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ON THE STRONG LAW OF LARGE NUMBERS FOR φ-SUB-GAUSSIAN RANDOM VARIABLES

ПРО ПОСИЛЕНИЙ ЗАКОН ВЕЛИКИХ ЧИСЕЛ ДЛЯ φ-СУБГАУССОВИХ ВИПАДКОВИХ ВЕЛИЧИН

For $p \geq 1$ let $\varphi_p(x) = x^2/2$ if $|x| \leq 1$ and $\varphi_p(x) = 1/p|x|^p - 1/p + 1/2$ if $|x| > 1$. For a random variable ξ let $\tau_{\varphi_p}(\xi)$ denote $\inf\{a \geq 0 : \forall_{\lambda \in \mathbb{R}} \ \ln \mathbb{E} \exp(\lambda\xi) \leq \varphi_p(a\lambda)\}$; τ_{φ_p} is a norm in a space $\mathrm{Sub}_{\varphi_p} = \{\xi : \tau_{\varphi_p}(\xi) < \infty\}$ of φ_p-sub-Gaussian random variables. We prove that if for a sequence $(\xi_n) \subset \mathrm{Sub}_{\varphi_p}$, $p > 1$, there exist positive constants c and α such that for every natural number n the inequality $\tau_{\varphi_p}\Big(\sum_{i=1}^n \xi_i\Big) \leq cn^{1-\alpha}$ holds, then $n^{-1}\sum_{i=1}^n \xi_i$ converges almost surely to zero as $n \to \infty$. This result is a generalization of the strong law of large numbers for independent sub-Gaussian random variables [see R. L. Taylor, T.-C. Hu, *Sub-Gaussian techniques in proving strong laws of large numbers*, Amer. Math. Monthly, **94**, 295 – 299 (1987)] to the case of dependent φ_p-sub-Gaussian random variables.

Нехай для $p \geq 1$ $\varphi_p(x) = x^2/2$, якщо $|x| \leq 1$, і $\varphi_p(x) = 1/p|x|^p - 1/p + 1/2$, якщо $|x| > 1$. Для випадкової величини ξ нехай $\tau_{\varphi_p}(\xi)$ позначає $\inf\{a \geq 0 : \forall_{\lambda \in \mathbb{R}} \ \ln \mathbb{E} \exp(\lambda\xi) \leq \varphi_p(a\lambda)\}$; τ_{φ_p} — норма у просторі $\mathrm{Sub}_{\varphi_p} = \{\xi : \tau_{\varphi_p}(\xi) < \infty\}$ φ_p-субгауссових випадкових величин. У цій роботі доведено наступне: якщо для послідовності $(\xi_n) \subset \mathrm{Sub}_{\varphi_p}$, $p > 1$, існують додатні сталі c і α такі, що для будь-якого натурального числа n виконується нерівність $\tau_{\varphi_p}\Big(\sum_{i=1}^n \xi_i\Big) \leq cn^{1-\alpha}$, то $n^{-1}\sum_{i=1}^n \xi_i$ збігається майже напевно до нуля при $n \to \infty$. Цей результат узагальнює посилений закон великих чисел для незалежних субгауссових випадкових величин [див. R. L. Taylor, T.-C. Hu, *Sub-Gaussian techniques in proving strong laws of large numbers*, Amer. Math. Monthly, **94**, 295 – 299 (1987)] у випадку, коли розглядаються залежні φ_p-субгауссові випадкові величини.

1. Introduction. The classical Kolmogorov strong laws of large numbers are dealt with independent variables. Investigations of limit theorems for dependent random variables are extensive and episodic. The strong law of large numbers (SLLN) for various classes of many type associated random variables one can find for instance in Bulinski and Shashkin [2] (Ch. 4). Most of them are considered in the spaces of integrable functions. It is also interested to describe general conditions under which the SLLN holds in other spaces of random variables than L_p-spaces. In this paper we investigate almost sure convergence of the arithmetic mean (but not only) sequences of φ-sub-Gaussian random variables.

The notion of sub-Gaussian random variables was introduced by Kahane in [8]. A random variable ξ is *sub-Gaussian* if its moment generating function is majorized by the moment generating function of some centered Gaussian random variable with variance σ^2 that is $\mathbb{E}\exp(\lambda\xi) \leq \leq \mathbb{E}\exp(\lambda g) = \exp(\sigma^2\lambda^2/2)$, where $g \sim \mathcal{N}(0, \sigma^2)$ (see [4] or [3], Ch. 1). In terms of the cumulant generating functions this condition takes a form $\ln \mathbb{E}\exp(\lambda\xi) \leq \sigma^2\lambda^2/2$.

One can generalize the notion of sub-Gaussian random variables to classes of φ-sub-Gaussian random variables (see [3], Ch. 2). A continuous even convex function $\varphi(x)$ $(x \in \mathbb{R})$ is called a *N-function*, if the following condition holds:

(a) $\varphi(0) = 0$ and $\varphi(x)$ is monotone increasing for $x > 0$,

(b) $\lim_{x\to 0} \varphi(x)/x = 0$ and $\lim_{x\to\infty} \varphi(x)/x = \infty$.

It is called a *quadratic N-function*, if in addition $\varphi(x) = cx^2$ for all $|x| \le x_0$ with $c > 0$ and $x_0 > 0$. The quadratic condition is needed to ensure nontriviality for classes of φ-sub-Gaussian random variables (see [3, p. 67]).

***Example* 1.1.** Let for $p \ge 1$

$$\varphi_p(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if} \quad |x| \le 1, \\ \dfrac{1}{p}|x|^p - \dfrac{1}{p} + \dfrac{1}{2}, & \text{if} \quad |x| > 1. \end{cases}$$

The function φ_p is an example of the quadratic N-function which is some standardization of the function $|x|^p$ (see [10], Lemma 2.5). Let us emphasize that for $p = 2$ we have the case of sub-Gaussian random variables.

Let φ be a quadratic N-function. A random variable ξ is said to be *φ-sub-Gaussian* if there is a constant $a > 0$ such that $\ln \mathbb{E} \exp(\lambda\xi) \le \varphi(a\lambda)$. The *$\varphi$-sub-Gaussian standard (norm)* $\tau_\varphi(\xi)$ is defined as

$$\tau_\varphi(\xi) = \inf\big\{a \ge 0 \colon \; \forall_{\lambda\in\mathbb{R}} \; \ln \mathbb{E} \exp(\lambda\xi) \le \varphi(a\lambda)\big\};$$

a space $\mathrm{Sub}_\varphi = \{\xi \colon \tau_\varphi(\xi) < \infty\}$ with the norm τ_φ is a Banach space (see [3], Ch. 2, Theorem 4.1)

Let $\varphi(x)$, $x \in \mathbb{R}$, be a real-valued function. The function $\varphi^*(y)$ $(y \in \mathbb{R})$ defined by $\varphi^*(y) = \sup_{x\in\mathbb{R}}\{xy - \varphi(x)\}$ is called the *Young–Fenchel transform* or the *convex conjugate* of φ (in general, φ^* may take value ∞). It is known that if φ is a quadratic N-function, then φ^* is quadratic N-function too. For instance, since our φ_p is a differentiable (even at ± 1) function one can easy check (see [10], Lemma 2.6) that $\varphi_p^* = \varphi_q$ for $p, q > 1$, if $1/p + 1/q = 1$.

***Remark* 1.1.** One can define the space Sub_{φ_p} by using the Luxemburg norm of the form

$$\|\xi\|_{\psi_q} = \inf\big\{K > 0 \colon \mathbb{E} \exp|\xi/K|^q \le 2\big\}, \quad q = p/(p-1),$$

and then $\mathrm{Sub}_{\varphi_p} = \{\xi \colon \|\xi\|_{\psi_q} < \infty \text{ and } \mathbb{E}\xi = 0\}$ (compare [10], Theorem 2.7), the space $L^0_{\psi_q} = \mathrm{Sub}_{\varphi_p}$). Note that $\|\mathbb{E}\xi\|_{\psi_q} = \|\xi\|_{\psi_q}$, and we get that if $\|\xi\|_{\psi_q} < \infty$, then $\xi - \mathbb{E}\xi \in \mathrm{Sub}_{\varphi_p}$.

***Example* 1.2.** The standard normal random variable g belongs to Sub_{φ_2} and $\tau_{\varphi_2}(g) = 1$, since

$$\mathbb{E} \exp(tg) = \exp\big(t^2/2\big) = \exp(\varphi_2(t)).$$

Because g^2 has χ_1^2-distribution with one degree of freedom whose moment generating function is $\mathbb{E} \exp(tg) = (1 - 2t)^{-1/2}$ for $t < 1/2$, then

$$\mathbb{E} \exp\big(g^2/K^2\big) = \big(1 - 2/K^2\big)^{-1/2},$$

which is less or equal 2 if $K \ge \sqrt{8/3}$. It gives that $\|\xi\|_{\psi_2} = \sqrt{8/3}$. Let us observe that ψ_2-norm of $|g|$ is equal to ψ_2-norm of g. It implies that $|g| - \mathbb{E}|g| \in \mathrm{Sub}_{\varphi_2}$. Similarly as above one can show that

$$\||g|^{2/q}\|_{\psi_q} = \inf\big\{K > 0; \; \mathbb{E} \exp(g^2/K^q) \le 2\big\} = (8/3)^{1/q} < \infty.$$

Thus we get that $|g|^{2/q} - \mathbb{E}|g|^{2/q} \in \mathrm{Sub}_{\varphi_p}$, where $1/p + 1/q = 1$.

Let us recall that the convex conjugate is order-reversing and possesses some scaling property. If $\varphi_1 \geq \varphi_2$, then $\varphi_1^* \leq \varphi_2^*$. Let for $a > 0$ and $b \neq 0$ $\psi(x) = a\varphi(bx)$, then $\psi^*(y) = a\varphi^*(y/(ab))$ (see, e.g., [6], Ch. X, Proposition 1.3.1).

The convex conjugate of the cumulant generating function can be served to estimate of 'tails' distribution of a centered random variable. Let $\mathbb{E}\xi = 0$ and ψ_ξ denote the cumulant generating function of ξ, i.e., $\psi_\xi(\lambda) = \ln \mathbb{E}\exp(\lambda\xi)$, then for $\varepsilon > 0$

$$\mathbb{P}(\xi \geq \varepsilon) \leq \exp(-\psi_\xi^*(\varepsilon)).$$

Let us observe that for $\xi \in \mathrm{Sub}_\varphi$, by the definition of $\tau_\varphi(\xi)$, we have the inequality $\psi_\xi(\lambda) \leq \leq \varphi(\tau_\varphi(\xi)\lambda)$ and, by the order-reversing and the scaling property, we get $\psi_\xi^*(\varepsilon) \geq \varphi^*(\varepsilon/\tau_\varphi(\xi))$.

Now we can obtain some weaker form of the above estimation but with using the general function φ:

$$\mathbb{P}(|\xi| \geq \varepsilon) \leq 2\exp\left(-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right) \tag{1}$$

(see [3], Ch. 2, Lemma 4.3).

2. Results. First we show that if we have some upper estimate for τ_φ, then in (1) we can substitute this estimate instead of τ_φ.

Lemma 2.1. *If $\tau_\varphi(\xi) \leq C(\xi)$ for every $\xi \in \mathrm{Sub}_\varphi$, then*

$$\mathbb{P}(|\xi| \geq \varepsilon) \leq 2\exp\left(-\varphi^*\left(\frac{\varepsilon}{C(\xi)}\right)\right).$$

Proof. Since φ is even and increasing monotonic for $x > 0$, we get

$$\varphi(\tau_\varphi(\xi)x) = \varphi(\tau_\varphi(\xi)|x|) \leq \varphi(C(\xi)|x|) = \varphi(C(\xi)x).$$

And again by the order-reversing and the scaling property we obtain

$$\varphi^*\left(\frac{y}{\tau_\varphi(\xi)}\right) \geq \varphi^*\left(\frac{y}{C(\xi)}\right),$$

which combined with (1) establishes the inequality.

With these preliminaries accounted for, we can prove the main result of the paper.

Theorem 2.1. *Let $(\xi_n) \subset \mathrm{Sub}_{\varphi_p}$ for some $p > 1$. If there exist positive constants c and α such that for every natural number n the condition $\tau_{\varphi_p}\left(\sum_{i=1}^n \xi_i\right) \leq cn^{1-\alpha}$ holds, then the term $n^{-1}\sum_{i=1}^n \xi_i$ converges almost surely to zero as $n \to \infty$.*

Proof. Since $\varphi_p^* = \varphi_q$, by Lemma 2.1 and the condition of the theorem we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n \xi_i\right| \geq n\varepsilon\right) \leq 2\exp\left(-\varphi_q\left(\frac{n^\alpha\varepsilon}{c}\right)\right).$$

For sufficiently large n ($n > (c/\varepsilon)^{1/\alpha}$) we have $n^\alpha\varepsilon/c > 1$ and, in consequence,

$$\varphi_q\left(\frac{n^\alpha\varepsilon}{c}\right) = n^{q\alpha}\frac{1}{q}\left(\frac{\varepsilon}{c}\right)^q - \frac{1}{q} + \frac{1}{2}.$$

Thus, we get the estimate

$$\mathbb{P}\left(\left|\sum_{i=1}^{n}\xi_i\right| \geq n\varepsilon\right) \leq 2\exp\left(\frac{1}{q}-\frac{1}{2}\right)\exp\left(-n^{q\alpha}\frac{1}{q}\left(\frac{\varepsilon}{c}\right)^{q}\right)$$

for every ε and $n > (c/\varepsilon)^{1/\alpha}$. Thus, by the integral test, we obtain convergence of the series $\sum_{n=1}^{\infty}\mathbb{P}\left(\left|\sum_{i=1}^{n}\xi_i\right| \geq n\varepsilon\right)$. It follows the completely and, in consequence, almost sure convergence of $n^{-1}\sum_{i=1}^{n}\xi_i$ to zero.

***Remark* 2.1.** Let us emphasize that the above theorem is a generalization of the theorem (SLLN) (see [9, p. 297]) to the case of φ_p-sub-Gaussian random variables, not only sub-Gaussian ones. Moreover, we do not assume their independence. For this reason we used a modified condition for a behavior of the norm τ_p than Taylor and Hu, which describes below.

Since τ_φ is a norm, we obtain

$$\tau_\varphi\left(\sum_{i=1}^{n}\xi_i\right) \leq \sum_{i=1}^{n}\tau_\varphi(\xi_i).$$

If for instance ξ_i, $i = 1,\ldots,n$, are copies of the same variable ξ, then in the above the equality holds and $\tau_\varphi\left(\sum_{i=1}^{n}\xi_i\right) = n\tau_\varphi(\xi)$. Let us observe that in this case the assumption of Theorem 2.1 is not satisfied. Additionally informations about form of dependence (or independence) sometime allow us to improve this estimate. So, for an independence sequence (ξ_n), if there is some $r \in (0,2]$ such that $\varphi(|x|^{1/r})$ is convex, then

$$\tau_\varphi\left(\sum_{i=1}^{n}\xi_i\right)^{r} \leq \sum_{i=1}^{n}\tau_\varphi(\xi_i)^{r} \tag{2}$$

(see [3], Sec. 2, Theorem 5.2). If r is bigger then the estimate is better. For the function φ_p we can always take $r = \min\{p,2\}$. In Taylor's and Hu's SLLN variables ξ_n were sub-Gaussian and independent and it was taken $p = 2$. Let us emphasize that in this case if, in addition, $\xi_1,\ldots,\xi_n$, have the same distribution as ξ then $\tau_\varphi\left(\sum_{i=1}^{n}\xi_i\right) \leq \sqrt{n}\tau_\varphi(\xi)$ and the condition of Theorem 2.1 is satisfied ($c = \tau_\varphi(\xi)$ and $\alpha = 1/2$).

Let us emphasize that another assumptions on dependence of $\xi_1,\ldots,\xi_n$ can give the same estimate of the norm of $\tau_\varphi\left(\sum_{i=1}^{n}\xi_i\right)$. In the paper Giuliano Antonini et al. [5] (Lemma 3) it was proved that for φ-sub-Gaussian acceptable random variables the inequality (2) holds, if $\varphi(|x|^{1/r})$ is convex. The definition of acceptability of sequence of random variable one can find therein. For us it is the most important that these estimates are the same. In this article there is some version of the Marcinkiewicz – Zygmund law of large numbers for φ-sub-Gaussian random variables as a corollary of much more general theorem. We give an independent proof of this corollary but under modified assumptions.

Proposition 2.1. *Let (ξ_n), $p > 1$, be a bounded sequence of φ_p-sub-Gaussian random variables and let $r = \min\{p,2\}$. If, in addition,*

$$\tau_{\varphi_p}\left(\sum_{i=1}^{n}\xi_i\right)^{r} \leq \sum_{i=1}^{n}\tau_{\varphi_p}(\xi_i)^{r}, \tag{3}$$

then $n^{-1/s}\sum_{i=1}^{n}\xi_i \to 0$ almost surely for any $0 < s < r$.

***Remark* 2.2.** Since $\varphi_p(|x|^{1/r})$ is convex, the estimate (3) is satisfied by sequences of independent or acceptable random variables, for instance.

Proof. Let $b = \sup_{n\geq 1}\tau_{\varphi_p}(\xi_n)$, then $\sum_{i=1}^n \tau_{\varphi_p}(\xi_i)^r \leq nb^r$ and, in consequence, $\tau_{\varphi_p}\left(\sum_{i=1}^n \xi_i\right) \leq n^{1/r}b$. For positive number s less than r, by Lemma 2.1, we obtain

$$\mathbb{P}\left(\left|\sum_{i=1}^n \xi_i\right| \geq n^{\frac{1}{s}}\varepsilon\right) \leq 2\exp\left(-\varphi_q\left(\frac{n^{1/s}\varepsilon}{n^{1/r}b}\right)\right) = 2\exp\Big(-\varphi_q\Big(n^{(1/s-1/r)}\frac{\varepsilon}{b}\Big)\Big).$$

For $n > (b/\varepsilon)^{(1/s-1/r)^{-1}}$, we have

$$\varphi_q\Big(n^{(1/s-1/r)}\frac{\varepsilon}{b}\Big) = n^{q(1/s-1/r)}\frac{1}{q}\Big(\frac{\varepsilon}{b}\Big)^q - \frac{1}{q} + \frac{1}{2}$$

and, in consequence,

$$\sum_{n=1}^\infty \exp\Big(-\varphi_q\Big(n^{(1/s-1/r)}\frac{\varepsilon}{b}\Big)\Big) < \infty,$$

which, in view of Borel – Cantelli lemma, completes the proof.

***Remark* 2.3.** Because we apply the function $\varphi_p(x)$ instead of $|x|^p$, then we must not restrict p to be less or equal 2 to ensure the fulfillment of the quadratic condition for the function $|x|^p$. Moreover, we use the metric property (3) instead of assumptions on some form of dependence random variables (compare [5], Corollary 7).

***Example* 2.1.** The proof of Hoeffding – Azuma's inequality for a sequence (ξ_n) of bounded random variables such that $|\xi_n| \leq d_n$ a.s. and $\mathbb{E}\xi_n = 0$ is based on an estimate of the moment generating function of the partial sum $\sum_{i=1}^n \xi_i$. Under assumptions that ξ_n are independent (Hoeffding) or ξ_n are martingales increments (Azuma) the following inequality holds:

$$\mathbb{E}\exp\left(\lambda\sum_{i=1}^n \xi_i\right) \leq \exp\left(\frac{\lambda^2\sum_{i=1}^n d_i^2}{2}\right) \tag{4}$$

(see [7, 1]). Let us emphasize that in [1] Azuma has proved the above estimate under more general assumptions on (ξ_n) which satisfy centered bounded martingales increments. The inequality (4) means that

$$\tau_{\varphi_2}\left(\sum_{i=1}^n \xi_i\right) \leq \left(\sum_{i=1}^n d_i^2\right)^{1/2}.$$

If we take $d_n = 1$ for $n = 1, 2, \ldots$, then we get the condition

$$\tau_{\varphi_2}\left(\sum_{i=1}^n \xi_i\right) \leq \sqrt{n},$$

which follows that the sequence (ξ_n) satisfies the assumptions of Proposition 2.1 with $p = r = 2$ and the norm $\tau_{\varphi_2}(\xi_n) \leq 1$ and we get the almost sure convergence $n^{-1/s}\sum_{i=1}^n \xi_i$ to 0 for any $0 < s < 2$. Let us note that for $s = 1$ we obtain SLLN for this sequence.

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