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SCATTERED SUBSETS OF GROUPS

РОЗРІДЖЕНІ ПІДМНОЖИНИ ГРУП

We define scattered subsets of a group as asymptotic counterparts of the scattered subspaces of a topological space and prove that a subset A of a group G is scattered if and only if A does not contain any piecewise shifted IP-subsets. For an amenable group G and a scattered subspace A of G, we show that $\mu(A) = 0$ for each left invariant Banach measure μ on G. It is shown that every infinite group can be split into $\aleph_0$ scattered subsets.

Розріджені підмножини групи визначено, як асимптотичні аналоги розріджених підпросторів топологічного простору. Доведено, що підмножина A групи G є розрідженою тоді і тільки тоді, коли A не містить кусково-зсунутих IP-підмножин. Показано, що для аменабельної групи G та розрідженого підпростору A групи G рівність $\mu(A) = 0$ виконується для кожної лівої інваріантної банахової міри μ на G. Встановлено, що кожну нескінченну групу можна розбити на $\aleph_0$ розріджених підмножин.

1. Introduction. Given a discrete space X, we take the points of βX, the Stone – Čech compactification of X, to be the ultrafilters on X, with the points of X identified with the principal ultrafilters on X. The topology on βX can be defined by stating that the sets of the form $\overline{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, form a base for the open sets. We note that the sets of this form are clopen and that, for any $p \in \beta X$ and $A \subseteq X$, $A \in p$ if and only if $p \in \overline{A}$. For any $A \subseteq X$ we denote $A^* = \overline{A} \cap G^*$, where $G^* = \beta G \setminus G$. The universal property of βG states that every mapping $f : X \to Y$, where Y is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \to Y$.

Now let G be a discrete group. Using the universal property of βG, we can extend the group multiplication from G to βG in two steps. Given $g \in G$, the mapping

$$x \mapsto gx \colon G \to \beta G$$

extends to the continuous mapping

$$q \mapsto gq \colon \beta G \to \beta G.$$

Then, for each $q \in \beta G$, we extend the mapping $g \mapsto gq$ defined from G into βG to the continuous mapping

$$p \mapsto pq \colon \beta G \to \beta G.$$

The product pq of the ultrafilters p, q can also be defined by the rule: given a subset $A \subseteq G$,

$$A \in pq \leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

To describe the base for pq, we take any element $P \in p$ and, for every $x \in P$, choose some element $Q_x \in q$. Then $\cup_{x \in p} xQ_x \in pq$, and the family of subsets of this form is a base for the ultrafilter pq.

By the construction, the binary operation $(p, q) \mapsto pq$ is associative, so βG is a semigroup, and G^* is a subsemigroup of βG. For each $q \in \beta G$, the right shift $x \mapsto xq$ is continuous, and the left shift $x \mapsto xq$ is continuous for each $g \in G$.

For the structure of a compact right topological semigroup βG and plenty of its applications to combinatorics, topological algebra and functional analysis see [1 – 5].

Given a subset A of a group G and an ultrafilter $p \in G^*$, we define a *p-companion* of A by

$$\Delta_p(A) = A^* \cap Gp = \{gp \colon g \in G, A \in gp\},$$

and say that a subset S of G^* is an *ultracompanion* of A if $S = \Delta_p(A)$ for some $p \in G^*$. For ultracompanions of subsets of groups and metric spaces see [6, 7].

Clearly, A is finite if and only if $\Delta_p(A) = \varnothing$ for each $p \in G^*$.

We say that a subset A of a group G is

thin if $|\Delta_p(A)| \leq 1$ for each $p \in G^*$;

n-thin, $n \in \mathbb{N}$ if $|\Delta_p(A)| \leq n$ for each $p \in G^*$;

sparse if each ultracompanion of A is finite;

disparse if each ultracompanion of A is discrete;

scattered if, for each infinite subset Y of A, there is $p \in Y^*$ such that $\Delta_p(Y)$ is finite.

We denote by $[G]^{<\omega}$ the family of all finite subsets of G. Given any $F \in [G]^{<\omega}$ and $g \in G$, we put

$$B(g, F) = Fg \cup \{g\}$$

and, following [8], say that $B(g, F)$ is a *ball of radius F around g*. For a subset Y of G, we put $B_Y(g, F) = Y \cap B(g, F)$. By [6] (Proposition 4), Y is n-thin if and only if for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $|B_Y(y, F)| \leq n$ for each $y \in Y \setminus H$. For thin subsets of a group, their applications and modifications see [9 – 19].

By [6] (Proposition 5) and [20] (Theorems 3 and 10), for a subset A of a group G, the following statements are equivalent:

(1) A is sparse;

(2) for every infinite subset X of G, there exists finite subset $F \subset G$ such that $\bigcap_{g \in F} gA$ is finite;

(3) for every infinite subset Y of A, there exists $F \in [G]^{<\omega}$ such that, for every $H \in [G]^{<\omega}$, we have

$$\{y \in Y \colon B_A(y, H) \setminus B_A(y, F) = \varnothing\} \neq \varnothing;$$

(4) A has no subsets asymorphic to the subset $W_2 = \{g \in \oplus_\omega \mathbb{Z}_2 \colon \operatorname{supt} g \leq 2\}$ of the group $\oplus_\omega \mathbb{Z}_2$, where $\operatorname{supt} g$ is the member of nonzero coordinates of g.

The notion of asymorphisms and coarse equivalence will be defined in the next section. The sparse sets were introduced in [21] in order to characterise strongly prime ultrafilters in G^*, the ultrafilters from $G^* \setminus \overline{G^* G^*}$. More on sparse subsets can be find in [10, 11, 16, 22].

In this paper, answering Question 4 from [6], we prove that a subset A of a group G is scattered if and only if A is disparse, and characterize the scattered subsets in terms of prohibited subsets. We answer also Question 2 from [6] proving that each scattered subset of an amenable group is absolute null. We prove that every infinite group G can be partitioned into $\aleph_0$ scattered subsets. The results are exposed in Section 2, their proofs in Section 3.

2. Results. Our first statement shows that, from the asymptotic point of view [23], the scattered subsets of a group can be considered as the counterparts of the scattered subspaces of a topological space.

Proposition 1. *For a subset A of a group G, the following two statements are equivalent*:

(i) *A is scattered*;

(ii) *for every infinite subset Y of A, there exists $F \in [G]^{<\omega}$such that, for every $H \in [G]^{<\omega}$, we have*

$$\{y \in Y : B_Y(y, H) \setminus B_Y(y, F) = \varnothing\} \neq \varnothing.$$

Proposition 2. *A subset A of a group G is scattered if and only if, for every countable subgroup H of G, $A \cap H$ is scattered in H.*

Let A be a subset of a group G, $K \in [G]^{<\omega}$. A sequence $a_0, \ldots, a_n$ in A is called *K-chain* from a_0 to a_n if $a_{i+1} \in B(a_i, K)$ for each $i \in \{0, \ldots, n-1\}$. For every $a \in A$, we denote

$$B^{\square}_A(a, K) = \{b \in A : \text{ there is a } K\text{-chain from } a \text{ to } b\}$$

and, following [24] (Chapter 3), say that A is *cellular* (or *asymptotically zero-dimentional*) if, for every $K \in [G]^{<\omega}$, there exists $K' \in [G]^{<\omega}$ such that, for each $a \in A$,

$$B^{\square}_A(a, K) \subseteq B_A(a, K').$$

Now we need some more asymptology (see [24], Chapter 1). Let G, H be groups, $X \subseteq G$, $Y \subseteq H$. A mapping $f : X \to Y$ is called a *$\prec$-mapping* if, for every $F \in [G]^{<\omega}$, there exists $K \in [G]^{<\omega}$ such that, for every $x \in X$,

$$f(B_X(x, F)) \subseteq B_Y(f(x), K).$$

If f is a bijection such that f and f^{-1} are $\prec$-mappings, we say that f is an *asymorphism*. The subsets X and Y are called *coarse equivalent* if there exist asymorphic subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that $X \subseteq B_X(X', F)$, $Y \subseteq B_Y(Y', K)$ for some $F \in [G]^{<\omega}$ and $K \in [H]^{<\omega}$.

Following [23], we say, that the set Y of G *has no asymptotically isolated balls* if Y does not satisfy Proposition 1(ii): for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $B_Y(y, H) \setminus B_Y(y, F) \neq \varnothing$ for each $y \in Y$.

By [23], a countable cellular subset Y of G with no asymptotically isolated balls is coarsely equivalent to the group $\oplus_\omega \mathbb{Z}_2$.

Proposition 3. *Let X be a countable subset of a group G. If X is not cellular, then X contains a subset Y coarsely equivalent to $\oplus_\omega \mathbb{Z}_2$.*

Let $(g_n)_{n<\omega}$ be an injective sequence in a group G. The set

$$\{g_{i_1} g_{i_2} \ldots g_{i_n} : 0 \leq i_1 < i_2 < \ldots < i_n < \omega\}$$

is called an *IP-set* [1, p. 406], the abriviation for "infinite dimensional parallelepiped".

Given a sequence $(b_n)_{n<\omega}$ in G, we say that the set

$$\{g_{i_1} g_{i_2} \ldots g_{i_n} b_{i_n} : 0 \leq i_1 < i_2 < \ldots < i_n < \omega\}$$

is a *piecewise shifted IP-set*.

Theorem 1. *For a subset A of a group G, the following statements are equivalent*:
(i) *A is scattered*;
(ii) *A is disparse*;
(iii) *A contains no subsets coarsely equivalent to the group $\oplus_\omega \mathbb{Z}_2$*;
(iv) *A contains no piecewise shifted IP-sets.*

By the equivalence (i) $\Leftrightarrow$ (ii) and Propositions 10 and 12 from [6], the family of all scattered subsets of an infinite group G is a translation invariant ideal in the Boolean algebra of all subsets of G strictly contained in the ideal of all small subsets.

Now we describe some relationships between the left invariant ideals Sp_G, Sc_G of all sparse and scattered subsets of a group G on one hand, and closed left ideals of the semigroup βG.

Let J be a left invariant ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of a group G. We set

$$\hat{J} = \{p \in \beta G : G \setminus A \in p \text{ for each } A \in J\}$$

and note that $\hat{J}$ is a closed left ideal of the semigroup βG. On the other hand, for a closed left ideal L of βG, we set

$$\check{L} = \{A \subseteq G : A \notin p \text{ for each } p \in L\}$$

and note that $\check{L}$ is a left invariant ideal in $\mathcal{P}_G$. Moreover, $\check{\hat{J}} = J$ and $\hat{\check{L}} = L$.

Clearly, $\widehat{[G]^{<\omega}} = G^*$ and by Theorem 1,

$$\hat{Sc_G} = \mathrm{cl}\{p \in \beta G : Gp \text{ is discrete in } \beta G\} =$$

$$= \mathrm{cl}\{p \in \beta G : p = \varepsilon p \text{ for some idempotent } \varepsilon \in G^*\}. \tag{1}$$

Given a left invariant ideal J in $\mathcal{P}_G$ and following [11], we define a left invariant ideal $\sigma(J)$ by the rule: $A \in \sigma(J)$ if and only if $\Delta_p(A)$ is finite for every $p \in \hat{J}$. Equivalently, $\sigma(J) = \mathrm{cl}(\check{G}^* \hat{J})$. Thus, we have

$$\hat{Sp_G} = \mathrm{cl}(G^* G^*).$$

We say that a left invariant ideal J in $\mathcal{P}_G$ is *sparse-complete* if $\sigma(J) = J$ and denote by $\sigma^*(J)$ the intersection of all sparse-complete ideals containing J. Clearly, the sparse-completion $\sigma^*(J)$ is the smallest sparse-complete ideal such that $J \subseteq \sigma^*(J)$. By [11] (Theorem 4(1)), $\sigma^*(J) = \bigcup_{n \in \omega} \sigma^n(J)$, where $\sigma^0(J) = J$ and $\sigma^{n+1}(J) = \sigma(\sigma^n(J))$. We can prove that $A \in \sigma^n([G]^{<\omega})$ if and only if A has no subsets asymorphic to $W_n = \{g \in \oplus_\omega \mathbb{Z}_2 : \operatorname{supt} g \le n\}$.

By [11] (Theorem 4(2)), the ideal Sp_G is not sparse complete. By (1), the ideal Sc_G is sparse-complete. Hence $\sigma^*([G]^{<\omega}) \subseteq Sc_G$ but $\sigma^*([G]^{<\omega}) \neq Sc_G$.

Recall that a subset A of an amenable group G is *absolute null* if $\mu(A) = 0$ for each left invariant Banach measure μ on G. For sparse subsets, the following theorem was proved in [10] (Theorem 5.1).

Theorem 2. *Every scattered subset A of an amenable group G is absolute null.*

Let A be a subset of $\mathbb{Z}$. The *upper density* $\overline{d}(A)$ is denoted by

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{-n, -n+1, \ldots, n-1, n\}|}{2n+1}.$$

By [25] (Theorem 11.11), if $\overline{d}(A) > 0$, then A contains a piecewise shifted IP-set. We note that Theorem 2 generalizes this statement because there exists a Banach measure μ on $\mathbb{Z}$ such that $\overline{d}(A) = \mu(A)$.

In connection with Theorem 1, one may ask if it possible to replace piecewise shifted IP-sets to (left or right) shifted IP-sets. By Theorem 2 and [25] (Theorem 11.6), this is impossible.

By Theorem 1 and Proposition 13 from [6] an infinite group G can be partitioned into $\aleph_0$ scattered subsets provided that G is embeddable into a direct product of countable groups, in particular if G is Abelian.

Theorem 3. *Every infinite group G can be partitioned into $\aleph_0$ scattered subsets.*

We note that Theorem 3 does not hold with sparse subsets in place of scattered subsets [22].

By Theorems 1 and 3, every infinite groups admits a countable partition such that each cell has no piecewise-shifted IP-sets.

We recall that a free ultrafilter p on a set X is *countably complete* if, for every countable partition of X, one cell of the partition is a member of p. In this case $|X|$ should be Ulam-measurable. Let p be a countably complete ultrafilter on a group G. Applying Theorem 3, we conclude that the orbit Gp is discrete in G^*.

3. Proofs. ***Proof of Proposition* 1.** (i) $\Rightarrow$ [(ii). We take $p \in Y^*$ such that $\Delta_p(Y)$ is finite, so $\Delta_p(Y) = Fp$ for some $F \in [G]^{<\omega}$. Given any $H \in [G]^{\omega}$, we have $hp \notin \Delta_p(Y)$ for each $h \in H \setminus F$. Hence $hP_h \cap Y = \varnothing$ for some $P_h \in p$. We put $P = \bigcap_{h \in H \setminus P} P_h$ and note that

$$P \subseteq \{y \in Y : B_Y(y, H) \setminus B_Y(y, F) = \varnothing\}.$$

(ii) $\Rightarrow$ (i). We take an infinite subset Y of A, choose corresponding $F \in [G]^{<\omega}$ and, for each $H \in [G]^{<\omega}$, denote

$$P_H = \{y \in Y : B_Y(y, H) \setminus B_Y(y, F) = \varnothing\}.$$

By (ii), the family $\{P_H : H \in [G]^{<\omega}\}$ has a finite intersection property and $\bigcap_{H \in [G]^{<\omega}} P_H = \varnothing$. Hence $\{P_H : H \in [G]^{<\omega}\}$ is contained in some ultrafilter $p \in Y^*$. By the choice of p, we have $gp \notin \Delta_p(Y)$ for each $g \in G \setminus (F \cup \{e\})$, e is the identity of G. It follows that $\Delta_p(Y)$ is finite so A is scattered.

***Proof of Proposition* 2.** Assume that A is not scattered and choose a subset Y of A which does not satisfy the condition (ii) of Proposition 1. We take an arbitrary $a \in A$ and put $F_0 = \{e, a\}$. Then we choose inductively a sequence $(F_n)_{n \in \omega}$ in $[G]^{<\omega}$ such that

(1) $F_n F_n^{-1} \subset F_{n+1}$;

(2) $B_Y(y, F_{n+1}) \setminus B_Y(y, F_n) \neq \varnothing$ for every $y \in Y$.

After ω steps, we put $H = \bigcup_{n \in \omega} F_n$. By the choice of F_0, $Y \cap H \neq \varnothing$. By (1), H is a subgroup. By (2), $(Y \cap H)$ is not scattered in H.

***Proof of Proposition* 3.** Replacing G by by the subgroup generating by X, we assume that G is countable. We write G as an union of an increasing chain F_n of finite subsets such that $F_0 = \{e\}$, $F_n = F_n^{-1}$. In view of [23], it suffices to find a cellular subset Y of X with no asymptotically isolated balls.

Since X is not cellular, there exists $F \in [G]^{<\omega}$ such that

(1) for every $n \in \mathbb{N}$, there is $x \in X$ such that

$$B_X^{\square}(x, F) \setminus B_X(x, F_n) \neq \varnothing.$$

We assume that G is finitely generated and choose a system of generators $K \in [G]^{<\omega}$ such that $K = K^{-1}$ and $F \subseteq K$. Then we consider the Cayley graph $\Gamma = \mathrm{Cay}(G, K)$ with the set of vertices G and the set of edges $\{\{g, h\} : g^{-1}h \in K\}$. We endow Γ with the path metric d and say that a sequence $a_0, \ldots, a_n \in G$ is a geodesic path if $a_0, \ldots, a_n$ is the shortest path from a_0 to a_n, in particular, $d(a_0, a_n) = n$. Using (1), for each $n \in \mathbb{N}$, we choose a geodesic path L_n of length 3^n such that $L_n \subset X$ and

(2) $B_G(L_n, F_n \cap B_G(L_{n+1}, F_{n+1}) = \varnothing$ for every $n \in \mathbb{N}$. Let $L_n = \{a_{n0}, \ldots, a_{n3^n}\}$. For each $i \in \{0, \ldots, 3^n\}$, we take a tercimal decomposition of i and denote by Y_n the subset of all $a_i \in L$ such that i has no 1-s in its decomposition based on $\{0, 1, 2\}$ (see [26]). By [2] and the construction of Y_n, the set $Y = \bigcup_{n\in\mathbb{N}} Y_n$ is cellular and has no asymptotically isolated balls.

Now let G be an arbitrary countable group. We consider a subgroup H of G generated by F and decompose G into left cosets by H. If X meets only finite number of these cosets, then X is contained in some finitely generated subgroup of G and we arrive in the previous case. At last, let $\{Hx_n : n \in \mathbb{N}\}$ be a decomposition of G into left cosets by H and X meets infinitely many of them. We endow each Hx_n with the structure of a graph Γ_n naturally isomorphic to the Cayley graph $\mathrm{Cay}(H, F)$. Then we use (1) to choose an increasing sequence $(m_n)_{n\in\omega}$ and a sequence $(L_n)_{n\in\mathbb{N}}$ of geodesic paths of length 3^n satisfying (2) and such that $L_n \subset X$. For each $n \in \mathbb{N}$, we define a subset Y_n of L_n as before, and put $Y = \bigcup_{n\in\mathbb{N}} Y_n$. By (2) and the construction of Y_n, Y is cellular and has no asymptotically isolated balls.

***Proof of Theorem* 1.** We follow the tour (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv). We prove that a piecewise shifted IP-subset

$$A = \{g_{i_1} g_{i_2} \ldots g_{i_n} b_{i_n} : 0 \le i_1 < \ldots < i_n < \omega\}$$

of G is not scattered. For each $m \in \omega$, let

$$A_m = \{g_{i_1} g_{i_2} \ldots g_{i_n} b_{i_n} : m < i_1 < \ldots < i_n < \omega\}.$$

We take an arbitrary $p \in A^*$ and show that $\Delta_p(A)$ is infinite.

If $A_m \in p$ for every $m \in \omega$, then $g_n p \in A^*$ for each $n \in \omega$. Otherwise, there exists $m \in \omega$ such that

$$\{g_m g_{i_1} \ldots g_{i_n} b_{i_n} : m < i_1 < \ldots < i_n < \omega\} \in p.$$

Then $g_m^{-1} p \in A^*$ and we repeat the arguments for $g_m^{-1} p$.

(iv) $\Rightarrow$ (ii). Assume that A is not disparse and take $p \in A^*$ such that p is not isolated in $\Delta_p(A)$. Then $p = qp$ for some $q \in G^*$. The set $\{x \in G^* : xp = p\}$ is a closed subsemigroup of G^* and, by [1] (Theorem 2.5), there is an idempotent $r \in G^*$ such that $p = rp$. We take $R \in r$ and $P_g \in p$, $g \in R$ such that $\bigcup_{g\in R} gP_g \subseteq A$.Since r is an idempotent, by [1] (Theorem 5.8), there is an injective sequence $(g_n)_{n\in\omega}$ in G such that

$$\{g_{i_1} \ldots g_{i_n} : 0 \le i_1 < \ldots < i_n < \omega\} \subseteq R.$$

For each $n \in \omega$, we pick $b_n \in \bigcap\{P_g : g = g_{i_1} \ldots g_{i_n} : 0 \le i_1 < \ldots < i_n < \omega\}$ and note that

$$\{g_{i_1} \ldots g_{i_n} b_{i_n} : 0 \le i_1 < \ldots < i_n < \omega\} \subseteq A.$$

(ii) $\Rightarrow$ (iii). We assume that A contains a subset coarsely equivalent to the group $B = \oplus_\omega \mathbb{Z}_2$. Then there exist a subset X of B, $H \in [B]^{<\omega}$ such that $B = H + X$, and an injective $\prec$-mapping $f : X \to A$. We take an arbitrary idempotent $r \in B^*$, pick $h \in H$ such that $h + X \in r$ and put $p = r - h$. Since $r + p = r$, we see that p is not isolated in $\Delta_p(X)$. We denote $q = f^\beta(p)$. Let $b \in B$, $b \ne 0$ and $b + p \in X^*$. Since f is an injective $\prec$-mapping, there is $g \in G \setminus \{e\}$ such that $f^\beta(b + p) = g + q$. It follows that q is not isolated in $\Delta_q(A)$. Hence A is not disparse.

(iii) $\Rightarrow$ (i). Let X be a countable subset of A. By Proposition 3, X is cellular. By [23], X satisfies Proposition 1(ii). Hence X is scattered. By Proposition 2, A is scattered.

***Proof of Theorem* 2.** We assume that $\mu(A) > 0$ for some Banach measure μ on G. We use the arguments from [10, p. 506, 507] to choose a decreasing sequence $(A_n)_{n\in\omega}$ of subsets of G and an injective sequence $(g_n)_{n\in\omega}$ in G such that $A_0 = A$, $g_n A_{n+1} \subseteq A_n$ and $\mu(A_n) > 0$ for each $n \in \omega$. We pick $x_n \in A_{n+1}$ and put

$$X = \{g_0^{\varepsilon_0} \ldots g_n^{\varepsilon_n} x_n : n \in \omega \, , \, \varepsilon_i \in \{0, 1\}\}.$$

By the construction X is a piecewise shifted IP-sets and $X \subseteq A$. By Theorem 1, X is not scattered.

Theorem 2 is proved.

To prove Theorem 3, we need some definitions and notations.

Let G be an infinite group with the identity e, $\varkappa$ be an infinite cardinal. A family $\{G_\alpha \colon \alpha < \varkappa\}$ of subgroups of G is called a *filtration* if the following conditions hold:

(1) $G_0 = \{e\}$ and $G = \bigcup_{\alpha<\varkappa} G_\alpha$;

(2) $G_\alpha \subset G_\beta$ for all $\alpha < \beta < \varkappa$;

(3) $\bigcup_{\alpha<\beta} G_\alpha = G_\beta$ for each limit ordinal $\beta < \varkappa$.

Clearly, a countable group G admits a filtration if and only if G is not finitely generated. Every uncountable group G of cardinality $\varkappa$ admits a filtration satisfying the additional condition $|G_\alpha| < \varkappa$ for each $\alpha < \kappa$.

Following [27], for each $0 < \alpha < \kappa$, we decompose $G_{\alpha+1} \setminus G_\alpha$ into right cosets by G_α and choose some system X_α of representatives so $G_{\alpha+1} \setminus G_\alpha = G_\alpha X_\alpha$. We take an arbitrary element $g \in G \setminus \{e\}$ and choose the smallest subgroup G_α with $g \in G_\alpha$. By (3), $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < \varkappa$. Hence, $g \in G_{\alpha+1} \setminus G_{\alpha_1}$ and there exist $g_1 \in G_{\alpha_1}$ and $x_{\alpha_1} \in X_{\alpha_1}$ such that $g = g_1 x_{\alpha_1}$. If $g_1 \neq e$, we choose the ordinal α_2 and elements $g_2 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$ and $x_{\alpha_2} \in X_{\alpha_2}$ such that $g_1 = g_2 x_{\alpha_2}$. Since the set of ordinals $\{\alpha \colon \alpha < \varkappa\}$ is well-ordered, after finite number $s(g)$ of steps, we get the representation

$$g = x_{\alpha_{s(g)}} x_{\alpha_{s(g)-1}} \ldots x_{\alpha_2} x_{\alpha_1}, \quad x_{\alpha_i} \in X_{\alpha_i}.$$

We note that this representation is unique. For $n \in \mathbb{N}$, we denote

$$D_n = \{g \in G \setminus \{e\} \colon s(g) = n\}.$$

Given any $g = x_{\alpha_{s(g)}} \ldots x_{\alpha_2} x_{\alpha_1}$ and $m \in \{1, \ldots, s(g)\}$, we put

$$g(m) = x_{\alpha_{s(g)}} \ldots x_{\alpha_m}, \qquad \max g = \alpha_1.$$

For a subset $P \subseteq D_n$ and $m \in \{1, \ldots, n\}$, we denote

$$P(m) = \{g(m) \colon g \in P\},$$

$$\max P = \{\max g \colon g \in P\}.$$

Let p be an ultrafilter on G such that $D_n \in p$ and $m \in \{1, \ldots, n\}$. We denote by $\max p$ the ultrafilter on $\varkappa$ with the base $\{\max P \colon P \in p, P \subseteq D_n\}$, and by $p(m)$ the ultrafilter on G with the base $\{P(m) \colon P \in p, P \subseteq D_n\}$.

***Proof of Theorem* 3.** We use an induction by the cardinality of G. If G is countable, the statement is evident because each singleton is scattered. Assume that we have proved the theorem for all groups of cardinality less than $\varkappa$, $\varkappa > \aleph_0$ and take an arbitrary group G of cardinality $\varkappa$. We fix a filtration $\{G_\alpha\colon \alpha < \varkappa\}$ of G such that $|G_\alpha| < \varkappa$ for each $\alpha < \varkappa$.

For every $\alpha < \varkappa$, we use the inductive hypothesis to define a mapping $\chi_\alpha\colon G_{\alpha+1} \setminus G_\alpha \to \mathbb{N}$ such that $\chi_\alpha^{-1}(i)$ is scattered in $G_{\alpha+1}$ for every $i \in \mathbb{N}$. Then we take $g \in G \setminus \{e\}$, $g = x_{\alpha_n} \ldots x_{\alpha_1}$ and put

$$\chi(g) = (\chi_{\alpha_n}(x_{\alpha_n}), \chi_{\alpha_{n-1}}(x_{\alpha_n} x_{\alpha_{n-1}}), \ldots, \chi_{\alpha_1}(x_{\alpha_n} x_{\alpha_{n-1}} \ldots x_{\alpha_1})).$$

Thus, we have defined a mapping $\chi\colon G \setminus \{e\} \to \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$. In view of Theorem 1, it suffices to verify that $\chi^{-1}(m)$ is disparse for each $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$. We shall prove this statement by induction on n. Let $m = (m_1)$ and let p be an ultrafilter on G such that $\chi^{-1}(m) \in p$. We endow $\varkappa$ with the interval topology and denote $\lambda = \lim \max p$. If p is a principal ultrafilter then, by the definition of χ_λ, Gp is discrete. Otherwise, we take $P \in p$ such that $\max p \leq \lambda$. Then $P \notin gp$ for each $g \in G \setminus \{e\}$.

Suppose that we have proved that $\chi^{-1}(m')$ is disparse for each $m' = (m'_1, \ldots, m'_i)$, $i < n$ and let $m = (m_1, \ldots, m_n)$. We take an arbitrary ultrafilter p on G such that $\chi^{-1}(m) \in p$. If $\max p$ is a principal, we use the inductive hypothesis. Otherwise, we denote $\lambda_1 = \lim \max p(1), \ldots, \lambda_n = = \lim \max p(n)$. We choose $k \in \{1, \ldots, n\}$ such that $\lambda_1 = \ldots = \lambda_k$, $\lambda_{k+1} < \lambda_k$. By the inductive hypothesis, there exists $P \in p$ such that $P(k+1) \notin gp(k+1)$ for every $g \in G \setminus \{e\}$ such that $\max g \leq \lambda_{k+1}$. Then we choose $Q \in p$, $Q \subseteq D_n$ such that $\max q(k+1) \leq \lambda_{k+1}$, $\max q(k) > \lambda_{k+1}$ for each $q \in Q$. Then $Gp \cap \overline{(P \cap Q)} = \{p\}$, so Gp is discrete.

Theorem 3 is proved.

The referee pointed out that, with another definition, the scattered subsets appeared in [28].

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