x^1, x^2, x^3, x^4), $\psi = \psi(x^3, x^4)$ and $\phi = \phi(x^3, x^4)$ must be smooth functions of x^3 and x^4 such that $\psi(x^3, x^4) \neq 0$ and $\phi(x^3, x^4) \neq 0$ for suitable points (x^3, x^4), and $x^1 \neq -\dfrac{\psi_4}{\psi^2\psi_3}x^2 - \dfrac{\phi}{\psi\psi_3}$ must be satisfied for suitable points (x^1, x^2). Such a case is classified as type φ_C. (End of the third step.)

Theorem 3 is proved.

5. Paraholomorphic Norden – Walker (para-Kähler – Norden – Walker) metrics on (M_4, φ, g). Let (M_4, φ, g) be an almost para-Norden – Walker manifold. If

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im}\partial_j \varphi_k^m = 0, \tag{17}$$

then by virtue of Theorem 1 φ is integrable and the triple (M_4, φ, g) is called a paraholomorphic Norden – Walker or a para-Kähler – Norden – Walker manifold. Taking account of Remark 1, we see that almost para-Kähler – Norden – Walker manifold with condition $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$ does not exist.

We will write (4) and (7) in (17). By explicit calculation, we have the following theorem.

Theorem 4. *The triple (M_4, φ, g) is para-Kähler – Norden – Walker if and only if the following PDEs hold*:

$$K_1 = 0, \quad K_2 = 0, \quad a_2 = a_4 = b_1 = b_3 = 0, \quad Ka_1 - 2K_3 = 0, \quad Kb_2 + 2K_4 = 0.$$

***Remark* 2.** In a recent paper [20], a proper almost paracomplex structure on a Walker 4-manifold is defined and analyzed. The almost paracomplex structure φ defined in (6) coincides with that defined in [20] ((3)) in each of the cases (a) $c = 0$ and $a = b$, and case (b) $c = 0$ and $a = -b$. Note that in the former case (a), φ is integrable (cf. [20], Theorem 2). In fact, such happens in the following two situations:

i) $a\left(x^3, x^4\right) = b\left(x^3, x^4\right)$ in type φ_A (in Theorem 3),

ii) $a\left(x^1, x^2, x^3, x^4\right) = b\left(x^1, x^2, x^3, x^4\right)$ in type φ_B (in Theorem 3).

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