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CONVERGENCE OF MULTIPLE FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION*

ЗБІЖНІСТЬ КРАТНИХ РЯДІВ ФУР'Є ФУНКЦІЙ З ОБМЕЖЕНОЮ УЗАГАЛЬНЕНОЮ ВАРІАЦІЄЮ

The paper introduces a new concept of Λ-variation of multivariable functions and studies its relationship with the convergence of multidimensional Fourier series.

Введено нову концепцію Λ-варіації функцій багатьох змінних та вивчено її зв'язок зі збіжністю багатовимірних рядів Фур'є.

1. Classes of functions of bounded generalized variation. In 1881 Jordan [11] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation etc., see [2, 12, 15, 17]). In two-dimensional case the class BV of functions of bounded variation was introduced by Hardy [10].

For an interval $T = [a, b] \subset R$ we denote by $T^d = [a, b]^d$ the d-dimensional cube in R^d.

Consider a function $f(x)$ defined on T^d and a collection of intervals

$$J^k = (a^k, b^k) \subset T, \quad k = 1, 2, \ldots, d.$$

For $d = 1$ we set

$$f(J^1) := f(b^1) - f(a^1).$$

If for any function of $d-1$ variables the expression $f(J^1 \times \ldots \times J^{d-1})$ is already defined, then for a function of d variables the *mixed difference* is defined as follows:

$$f\left(J^1 \times \ldots \times J^d\right) := f\left(J^1 \times \ldots \times J^{d-1}, b^d\right) - f\left(J^1 \times \ldots \times J^{d-1}, a^d\right).$$

Let $E = \{I_k\}$ be a collection of nonoverlapping intervals from T ordered in arbitrary way and let $\Omega = \Omega(T)$ be the set of all such collections E. We denote by $\Omega_n = \Omega_n(T)$ set of all collections of n nonoverlapping intervals $I_k \subset T$.

For sequences of positive numbers

$$\Lambda^j = \{\lambda_n^j\}_{n=1}^{\infty}, \qquad \lim_{n\to\infty} \lambda_n^j = \infty, \quad j = 1, 2, \ldots, d,$$

and for a function $f(x)$, $x = (x_1, \ldots, x_d) \in T^d$ the $(\Lambda^1, \ldots, \Lambda^d)$-*variation of f with respect to the index set $D := \{1, 2, \ldots, d\}$* is defined as follows:

$$\{\Lambda^1, \ldots, \Lambda^d\} V^D\big(f, T^d\big) := \sup_{\{I_{i_j}^j\}\in\Omega} \sum_{i_1,\ldots,i_d} \frac{\left|f(I_{i_1}^1 \times \ldots \times I_{i_d}^d)\right|}{\lambda_{i_1}^1 \ldots \lambda_{i_d}^d}.$$

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For an index set $\alpha = \{j_1, \ldots, j_p\} \subset D$ and any $x = (x_1, \ldots, x_d) \in R^d$ we set $\widetilde{\alpha} := D \setminus \alpha$ and denote by x_α the vector of R^p consisting of components x_j, $j \in \alpha$, i.e.,

$$x_\alpha = (x_{j_1}, \ldots, x_{j_p}) \in R^p.$$

By

$$\{\Lambda^{j_1}, \ldots, \Lambda^{j_p}\} V^\alpha \big(f, x_{\widetilde{\alpha}}, T^d\big) \qquad \text{and} \qquad f\left(I^1_{i_{j_1}} \times \ldots \times I^p_{i_{j_p}}, x_{\widetilde{\alpha}}\right)$$

we denote respectively the $(\Lambda^{j_1}, \ldots, \Lambda^{j_p})$-variation over the p-dimensional cube T^p and mixed difference of f as a function of variables $x_{j_1}, \ldots, x_{j_p}$ with fixed values $x_{\widetilde{\alpha}}$ of other variables. The $(\Lambda^{j_1}, \ldots, \Lambda^{j_p})$*-variation of with respect to the index set* α is defined as follows:

$$\big\{\Lambda^{j_1}, \ldots, \Lambda^{j_p}\big\} V^\alpha (f, T^p) = \sup_{x_{\widetilde{\alpha}} \in T^{d-p}} \big\{\Lambda^{j_1}, \ldots, \Lambda^{j_p}\big\} V^\alpha \big(f, x_{\widetilde{\alpha}}, T^d\big).$$

Definition 1. *We say that the function f has total bounded $(\Lambda^1, \ldots, \Lambda^d)$-variation on T^d and write $f \in \{\Lambda^1, \ldots, \Lambda^d\} BV(T^d)$, if*

$$\{\Lambda^1, \ldots, \Lambda^d\} V(f, T^d) := \sum_{\alpha \subset D} \{\Lambda^1, \ldots, \Lambda^d\} V^\alpha \big(f, T^d\big) < \infty.$$

Definition 2. *We say that the function f is continuous in $(\Lambda^1, \ldots, \Lambda^d)$-variation on T^d and write $f \in C\{\Lambda^1, \ldots, \Lambda^d\} V(T^d)$, if*

$$\lim_{n\to\infty} \big\{\Lambda^{j_1}, \ldots, \Lambda^{j_{k-1}}, \Lambda^{j_k}_n, \Lambda^{j_{k+1}}, \ldots, \Lambda^{j_p}\big\} V^\alpha \big(f, T^d\big) = 0, \quad k = 1, 2, \ldots, p,$$

for any $\alpha \subset D$, $\alpha := \{j_1, \ldots, j_p\}$, where $\Lambda^{j_k}_n := \{\lambda^{j_k}_s\}_{s=n}^{\infty}$.

Definition 3. *We say that the function f has bounded partial $(\Lambda^1, \ldots, \Lambda^d)$-variation and write $f \in P\{\Lambda^1, \ldots, \Lambda^d\} BV(T^d)$ if*

$$P\{\Lambda^1, \ldots, \Lambda^d\} V(f, T^d) := \sum_{i=1}^{d} \Lambda^i V^{\{i\}} \big(f, T^d\big) < \infty.$$

In the case $\Lambda^1 = \ldots = \Lambda^d = \Lambda$ we set

$$\Lambda BV(T^d) := \{\Lambda^1, \ldots, \Lambda^d\} BV(T^d),$$

$$C\Lambda V(T^d) := C\{\Lambda^1, \ldots, \Lambda^d\} V(T^d),$$

$$P\Lambda BV(T^d) := P\{\Lambda^1, \ldots, \Lambda^d\} BV(T^d).$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$.

When $\lambda_n = n$ for all $n = 1, 2, \ldots$ we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ, i.e., HBV, $PHBV$, CHV, etc.

For two variable functions Dyachenko and Waterman [5] introduced another class of functions of generalized bounded variation.

Denoting by Γ the the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times \times [\gamma_k, \delta_k] \subset T^2$, for a function $f(x, y)$, $x, y \in T$, we set

$$\Lambda^* V(f, T^2) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

Definition 4 (Dyachenko, Waterman). *We say that* $f \in \Lambda^* BV(T^2)$ *if*

$$\Lambda V(f, T^2) := \Lambda V_1(f, T^2) + \Lambda V_2(f, T^2) + \Lambda^* V(f, T^2) < \infty.$$

In this paper we introduce a new classes of functions of generalized bounded variation and investigate the convergence of Fourier series of function of that classes.

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we denote

$$\Lambda^{\#} V_s\big(f, T^d\big) := \sup_{\{x^i\{s\}\} \subset T^{d-1}} \sup_{\{I_i^s\} \in \Omega} \sum_i \frac{\big|f(I_i^s, x^i\{s\})\big|}{\lambda_i},$$

where

$$x^i\{s\} := \big(x_1^i, \ldots, x_{s-1}^i, x_{s+1}^i, \ldots, x_d^i\big) \qquad \text{for} \quad x^i := \big(x_1^i, \ldots, x_d^i\big). \tag{1}$$

Definition 5. *We say that the function* f *belongs to the class* $\Lambda^{\#} BV(T^d)$, *if*

$$\Lambda^{\#} V\big(f, T^d\big) := \sum_{s=1}^{d} \Lambda^{\#} V_s\big(f, T^d\big) < \infty.$$

The notion of Λ-variation was introduced by Waterman [15] in one-dimensional case, by Sahakian [14] in two-dimensional case and by Sablin [13] in the case of higher dimensions. The notion of bounded partial variation (class PBV) was introduced by Goginava in [7]. These classes of functions of generalized bounded variation play an important role in the theory Fourier series.

***Remark* 1.** It is not hard to see that $\Lambda^{\#} BV(T^d) \subset P\Lambda BV(T^d)$ for any $d > 1$ and $\Lambda^* BV(T^2) \subset \subset \Lambda^{\#} BV(T^2)$.

We prove that the following theorem is true.

Theorem 1. *Let* $d \geq 2$ *and* $T = (t_1, t_2) \subset R$. *If*

$$\Lambda = \{\lambda_n\} \qquad \textit{with} \quad \lambda_n = \frac{n}{\log^{d-1}(n+1)}, \quad n = 1, 2, \ldots, \tag{2}$$

then

$$HV\big(f, T^d\big) \leq M(d) \Lambda^{\#} V\big(f, T^d\big). \tag{3}$$

Proof. We have to prove that for any $\alpha := \{j_1, \ldots, j_p\} \subset D$

$$\sup_{\{I_{i_j}^j\} \in \Omega} \sum_{i_1, \ldots, i_p} \frac{|f(I_{i_1}^1 \times \ldots \times I_{i_p}^p, x_{\widetilde{\alpha}})|}{i_1 \ldots i_p} \leq M(d) \sum_{s=1}^{d} \Lambda^{\#} V_s(f, T^d). \tag{4}$$

To this end, observe that

$$\sum_{i_1,\ldots,i_p} \frac{|f(I^1_{i_1}\times\ldots\times I^p_{i_p}, x_{\widetilde{\alpha}})|}{i_1\ldots i_p} = \sum_{\sigma}\sum_{i_{\sigma(1)}\leq\ldots\leq i_{\sigma(p)}} \frac{|f(I^1_{i_1}\times\ldots\times I^p_{i_p}, x_{\widetilde{\alpha}})|}{i_1\ldots i_p}, \tag{5}$$

where the sum is taken over all rearrangements $\sigma = \big\{\sigma(k)\big\}_{k=1}^{p}$ of the set $\{1,2,\ldots,p\}$.

Next, we have

$$\sum_{i_1\leq\ldots\leq i_p} \frac{\big|f(I^1_{i_1}\times\ldots\times I^p_{i_p}, x_{\widetilde{\alpha}})\big|}{i_1\ldots i_p} = \sum_{i_p}\frac{1}{i_p}\sum_{i_1\leq\ldots\leq i_p} \frac{\big|f(I^1_{i_1}\times\ldots\times I^p_{i_p}, x_{\widetilde{\alpha}})\big|}{i_1\ldots i_{p-1}}. \tag{6}$$

Taking into account that for the fixed i_p, $i_1\leq\ldots\leq i_p$, there exists $x_1^{i_p},\ldots,x_{p-1}^{i_p}\in T$ such that

$$\big|f(I^1_{i_1}\times\ldots\times I^p_{i_p}, x_{\widetilde{\alpha}})\big| \leq 2^d\left|f\Big(I^p_{i_p}, x_1^{i_p},\ldots,x_{p-1}^{i_p}, x_{\widetilde{\alpha}}\Big)\right|$$

from (6) we obtain

$$\sum_{i_1\leq\ldots\leq i_p} \frac{\big|f(I^1_{i_1}\times\ldots\times I^p_{i_p}, x_{\widetilde{\alpha}})\big|}{i_1\ldots i_p} \leq 2^d\sum_{i_p}\frac{\big|f(I^p_{i_p}, x_1^{i_p},\ldots,x_{p-1}^{i_p}, x_{\widetilde{\alpha}})\big|}{i_p}\sum_{i_1\leq\ldots\leq i_p}\frac{1}{i_1\ldots i_{p-1}}\leq$$

$$\leq M(d)\sum_{i_p}\frac{\log^{d-1}(i_p+1)}{i_p}\big|f(I^p_{i_p}, x_1^{i_p},\ldots,x_{p-1}^{i_p}, x_{\widetilde{\alpha}})\big|\leq$$

$$\leq M(d)\Lambda^{\#}V_{i_p}\big(f,T^d\big)\leq M(d)\Lambda^{\#}V\big(f,T^d\big).$$

Similarly one can obtain bounds for other summands in the right-hand side of (5), which imply (3).

Theorem 1 is proved.

***Corollary* 1.** *If the sequence* Λ *is defined by* (2), *then* $\Lambda^{\#}BV(T^d)\subset HBV(T^d)$.

Now, we denote

$$\Delta := \big\{\delta=(\delta_1,\ldots,\delta_d)\colon \delta_i=\pm1,\ i=1,2,\ldots,d\big\} \tag{7}$$

and

$$\pi_{\varepsilon\delta}(x) := (x_1,\, x_1+\varepsilon\delta_1)\times\ldots\times(x_d,\, x_d+\varepsilon\delta_d),$$

for $x=(x_1,\ldots,x_d)\in R^d$ and $\varepsilon>0$. We set $\pi_\delta(x) := \pi_{\varepsilon\delta}(x)$, if $\varepsilon=1$.

For a function f defined in some neighbourhood of a point x and $\delta\in\Delta$ we set

$$f_\delta(x) := \lim_{t\in\pi_\delta(x),\, t\to x} f(t), \tag{8}$$

if the last limit exists.

Theorem 2. *Suppose* $f\in\Lambda^{\#}BV(T^d)$ *for some sequence* $\Lambda=\{\lambda_n\}$.

(a) *If the limit* $f_\delta(x)$ *exists for some* $x=(x_1,\ldots,x_d)\in T^d$ *and some* $\delta=(\delta_1,\ldots,\delta_d)\in\Delta$, *then*

$$\lim_{\varepsilon\to0}\Lambda^{\#}V(f,\pi_{\varepsilon\delta}(x))=0. \tag{9}$$

(b) *If f is continuous on some compact $K \subset T^d$, then*

$$\lim_{\varepsilon\to 0} \Lambda^{\#}V\big(f,[x_1-\varepsilon,x_1+\varepsilon]\times\ldots\times[x_d-\varepsilon,x_d+\varepsilon]\big)=0 \tag{10}$$

uniformly with respect to $x=(x_1,\ldots,x_d)\in K$.

Proof. According to Definition 5, we need to prove that

$$\lim_{\varepsilon\to 0} \Lambda^{\#}V_s\big(f,\pi_{\varepsilon\delta}(x)\big)=0 \tag{11}$$

for any $s=1,2,\ldots,d$. Without loss of generality we can assume that $s=1$ and $\delta_i=1$ for $i=1,2,\ldots,d$. Assume to the contrary that (11) does not holds:

$$\lim_{\varepsilon\to 0} \Lambda^{\#}V_1\big(f,\pi_{\varepsilon\delta}(x)\big)\neq 0.$$

Then there exists a number α such that

$$\Lambda^{\#}V_1\big(f,\pi_{\varepsilon\delta}(x)\big)>\alpha>0 \tag{12}$$

for any $\varepsilon>0$.

Using induction on $k=1,2,\ldots,$ we construct positive numbers ε_k and the sequences of collections of nonoverlapping intervals

$$I_i^1\subset(x_1+\varepsilon_{k+1},x_1+\varepsilon_k),\quad i=n_k+1,\ldots,n_{k+1}, \tag{13}$$

and vectors

$$\beta^i=(\beta_1^i,\ldots,\beta_d^i)\in\pi_{\varepsilon_k\delta}(x),\quad i=n_k+1,\ldots,n_{k+1}, \tag{14}$$

as follows. By (12), for a fixed number $\varepsilon_1>0$ we find a collection of nonoverlapping intervals

$$I_i^1\subset(x_1,x_1+\varepsilon_1),\quad i=1,\ldots,n_1,$$

and vectors

$$\beta^i=(\beta_1^i,\ldots,\beta_d^i)\in\pi_{\varepsilon_1\delta}(x),\quad i=1,\ldots,n_1,$$

such that

$$\sum_{i=1}^{n_1}\frac{\big|f(I_i^1;\beta_2^i,\ldots,\beta_d^i)\big|}{\lambda_i}>\alpha. \tag{15}$$

Now, suppose the number ε_k, intervals (13) and the vectors (14) for some $k=1,2,\ldots$ are constructed. Since the limit $f_\delta(x)$ exists, we can choose ε_{k+1} satisfying

$$0<\varepsilon_{k+1}<\varepsilon_k,\qquad (x_1,x_1+\varepsilon_{k+1})\cap\left(\bigcup_{i=1}^{n_k}I_i^1\right)=\varnothing \tag{16}$$

and

$$\sum_{i=1}^{n_k}\frac{\big|f(J_i^1;\gamma_2^i,\ldots,\gamma_d^i)\big|}{\lambda_i}<\frac{\alpha}{2} \tag{17}$$

for any collection of nonoverlapping intervals

$$J_i^1 \subset (x_1, x_1 + \varepsilon_{k+1}), \quad i = 1, \ldots, n_k,$$

and for any vectors

$$\gamma^i = (\gamma_1^i, \ldots, \gamma_d^i) \in \pi_{\varepsilon_{k+1}\delta}(x), \quad i = 1, \ldots, n_k.$$

Further, according to (12) there is a collection of nonoverlapping intervals

$$J_i^1 \subset (x_1, x_1 + \varepsilon_{k+1}), \quad i = 1, \ldots, n_{k+1}, \tag{18}$$

and vectors

$$\gamma^i = (\gamma_1^i, \ldots, \gamma_d^i) \in \pi_{\varepsilon_{k+1}\delta}(x), \quad i = 1, \ldots, n_{k+1},$$

such that

$$\sum_{i=1}^{n_{k+1}} \frac{\left|f(J_i^1; \gamma_2^i, \ldots, \gamma_d^i)\right|}{\lambda_i} > \alpha. \tag{19}$$

Now, denoting

$$I_i^1 = J_i^1, \qquad \beta^i = \gamma^i \qquad \text{for} \quad i = n_k + 1, \ldots, n_{k+1}, \tag{20}$$

from (17) and (19) we get

$$\sum_{i=n_k+1}^{n_{k+1}} \frac{\left|f(I_i^1; \beta_2^i, \ldots, \beta_d^i)\right|}{\lambda_i} > \frac{\alpha}{2}. \tag{21}$$

Intervals (13) and vectors (14) for $k = 1, 2, \ldots,$ are constructed.

By (16), (18) and (20), the intervals I_i^1 are nonoverlapping for $i = 1, 2, \ldots,$ while according to (21),

$$\sum_{i=1}^{\infty} \frac{\left|f(I_i^1; \beta_2^i, \ldots, \beta_d^i)\right|}{\lambda_i} = \infty.$$

Consequently, $\Lambda^{\#}V_1(f, T^d) = \infty$. This contradiction completes proof of the statement (a) of Theorem 2.

To prove statement (b), observe that (a) obviously implies (10) for any point $x \in T^d$, where f is continuous. Hence, we have to prove that (10) holds uniformly with respect to $x \in K$, provided that f is continuous on the compact $K \subset T^d$.

To this end let us assume to the contrary that (10) does not hold uniformly on K. Then there exist $\delta > 0$ and sequences

$$x^i = (x_1^i, \ldots, x_d^i) \in K \qquad \text{and} \qquad \varepsilon_i > 0, \quad i = 1, 2, \ldots, \qquad \text{with} \quad \varepsilon_i \to 0$$

such that

$$\Lambda^{\#}V\big(f; \left[x_1^i - \varepsilon_i, x_1^i + \varepsilon_i\right] \times \ldots \times \left[x_d^i - \varepsilon_i, x_d^i + \varepsilon_i\right]\big) \geq \delta > 0.$$

Since K is compact we can assume without loss of generality that $x^i \to x$ for some $x = (x_1, \ldots, x_d) \in K$. Then obviously for each $\varepsilon > 0$ there is a number $i(\varepsilon)$ such that

$$[x_j^i - \varepsilon_i, x_j^i + \varepsilon_i] \subset [x_j - \varepsilon, x_j + \varepsilon], \quad j = 1, \ldots, d, \qquad \text{for} \quad i > i(\varepsilon).$$

Consequently,

$$\Lambda^{\#} V\big(f; [x_1 - \varepsilon, x_1 + \varepsilon] \times \ldots \times [x_d - \varepsilon, x_d + \varepsilon]\big) \geq \delta > 0,$$

for any $\varepsilon > 0$, which is a contradiction.

Theorem 2 is proved.

Next, we define

$$v_s^{\#}(f, n) := \sup_{\{x^i\}_{i=1}^n \subset T^d} \sup_{\{I_i^s\}_{i=1}^n \in \Omega_n} \sum_{i=1}^n \big|f(I_i^s, x^i\{s\})\big|, \qquad s = 1, \ldots, d, \quad n = 1, 2, \ldots,$$

where $x^i\{s\}$ is as in (1). The following theorem holds.

Theorem 3. *If the function $f(x)$, $x \in T^d$, satisfies the condition*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f, n) \log^{d-1}(n+1)}{n^2} < \infty, \quad s = 1, 2, \ldots, d,$$

then $f \in \left\{ \dfrac{n}{\log^{d-1}(n+1)} \right\}^{\#} BV(T^d)$.

Proof. Let $s = 1, \ldots, d$ be fixed. The for any collection of intervals $\{I_i^s\}_{i=1}^n \in \Omega_n$ and a sequence of vectors $\{x^i\}_{i=1}^n \in T^d$, using Abel's partial summation we obtain

$$\sum_{j=1}^n \frac{\big|f(I_j^s, x^j\{s\})\big| \log^{d-1}(j+1)}{j} =$$

$$= \sum_{j=1}^{n-1} \left(\frac{\log^{d-1}(j+1)}{j} - \frac{\log^{d-1}(j+2)}{j+1} \right) \sum_{k=1}^j \big|f\big(I_k^s, x^k\{s\}\big)\big| +$$

$$+ \frac{\log^{d-1}(n+1)}{n} \sum_{j=1}^n \big|f\big(I_j^s, x^j\{s\}\big)\big| \leq$$

$$\leq \sum_{j=1}^{n-1} \left(\frac{\log^{d-1}(j+1)}{j} - \frac{\log^{d-1}(j+2)}{j+1} \right) v_s^{\#}(f, j) + \frac{\log^{d-1}(n+1)}{n} v_s^{\#}(f, n). \tag{22}$$

Using the inequality

$$\frac{\log^{d-1}(n+1)}{n} v_s^{\#}(f, n) \leq \sum_{j=n}^{\infty} \left(\frac{\log^{d-1}(j+1)}{j} - \frac{\log^{d-1}(j+2)}{j+1} \right) v_s^{\#}(f, j), \tag{23}$$

from (22) we get

$$\left\{ \frac{n}{\log^{d-1}(n+1)} \right\}^{\#} V_s(f, T^d) \leq c \sum_{n=1}^{\infty} \frac{v_s^{\#}(f, n) \log^{d-1}(n+1)}{n^2} < \infty. \tag{24}$$

Theorem 3 is proved.

2. Convergence of multiple Fourier series. We suppose throughout this section, that $T = [0, 2\pi)$ and $T^d = [0, 2\pi)^d$, $d \geq 2$, stands for the d-dimensional torus.

We denote by $C(T^d)$ the space of continuous and 2π-periodic with respect to each variable functions with the norm
$$\|f\|_C := \sup_{(x_1,\ldots,x_d)\in T^d} |f(x_1, \ldots, x_d)|.$$

The Fourier series of the function $f \in L^1(T^d)$ with respect to the trigonometric system is the series
$$Sf(x_1, \ldots, x_d) := \sum_{n_1,\ldots,n_d=-\infty}^{+\infty} \widehat{f}(n_1, \ldots, n_d) e^{i(n_1x_1+\ldots+n_dx_d)},$$
where
$$\widehat{f}(n_1, \ldots, n_d) = \frac{1}{(2\pi)^d} \int\limits_{T^d} f(x^1, \ldots, x^d) e^{-i(n_1x_1+\ldots+n_dx_d)} dx_1 \ldots dx_d$$
are the Fourier coefficients of f.

In this paper we consider convergence of **only rectangular partial sums** (convergence in the sense of Pringsheim) of d-dimensional Fourier series. Recall that the rectangular partial sums are defined as follows:
$$S_{N_1,\ldots,N_d} f(x_1, \ldots, x_d) := \sum_{n_1=-N_1}^{N_1} \ldots \sum_{n_d=-N_d}^{N_d} \widehat{f}(n_1, \ldots, n_d) e^{i(n_1x^1+\ldots+n_dx^d)}.$$

We say that the point $x \in T^d$ is *a regular point* of a function f, if the limit $f_\delta(x)$ defined by (8) exists for any $\delta \in \Delta$ (see (7)). For the regular point x we denote
$$f^*(x) := \frac{1}{2^d} \sum_{\delta\in\Delta} f_\delta(x). \tag{25}$$

Definition 6. *We say that the class of functions $V \subset L^1(T^d)$ is a class of convergence on T^d, if for any function $f \in V$*

1) *the Fourier series of f converges to $f^*(x)$ at any regular point $x \in T^d$,*

2) *the convergence is uniform on a compact $K \subset T^d$, if f is continuous on K.*

The well known Dirichlet–Jordan theorem (see [18]) states that the Fourier series of a function $f(x)$, $x \in T$, of bounded variation converges at every point x to the value $[f(x + 0) + f(x - 0)]/2$. If f is in addition continuous on T, then the Fourier series converges uniformly on T.

Hardy [10] generalized the Dirichlet–Jordan theorem to the double Fourier series and proved that BV is a class of convergence on T^2.

The following theorem was proved by Waterman (for $d = 1$) and Sahakian (for $d = 2$).

Theorem WS (Waterman [15], Sahakian [14]). *If $d = 1$ or $d = 2$, then the class $HBV(T^d)$ is a class of convergence on T^d.*

In [1] Bakhvalov proved that the class HBV is not a class of convergence on T^d, if $d > 2$. On the other hand, he proved the following theorem.

Theorem B (Bakhvalov [1]). *The class $CHV(T^d)$ is a class of convergence on T^d for any $d = 1, 2, \ldots$.*

Convergence of spherical and other partial sums of double Fourier series of functions of bounded Λ-variation was investigated in deatails by Dyachenko [3, 4].

In [8, 9] Goginava and Sahakian investigated convergence of multiple Fourier series of functions of bounded partial Λ-variation. In particular, the following theorem was proved.

Theorem GS. (a) *If and* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1+\varepsilon}(n+1)}, \qquad n = 1, 2, \ldots, \quad d > 1,$$

for some $\varepsilon > 0$, *then the class* $P\Lambda BV(T^d)$ *is a class of convergence on* T^d.

(b) *If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1}(n+1)}, \qquad n = 1, 2, \ldots, \quad d > 1,$$

then the class $P\Lambda BV(T^d)$ *is not a class of convergence on* T^d.

In [5], Dyachenko and Waterman proved that the class $\Lambda^* BV(T^2)$ is a class convergence on T^2 for $\Lambda = \{\lambda_n\}$ with $\lambda_n = \dfrac{n}{\ln(n+1)}$, $n = 1, 2, \ldots$.

The main result of the present paper is the following theorem.

Theorem 4. (a) *If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n = \frac{n}{\log^{d-1}(n+1)}, \qquad n = 1, 2, \ldots, \quad d > 1, \tag{26}$$

then the class $\Lambda^{\#} BV(T^d)$ *is a class of convergence on* T^d.

(b) *If* $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ *with*

$$\lambda_n := \left\{ \frac{n\xi_n}{\log^{d-1}(n+1)} \right\}, \qquad n = 1, 2, \ldots, \quad d > 1, \tag{27}$$

where $\xi_n \to \infty$ *as* $n \to \infty$, *then there exists a continuous function* $f \in \Lambda^{\#} BV(T^d)$ *such that the cubical partial sums of* d*-dimensional Fourier series of* f *diverge unboundedly at* $(0, \ldots, 0) \in T^d$.

Proof. The proof of the part (a) is based on the following statement, that in the case $d = 2$ is proved by Sahakian (see formulaes (33) and (35) in [14]). For an arbitrary $d > 2$ the proof is similar.

Lemma S. *Suppose* $f \in HV(T^d)$ *and* $x \in T^d$. *If the limit* $f_\delta(x)$ *exists for any* $\delta \in \Delta$, *then for any* $\varepsilon > 0$

$$\left| S_{n_1,\ldots,n_d} f(x) - f^*(x) \right| \le M(d) \sum_{\delta \in \Delta} HV(f; \pi_{\varepsilon\delta}(x)) + o(1),$$

as $n_i \to \infty$, $i = 1, 2, \ldots, d$.

Moreover, the quantity $o(1)$ *tends to* 0 *uniformly on a compact* K, *if* f *is continuous on* K.

Now, if the sequence $\Lambda = \{\lambda_n\}$ is defined by (26) and $f \in \Lambda^{\#} BV(T^d)$, then Lemma S and Theorem 1 imply that for any $\varepsilon > 0$

$$\left| S_{n_1,\ldots,n_d} f(x) - f^*(x) \right| \le M(d) \sum_{\delta \in \Delta} \Lambda^{\#} V(f; \pi_{\varepsilon\delta}(x)) + o(1), \tag{28}$$

which combined with Theorem 2 completes the proof of (a).

To prove part (b) suppose that $\Lambda = \{\lambda_n\}$ is a sequence defined by (27). It is not hard to see that the class $C(T^d) \cap \Lambda^{\#}BV(T^d)$ is a Banach space with the norm

$$\|f\|_{\Lambda^{\#}BV} := \|f\|_C + \Lambda^{\#}BV(f).$$

Denoting

$$A_{i_1,\ldots,i_d} := \left[\frac{\pi i_1}{N+1/2}, \frac{\pi(i_1+1)}{N+1/2}\right) \times \ldots \times \left[\frac{\pi i_d}{N+1/2}, \frac{\pi(i_d+1)}{N+1/2}\right),$$

we consider the following functions:

$$g_N(x_1,\ldots,x_d) := \sum_{i_1,\ldots,i_d=1}^{N-1} 1_{A_{i_1,\ldots,i_d}}(x_1,\ldots,x_d) \prod_{s=1}^{d} \sin(N+1/2)x_s,$$

for $N = 2, 3, \ldots$, where $1_A(x_1,\ldots,x_d)$ is the characteristic function of a set $A \subset T^d$.

It is easy to check that

$$\left\{\frac{n\xi_n}{\log^{d-1}(n+1)}\right\}^{\#} V_s(g_N) \le c \sum_{i=1}^{N-1} \frac{\log^{d-1}(i+1)}{i\xi_i} = o(\log^d N)$$

and hence

$$\|g_N\|_{\Lambda^{\#}BV} = o(\log^d N) = \eta_N \log^d N,$$

where $\eta_N \to 0$ as $N \to \infty$. Now, setting

$$f_N := \frac{g_N}{\eta_N \log^d N}, \quad N = 2, 3, \ldots,$$

we obtain that $f_N \in \Lambda^{\#}BV(T^d)$ and

$$\sup_N \|f_N\|_{\Lambda^{\#}BV} < \infty. \tag{29}$$

Now, for the cubical partial sums of the d-dimensional Fourier series of f_N at $(0,\ldots,0) \in T^d$ we have that

$$\pi^d S_{N,\ldots,N} f_N(0,\ldots,0) =$$

$$= \frac{1}{\eta_N \log^d N} \sum_{i_1,\ldots,i_d=1}^{N-1} \int_{A_{i_1,\ldots,i_d}} \prod_{s=1}^{d} \frac{\sin^2(N+1/2)x_s}{2\sin(x_s/2)} dx_1 \ldots dx_d \ge$$

$$\ge \frac{c}{\eta_N \log^d N} \sum_{i_1,\ldots,i_d=1}^{N-1} \frac{1}{i_1 \ldots i_d} \ge \frac{c}{\eta_N} \to \infty \tag{30}$$

as $N \to \infty$. Applying the Banach–Steinhaus theorem, from (29) and (30) we conclude that there exists a continuous function $f \in \Lambda^{\#}BV(T^d)$ such that

$$\sup_N \left|S_{N,\ldots,N} f(0,\ldots,0)\right| = \infty.$$

Theorem 4 is proved.

The next theorem follows from Theorems 3 and 4.

Theorem 5. *For any $d > 1$ the class of functions $f(x)$ $x \in T^d$ satisfying the following condition*:

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f,n)\log^{d-1}(n+1)}{n^2} < \infty, \quad s = 1,\ldots,d,$$

is a class of convergence.

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