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Feng Qi (Inst. Math., Henan Polytechnic Univ. and College Sci., Tianjin Polytechnic Univ., China),
Tian-Yu Zhang, Bo-Yan Xi (College Math., Inner Mongolia Univ. Nat., China)

HERMITE – HADAMARD TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE OF CONVEXITY *

ІНТЕГРАЛЬНІ НЕРІВНОСТІ ТИПУ ЕРМІТА – АДАМАРА, ПЕРШІ ПОХІДНІ ЯКИХ МАЮТЬ ОПУКЛІСТЬ

We establish some new Hermite – Hadamard-type inequalities for functions whose first derivatives are of convexity and apply these inequalities to construct inequalities for special means.

Встановлено деякі нові нерівності типу Ерміта – Адамара для функцій, похідні яких мають опуклість. Ці нерівності застосовано при побудові нерівностей для спеціальних середніх.

1. Introduction. Throughout this paper, we will use I and I° to denote an interval on the real line $\mathbb{R}$ and the interior of I respectively.

In [4], the following Hermite – Hadamard type inequalities for continuously differentiable convex functions were proved.

Theorem 1.1 ([4], Theorem 2.2). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.2 ([4], Theorem 2.3). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $p > 1$. If the new mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}.$$

In [9], the above inequalities were generalized as follows.

Theorem 1.3 ([9], Theorems 1 and 2). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable on I°, $a, b \in I$ with $a < b$, and $q \geq 1$. If $|f'|^q$ is convex on $[a, b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}.$$

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In [7], the above inequalities were further generalized as follows.

Theorem 1.4 ([7], Theorems 2.3 and 2.4). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable on I°, $a, b \in I$ with $a < b$, and $p > 1$. If $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \times$$

$$\times\left\{\left[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\right]^{(p-1)/p} + \left[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right]^{(p-1)/p}\right\}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}(|f'(a)| + |f'(b)|).$$

In [5], an inequality similar to the above ones was given as follows.

Theorem 1.5 ([5], Theorem 3). *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p([a, b])$. Then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1/q}\|f'\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Recently, the following inequalities were obtained in [10, 11].

Theorem 1.6 [10]. *Let $I \subseteq \mathbb{R}$ be an open interval, with $a, b \in I$ and $a < b$, and $f : I \to \mathbb{R}$ be twice continuously differentiable mapping such that f'' is integrable. If $0 \le \lambda \le 1$ and $|f''|$ is a convex function on $[a, b]$, then*

$$\left|(\lambda - 1)f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2} + \int_a^b f(x)dx\right| \le$$

$$\le \begin{cases} \dfrac{(b-a)^2}{24}\left\{\left[\lambda^4 + (1+\lambda)(1-\lambda)^3 + \dfrac{5\lambda-3}{4}\right]|f''(a)| + \right. \\ \qquad \left. + \left[\lambda^4 + (2-\lambda)\lambda^3 + \dfrac{1-3\lambda}{4}\right]|f''(b)|\right\}, \quad 0 \le \lambda \le \dfrac{1}{2}, \\ \dfrac{(b-a)^2}{48}(3\lambda - 1)\big(|f''(a)| + |f''(b)|\big), \quad \dfrac{1}{2} \le \lambda \le 1. \end{cases}$$

Theorem 1.7 [11]. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L([a, b])$. If $|f'|^q$ is convex for $q \ge 1$ on $[a, b]$, then*

$$\left|\frac{1}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le$$

$$\le \frac{b-a}{12}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}\left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{1/q} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{1/q}\right]$$

and

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le$$

$$\le\frac{5(b-a)}{72}\left[\left(\frac{61|f'(a)|^q+29|f'(b)|^q}{90}\right)^{1/q}+\left(\frac{29|f'(a)|^q+61|f'(b)|^q}{90}\right)^{1/q}\right].$$

A function $f: I\subseteq\mathbb{R}\to[0,\infty)$ is said to be s-convex if the inequality

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y)$$

holds for all $x,y\in I$, $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$, and some fixed $s\in(0,1]$.

In [1], some inequalities of Hermite – Hadamard type for s-convex functions were established as follows.

Theorem 1.8 ([1], Theorems 2.3 and 2.4)**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously differentiable mapping on I° such that $f'\in L([a,b])$, where $a,b\in I$ with $a<b$.*

If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$ for $p>1$ and some fixed $s\in(0,1]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\times$$

$$\times\left\{\left[\left(2^{1-s}+s+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right]^{1/q}+\left[2^{1-s}|f'(a)|^q+\left(2^{1-s}+s+1\right)|f'(b)|^q\right]^{1/q}\right\},$$

where p is the conjugate of q, that is, $\frac{1}{p}+\frac{1}{q}=1.$

If $|f'|^q$ is s-convex on $[a,b]$ for $q\ge1$ and some fixed $s\in(0,1]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le\frac{b-a}{8}\left[\frac{2}{(s+1)(s+2)}\right]^{1/q}\times$$

$$\times\left\{\left[\left(2^{1-s}+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right]^{1/q}+\left[\left(2^{1-s}+1\right)|f'(b)|^q+2^{1-s}|f'(a)|^q\right]^{1/q}\right\}.$$

Some inequalities of Hermite – Hadamard type were also obtained in [2, 3, 6, 8, 12, 14 – 17, 19] and plenty of references therein.

In this paper we will establish some new Hermite – Hadamard type integral inequalities for functions whose first derivatives are of convexity and apply them to derive some inequalities of special means.

2. Lemmas. For establishing our new integral inequalities of Hermite – Hadamard type, we need the following lemmas.

Lemma 2.1. *Let I be an interval and $f: I \to \mathbb{R}$ be continuously differentiable on I°, with $a, b \in I$ and $a < b$, and $\lambda, \mu \in \mathbb{R}$. If $f' \in L([a,b])$, then*

$$(1-\mu)f(a) + \lambda f(b) + (\mu - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= (b-a)\left[\int_0^{1/2} (\lambda - t) f'(ta + (1-t)b)dt + \int_{1/2}^1 (\mu - t) f'(ta + (1-t)b)dt\right].$$

Proof. This follows from standard integration by parts.

Lemma 2.2 [18]. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable on I°, $a, b \in I$ with $a < b$. If $f' \in L([a,b])$ and $\lambda, \mu \in \mathbb{R}$, then*

$$\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{b-a}{4}\int_0^1 \left[(1-\lambda-t) f'\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu - t) f'\left(t\frac{a+b}{2} + (1-t)b\right)\right] dt.$$

Proof. This may be derived via standard integration by parts.

***Remark* 2.1.** Lemmas 2.1 and 2.2 are equivalent to each other.

3. New integral inequalities of Hermite–Hadamard type. Now we are in a position to establish some new integral inequalities of Hermite–Hadamard type for functions whose derivatives are of convexity.

Theorem 3.1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function on I°, $a, b \in I$ with $a < b$ and $0 \le \lambda \le \frac{1}{2} \le \mu \le 1$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|(1-\mu)f(a) + \lambda f(b) + (\mu - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{24}\big[\big(10 - 3\lambda +$$

$$+ 8\lambda^3 - 15\mu + 8\mu^3\big)|f'(a)| + \big(8 - 9\lambda + 24\lambda^2 - 8\lambda^3 - 21\mu + 24\mu^2 - 8\mu^3\big)|f'(b)|\big]. \qquad (3.1)$$

Proof. By Lemma 2.1 and the convexity of $|f'|$ on $[a,b]$, we have

$$\left|(1-\mu)f(a) + \lambda f(b) + (\mu - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le$$

$$\le (b-a)\left[\int_0^{1/2} |\lambda - t||f'(ta + (1-t)b)|dt + \int_{1/2}^1 |\mu - t||f'(ta + (1-t)b)|dt\right] \le$$

$$\leq (b-a)\left[\int\limits_0^{1/2} |\lambda - t|\big(t|f'(a)| + (1-t)|f'(b)|\big)dt + \int\limits_{1/2}^{1} |\mu - t|\big(t|f'(a)| + (1-t)|f'(b)|\big)dt\right].$$

Substituting equations

$$\int\limits_0^{1/2} |\lambda - t|\big(t|f'(a)| + (1-t)|f'(b)|\big)dt = \frac{1}{24}\left[\big(1 - 3\lambda + 8\lambda^3\big)|f'(a)| + \big(2 - 9\lambda + 24\lambda^2 - 8\lambda^3\big)|f'(b)|\right]$$

and

$$\int\limits_{1/2}^{1} |\mu - t|\big(t|f'(a)| + (1-t)|f'(b)|\big)dt =$$

$$= \frac{1}{24}\left[\big(9 - 15\mu + 8\mu^3\big)|f'(a)| + \big(6 - 21\mu + 24\mu^2 - 8\mu^3\big)|f'(b)|\right]$$

into the above inequality leads to (3.1).

Theorem 3.1 is proved.

Theorem 3.2. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function on* I°, $a, b \in I$ *with* $a < b$ *and* $0 \leq \lambda \leq \dfrac{1}{2} \leq \mu \leq 1$. *If* $|f'|^q$ *for* $q > 1$ *is convex on* $[a, b]$ *and* $q \geq p > 0$, *then*

$$\left|(1-\mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int\limits_a^b f(x)dx\right| \leq$$

$$\leq (b-a)\left(\frac{q-1}{2q-p-1}\right)^{1-1/q}\left[\frac{1}{(p+1)(p+2)}\right]^{1/q} \times$$

$$\times\left\{\left[\left(\frac{1}{2} - \lambda\right)^{(2q-p-1)/(q-1)} + \lambda^{(2q-p-1)/(q-1)}\right]^{1-1/q} \times\right.$$

$$\times\left(\left[\frac{1}{2}(p+1+2\lambda)\left(\frac{1}{2} - \lambda\right)^{p+1} + \lambda^{p+2}\right]|f'(a)|^q +\right.$$

$$\left. + \left[\frac{1}{2}(p+3-2\lambda)\left(\frac{1}{2} - \lambda\right)^{p+1} + (p+2-\lambda)\lambda^{p+1}\right]|f'(b)|^q\right)^{1/q} +$$

$$+ \left[\left(\mu - \frac{1}{2}\right)^{(2q-p-1)/(q-1)} + (1-\mu)^{(2q-p-1)/(q-1)}\right]^{1-1/q} \times$$

$$\times\left(\left[\frac{1}{2}(p+1+2\mu)\left(\mu - \frac{1}{2}\right)^{p+1} + (p+1+\mu)(1-\mu)^{p+1}\right]|f'(a)|^q +\right.$$

$$+\left[\frac{1}{2}(p+3-2\mu)\left(\mu-\frac{1}{2}\right)^{p+1}+(1-\mu)^{p+2}\right]|f'(b)|^q\Bigg)^{1/q}\Bigg\}.$$

Proof. By Lemma 2.1, the convexity of $|f'|^q$ on $[a,b]$, and Hölder's integral inequality, we obtain

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le$$

$$\le(b-a)\left[\int\limits_0^{1/2}|\lambda-t||f'(ta+(1-t)b)|dt+\int\limits_{1/2}^1|\mu-t||f'(ta+(1-t)b)|dt\right]\le$$

$$\le(b-a)\left[\left(\int\limits_0^{1/2}|\lambda-t|^{(q-p)/(q-1)}dt\right)^{1-1/q}\left(\int\limits_0^{1/2}|\lambda-t|^p|f'(ta+(1-t)b)|^q dt\right)^{1/q}+\right.$$

$$\left.+\left(\int\limits_{1/2}^1|\mu-t|^{(q-p)/(q-1)}dt\right)^{1-1/q}\left(\int\limits_{1/2}^1|\mu-t|^p|f'(ta+(1-t)b)|^q dt\right)^{1/q}\right]\le$$

$$\le(b-a)\left\{\left[\int\limits_0^{1/2}|\lambda-t|^{(q-p)/(q-1)}dt\right]^{1-1/q}\left[\int\limits_0^{1/2}|\lambda-t|^p\big(t|f'(a)|^q+(1-t)|f'(b)|^q\big)dt\right]^{1/q}+\right.$$

$$\left.+\left[\int\limits_{1/2}^1|\mu-t|^{(q-p)/(q-1)}dt\right]^{1-1/q}\left[\int\limits_{1/2}^1|\mu-t|^p\big(t|f'(a)|^q+(1-t)|f'(b)|^q\big)dt\right]^{1/q}\right\}. \qquad (3.2)$$

Furthermore, a straightforward computation gives

$$\int\limits_0^{1/2}|\lambda-t|^{(q-p)/(q-1)}dt=\frac{q-1}{2q-p-1}\left[\left(\frac{1}{2}-\lambda\right)^{(2q-p-1)/(q-1)}+\lambda^{(2q-p-1)/(q-1)}\right], \qquad (3.3)$$

$$\int\limits_{1/2}^1|\mu-t|^{(q-p)/(q-1)}dt=\frac{q-1}{2q-p-1}\left[\left(\mu-\frac{1}{2}\right)^{(2q-p-1)/(q-1)}+(1-\mu)^{(2q-p-1)/(q-1)}\right],$$

$$\int\limits_0^{1/2}|\lambda-t|^p\big(t|f'(a)|^q+(1-t)|f'(b)|^q\big)dt=\frac{1}{(p+1)(p+2)}\left\{\left[\frac{1}{2}(p+1+2\lambda)\left(\frac{1}{2}-\lambda\right)^{p+1}+\right.\right.$$

$$+\lambda^{p+2}\Bigg]|f'(a)|^q+\left[\frac{1}{2}(p+3-2\lambda)\left(\frac{1}{2}-\lambda\right)^{p+1}+(p+2-\lambda)\lambda^{p+1}\right]|f'(b)|^q\Bigg\},$$

and

$$\int\limits_{1/2}^{1}|\mu-t|^p\big(t|f'(a)|^q+(1-t)|f'(b)|^q\big)dt=\frac{1}{(p+1)(p+2)}\left\{\left[\frac{1}{2}(p+1+2\mu)\left(\mu-\frac{1}{2}\right)^{p+1}+\right.\right.$$

$$\left.\left.+(p+1+\mu)(1-\mu)^{p+1}\right]|f'(a)|^q+\left[\frac{1}{2}(p+3-2\mu)\left(\mu-\frac{1}{2}\right)^{p+1}+(1-\mu)^{p+2}\right]|f'(b)|^q\right\}. \tag{3.4}$$

Substituting the equations from (3.3) to (3.4) into (3.2) results in the inequality (3.1).

Theorem 3.2 is proved.

***Corollary* 3.1.** *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function on I°, $a,b\in I$ with $a<b$ and $0\le\lambda\le\dfrac{1}{2}\le\mu\le1$. If $|f'|^q$ for $q\ge1$ is convex on $[a,b]$, then*

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le\frac{b-a}{2}\left(\frac{1}{3}\right)^{1/q}\times$$

$$\times\left\{\left[\left(\frac{1}{2}-\lambda\right)^2+\lambda^2\right]^{1-1/q}\left(\left[(1+\lambda)\left(\frac{1}{2}-\lambda\right)^2+\lambda^3\right]|f'(a)|^q+\left[(2-\lambda)\left(\frac{1}{2}-\lambda\right)^2+\right.\right.\right.$$

$$\left.\left.+(3-\lambda)\lambda^2\right]|f'(b)|^q\right)^{1/q}+\left[\left(\mu-\frac{1}{2}\right)^2+(1-\mu)^2\right]^{1-1/q}\left(\left[(1+\mu)\left(\mu-\frac{1}{2}\right)^2+\right.\right.$$

$$\left.\left.\left.+(2+\mu)(1-\mu)^2\right]|f'(a)|^q+\left[(2-\mu)\left(\mu-\frac{1}{2}\right)^2+(1-\mu)^3\right]|f'(b)|^q\right)^{1/q}\right\}$$

and

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le$$

$$\le\frac{b-a}{2}\left[\frac{2}{(q+1)(q+2)}\right]^{1/q}\left\{\left(\left[\frac{1}{2}(q+1+2\lambda)\left(\frac{1}{2}-\lambda\right)^{q+1}+\lambda^{q+2}\right]|f'(a)|^q+\right.\right.$$

$$\left.+\left[\frac{1}{2}(q+3-2\lambda)\left(\frac{1}{2}-\lambda\right)^{q+1}+(q+2-\lambda)\lambda^{q+1}\right]|f'(b)|^q\right)^{1/q}+$$

$$+\left(\left[\frac{1}{2}(q+1+2\mu)\left(\mu-\frac{1}{2}\right)^{q+1}+(q+1+\mu)(1-\mu)^{q+1}\right]|f'(a)|^q+\right.$$

$$\left.\left.+\left[\frac{1}{2}(q+3-2\mu)\left(\mu-\frac{1}{2}\right)^{q+1}+(1-\mu)^{q+2}\right]|f'(b)|^q\right)^{1/q}\right\}.$$

Proof. This follows from Theorem 3.1 and setting $p=1$ and $p=q$ in Theorem 3.2.

***Corollary* 3.2.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function on I°, $a,b\in I$ with $a<b$, $m>0$, and $m\ge 2\ell\ge 0$. If $|f'|^q$ for $q>1$ is convex on $[a,b]$, and $q\ge p>0$, then*

$$\left|\frac{\ell}{m}[f(a)+f(b)]+\frac{m-2\ell}{m}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le$$

$$\le\frac{b-a}{4m^2}\left(\frac{q-1}{2q-p-1}\right)^{1-1/q}\left(\frac{1}{2m(p+1)(p+2)}\right)^{1/q}\times$$

$$\times[(2\ell)^{(2q-p-1)/(q-1)}+(m-2\ell)^{(2q-p-1)/(q-1)}]^{1-1/q}\times$$

$$\times\Big\{\big([(2\ell)^{p+2}+(mp+m+2\ell)(m-2\ell)^{p+1}]|f'(a)|^q+$$

$$+[(2mp+4m-2\ell)(2\ell)^{p+1}+(mp+3m-2\ell)(m-2\ell)^{p+1}]|f'(b)|^q\big)^{1/q}+$$

$$+\big([(2mp+4m-2\ell)(2\ell)^{p+1}+(mp+3m-2\ell)(m-2\ell)^{p+1}]|f'(a)|^q+$$

$$+[(2\ell)^{p+2}+(mp+m+2\ell)(m-2\ell)^{p+1}]|f'(b)|^q\big)^{1/q}\Big\}.$$

Proof. This follows from letting $\lambda=1-\mu=\dfrac{\ell}{m}$ in Theorem 3.2.

***Corollary* 3.3.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function on I°, a, $b\in I$ with $a<b$, $m>0$, and $m\ge 2\ell\ge 0$. If $|f'|^q$ for $q\ge 1$ is convex on $[a,b]$, then*

$$\left|\frac{\ell}{m}[f(a)+f(b)]+\frac{m-2\ell}{m}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le$$

$$\le\frac{b-a}{8m^2}\left(\frac{1}{3m}\right)^{1/q}[(m-2\ell)^2+(2\ell)^2]^{1-1/q}\Big\{\big([(m+\ell)(m-2\ell)^2+4\ell^3]|f'(a)|^q+$$

$$+[(2m-\ell)(m-2\ell)^2+4(3m-\ell)\ell^2]|f'(b)|^q\big)^{1/q}+\big([4(3m-\ell)\ell^2+$$

$$+(2m-\ell)(m-2\ell)^2]|f'(a)|^q+[(m+\ell)(m-2\ell)^2+4\ell^3]|f'(b)|^q\big)^{1/q}\Big\}$$

and

$$\left|\frac{\ell}{m}[f(a)+f(b)]+\frac{m-2\ell}{m}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\le$$

$$\le\frac{b-a}{4m}\left[\frac{1}{2m^2(q+1)(q+2)}\right]^{1/q}\Big\{\big([(2\ell)^{q+2}+(mq+m+2\ell)(m-2\ell)^{q+1}]|f'(a)|^q+$$

$$+[(2mq+4m-2\ell)(2\ell)^{q+1}+(mq+3m-2\ell)(m-2\ell)^{q+1}]|f'(b)|^q\big)^{1/q}+$$

$$+\big([(2mq+4m-2\ell)(2\ell)^{q+1}+(mq+3m-2\ell)(m-2\ell)^{q+1}]|f'(a)|^q+$$

$$+[(2\ell)^{q+2}+(mq+m+2\ell)(m-2\ell)^{q+1}]|f'(b)|^q\big)^{1/q}\Big\}.$$

Proof. This follows from setting $\lambda=1-\mu=\dfrac{\ell}{m}$ in Corollary 3.1.

4. Applications to special means. For two positive numbers $a>0$ and $b>0$, define

$$A(a,b)=\frac{a+b}{2},\quad G(a,b)=\sqrt{ab}\,,\quad H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a\ne b,\\ a, & a=b,\end{cases}\qquad L(a,b)=\begin{cases}\dfrac{b-a}{\ln b-\ln a}, & a\ne b,\\ a, & a=b,\end{cases}$$

and

$$L_s(a,b)=\begin{cases}\left[\dfrac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{1/s}, & s\ne 0,-1 \text{ and } a\ne b,\\ L(a,b), & s=-1 \text{ and } a\ne b,\\ I(a,b), & s=0 \text{ and } a\ne b,\\ a, & a=b.\end{cases}$$

It is well known that A, G, H, $L=L_{-1}$, $I=L_0$, and L_s are respectively called the arithmetic, geometric, harmonic, logarithmic, exponential, and generalized logarithmic means of two positive number a and b.

Theorem 4.1. *Let* $b>a>0$, $q>1$, $q\ge p>0$, $m>0$, $m\ge 2\ell\ge 0$, *and* $s\in\mathbb{R}$.

(1) *If either* $s>1$ *and* $(s-1)q\ge 1$ *or* $s<1$ *and* $s\ne 0$, *then*

$$\left|\frac{2\ell A\big(a^s,b^s\big)+(m-2\ell)[A(a,b)]^s}{m}-[L_s(a,b)]^s\right|\le\frac{b-a}{4m^2}|s|\left(\frac{q-1}{2q-p-1}\right)^{1-1/q}\times$$

$$\times\left[\frac{1}{2m(p+1)(p+2)}\right]^{1/q}\left[(2\ell)^{(2q-p-1)/(q-1)}+(m-2\ell)^{(2q-p-1)/(q-1)}\right]^{1-1/q}\times$$

$$\times\Big\{\big[\big((2\ell)^{p+2}+(mp+m+2\ell)(m-2\ell)^{p+1}\big)a^{(s-1)q}+$$

$$+\big((2mp+4m-2\ell)(2\ell)^{p+1}+(mp+3m-2\ell)(m-2\ell)^{p+1}\big)b^{(s-1)q}\big]^{1/q}+$$

$$+\big[\big((2mp+4m-2\ell)(2\ell)^{p+1}+(mp+3m-2\ell)(m-2\ell)^{p+1}\big)a^{(s-1)q}+$$

$$+\big((2\ell)^{p+2}+(mp+m+2\ell)(m-2\ell)^{p+1}\big)b^{(s-1)q}\big]^{1/q}\Big\}.$$

(2) *If* $s=-1$, *then*

$$\left|\frac{1}{m}\left[\frac{2\ell}{H(a,b)}+\frac{m-2\ell}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|\le\frac{b-a}{4m^2}\left(\frac{q-1}{2q-p-1}\right)^{1-1/q}\left[\frac{1}{2m(p+1)(p+2)}\right]^{1/q}\times$$

$$\times\big[(2\ell)^{(2q-p-1)/(q-1)}+(m-2\ell)^{(2q-p-1)/(q-1)}\big]^{1-1/q}\times$$

$$\times\left\{\left[\frac{(2\ell)^{p+2}+(mp+m+2\ell)(m-2\ell)^{p+1}}{a^{2q}}+\right.\right.$$

$$\left.+\frac{(2mp+4m-2\ell)(2\ell)^{p+1}+(mp+3m-2\ell)(m-2\ell)^{p+1}}{b^{2q}}\right]^{1/q}+$$

$$+\left[\frac{(2mp+4m-2\ell)(2\ell)^{p+1}+(mp+3m-2\ell)(m-2\ell)^{p+1}}{a^{2q}}+\right.$$

$$\left.\left.+\frac{(2\ell)^{p+2}+(mp+m+2\ell)(m-2\ell)^{p+1}}{b^{2q}}\right]^{1/q}\right\}.$$

Proof. Set $f(x)=x^s$ for $x>0$ and $s\ne 0,1$. Then it is easy to obtain that

$$f'(x)=sx^{s-1},\quad |f'(x)|^q=|s|^q x^{(s-1)q},\quad \big(|f'(x)|^q\big)''=(s-1)q[(s-1)q-1]|s|^q x^{(s-1)q-2}.$$

Hence, when $s>1$ and $(s-1)q\ge 1$, or when $s<1$ and $s\ne 0$, the function $|f'|^q$ is convex on $[a,b]$. From Corollary 3.2, Theorem 4.1 follows.

By the argument similar to Theorem 4.1, we further obtain the following conclusions.

Theorem 4.2. *Let* $b>a>0$, $q\ge 1$, $m>0$, $m\ge 2\ell\ge 0$, *and* $s\in\mathbb{R}$.

(1) *If either* $s>1$ *and* $(s-1)q\ge 1$ *or* $s<1$ *and* $s\ne 0$, *then*

$$\left|\frac{2\ell A(a^s,b^s)+(m-2\ell)[A(a,b)]^s}{m}-[L_s(a,b)]^s\right|\le\frac{b-a}{8m^2}\left(\frac{1}{3m}\right)^{1/q}\big[4\ell^2+(m-2\ell)^2\big]^{1-1/q}|s|\times$$

$$\times\Big\{\big[\big(4\ell^3+(m+\ell)(m-2\ell)^2\big)a^{(s-1)q}+\big(4(3m-\ell)\ell^2+(2m-\ell)(m-2\ell)^2\big)b^{(s-1)q}\big]^{1/q}+$$

$$+\big[\big(4(3m-\ell)\ell^2+(2m-\ell)(m-2\ell)^2\big)a^{(s-1)q}+\big(4\ell^3+(m+\ell)(m-2\ell)^2\big)b^{(s-1)q}\big]^{1/q}\Big\}$$

and

$$\left|\frac{2\ell A(a^s,b^s)+(m-2\ell)[A(a,b)]^s}{m}-[L_s(a,b)]^s\right|\le\frac{b-a}{4m}|s|\left[\frac{1}{2m^2(q+1)(q+2)}\right]^{1/q}\times$$

$$\times\Big\{\big[\big((2\ell)^{q+2}+(mq+m+2\ell)(m-2\ell)^{q+1}\big)a^{(s-1)q}+$$

$$+\big((2mq+4m-2\ell)(2\ell)^{q+1}+(mq+3m-2\ell)(m-2\ell)^{q+1}\big)b^{(s-1)q}\big]^{1/q}+$$

$$+\big[\big((2mq+4m-2\ell)(2\ell)^{q+2}+(mq+3m-2\ell)(m-2\ell)^{q+1}\big)a^{(s-1)q}+$$

$$+\big((2\ell)^{q+1}+(mq+m+2\ell)(m-2\ell)^{q+1}\big)b^{(s-1)q}\big]^{1/q}\Big\}.$$

(2) *If* $s=-1$, *then*

$$\left|\frac{1}{m}\left[\frac{2\ell}{H(a,b)}+\frac{m-2\ell}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|\leq\frac{b-a}{8m^2}\left(\frac{1}{3m}\right)^{1/q}\left[4\ell^2+(m-2\ell)^2\right]^{1-1/q}\times$$

$$\times\left\{\left[\frac{4\ell^3+(m+\ell)(m-2\ell)^2}{a^{2q}}+\frac{4(3m-\ell)\ell^2+(2m-\ell)(m-2\ell)^2}{b^{2q}}\right]^{1/q}+\right.$$

$$\left.+\left[\frac{4(3m-\ell)\ell^2+(2m-\ell)(m-2\ell)^2}{a^{2q}}+\frac{4\ell^3+(m+\ell)(m-2\ell)^2}{b^{2q}}\right]^{1/q}\right\}$$

and

$$\left|\frac{1}{m}\left[\frac{2\ell}{H(a,b)}+\frac{m-2\ell}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|\leq\frac{b-a}{4m}\left[\frac{1}{2m^2(q+1)(q+2)}\right]^{1/q}\times$$

$$\times\left\{\left[\frac{(2\ell)^{q+2}+(mq+m+2\ell)(m-2\ell)^{q+1}}{a^{2q}}+\right.\right.$$

$$\left.+\frac{(mq+3m-2\ell)(m-2\ell)^{q+1}+(2mq+4m-2\ell)(2\ell)^{q+1}}{b^{2q}}\right]^{1/q}+$$

$$+\left[\frac{(2mq+4m-2\ell)(2\ell)^{q+1}+(mq+3m-2\ell)(m-2\ell)^{q+1}}{a^{2q}}+\right.$$

$$\left.\left.+\frac{(2\ell)^{q+2}+(mq+m+2\ell)(m-2\ell)^{q+1}}{b^{2q}}\right]^{1/q}\right\}.$$

In particular, we have

$$\left|\frac{2\ell A(a^s,b^s)+(m-2\ell)[A(a,b)]^s}{m}-[L_s(a,b)]^s\right|\leq\frac{b-a}{4m^2}|s|\left[4\ell^2+(m-2\ell)^2\right]A\big(a^{s-1},b^{s-1}\big)$$

and

$$\left|\frac{1}{m}\left[\frac{2\ell}{H(a,b)}+\frac{m-2\ell}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|\leq\frac{b-a}{4m^2}\frac{4\ell^2+(m-2\ell)^2}{H(a^2,b^2)}.$$

Theorem 4.3. *Let $b > a > 0$, $q > 1$, $q \geq p > 0$, $m > 0$, and $m \geq 2\ell \geq 0$. Then*

$$\left|\frac{2\ell \ln G(a,b) + (m-2\ell)\ln A(a,b)}{m} - \ln I(a,b)\right| \leq \frac{b-a}{4m^2}\left(\frac{q-1}{2q-p-1}\right)^{1-1/q} \times$$

$$\times\left[\frac{1}{2m(p+1)(p+2)}\right]^{1/q}\left[(m-2\ell)^{(2q-p-1)/(q-1)} + (2\ell)^{(2q-p-1)/(q-1)}\right]^{1-1/q}\times$$

$$\times\left\{\left[\frac{(2\ell)^{p+2} + (mp+m+2\ell)(m-2\ell)^{p+1}}{a^q}+\right.\right.$$

$$\left.+\frac{(mp+3m-2\ell)(m-2\ell)^{p+1} + (2mp+4m-2\ell)(2\ell)^{p+1}}{b^q}\right]^{1/q}+$$

$$+\left[\frac{(2mp+4m-2\ell)(2\ell)^{p+1} + (mp+3m-2\ell)(m-2\ell)^{p+1}}{a^q}+\right.$$

$$\left.\left.+\frac{(2\ell)^{p+2} + (mp+m+2\ell)(m-2\ell)^{p+1}}{b^q}\right]^{1/q}\right\}.$$

Proof. This follows from taking $f(x) = \ln x$ for $x > 0$ in Corollary 3.2.

By the similar argument to Theorem 4.3, we can obtain the following inequalities.

Theorem 4.4. *Let $b > a > 0$, $q \geq 1$, $m > 0$, and $m \geq 2\ell \geq 0$. Then*

$$\left|\frac{2\ell \ln G(a,b) + (m-2\ell)\ln A(a,b)}{m} - \ln I(a,b)\right| \leq \frac{b-a}{4m}\left[\frac{1}{2m^2(q+1)(q+2)}\right]^{1/q} \times$$

$$\times\left\{\left[\frac{(2\ell)^{q+2} + (mq+m+2\ell)(m-2\ell)^{q+1}}{a^q}+\right.\right.$$

$$\left.+\frac{(mq+3m-2\ell)(m-2\ell)^{q+1} + (2mq+4m-2\ell)(2\ell)^{q+1}}{b^q}\right]^{1/q}+$$

$$+\left[\frac{(2mq+4m-2\ell)(2\ell)^{q+1} + (mq+3m-2\ell)(m-2\ell)^{q+1}}{a^q}+\right.$$

$$\left.\left.+\frac{(2\ell)^{q+2} + (mq+m+2\ell)(m-2\ell)^{q+1}}{b^q}\right]^{1/q}\right\}$$

and

$$\left|\frac{2\ell \ln G(a,b) + (m-2\ell)\ln A(a,b)}{m} - \ln I(a,b)\right| \leq \frac{b-a}{8m^2}\left(\frac{1}{3m}\right)^{1/q}\left[(m-2\ell)^2 + (2\ell)^2\right]^{1-1/q}\times$$

$$\times\left\{\left[\frac{4\ell^3 + (m+\ell)(m-2\ell)^2}{a^q} + \frac{(2m-\ell)(m-2\ell)^2 + 4(3m-\ell)\ell^2}{b^q}\right]^{1/q}+\right.$$

$$+\left[\frac{4(3m-\ell)\ell^2+(2m-\ell)(m-2\ell)^2}{a^q}+\frac{4\ell^3+(m+\ell)(m-2\ell)^2}{b^q}\right]^{1/q}\Bigg\}.$$

In particular, we have

$$\left|\frac{2\ell\ln G(a,b)+(m-2\ell)\ln A(a,b)}{m}-\ln I(a,b)\right|\le\frac{b-a}{4m^2}\frac{4\ell^2+(m-2\ell)^2}{H(a,b)}.$$

***Remark* 4.1.** This paper is a simplified version of the preprint [13].

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