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TWO-TERM DIFFERENTIAL EQUATIONS WITH MATRIX DISTRIBUTIONAL COEFFICIENTS

ДВОЧЛЕННІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ З МАТРИЧНИМИ КОЕФІЦІЄНТАМИ-РОЗПОДІЛАМИ

We propose a regularization of the formal differential expression

$$l(y) = i^m y^{(m)}(t) + q(t)y(t), \quad t \in (a,b),$$

of order $m \geqslant 2$ with matrix distribution q. It is assumed that $q = Q^{([m/2])}$, where $Q = (Q_{i,j})_{i,j=1}^s$ is a matrix function with entries $Q_{i,j} \in L_2[a,b]$ if m is even and $Q_{i,j} \in L_1[a,b]$, otherwise. In the case of Hermitian matrix q, we describe self-adjoint, maximal dissipative, and maximal accumulative extensions of the associated minimal operator and its generalized resolvents.

Запропоновано регуляризацію формального диференціального виразу порядку $m \geqslant 2$

$$l(y) = i^m y^{(m)}(t) + q(t)y(t), \quad t \in (a,b),$$

з матричною узагальненою функцією q. Припускається, що $q = Q^{([m/2])}$, де $Q = (Q_{i,j})_{i,j=1}^s$ — матрична функція з елементами $Q_{i,j} \in L_2[a,b]$ у випадку парного m і $Q_{i,j} \in L_1[a,b]$ для непарного m. У випадку ермітової матриці q описано самоспряжені максимальні дисипативні та максимальні акумулятивні розширення асоційованого мінімального оператора та його узагальнені резольвенти.

1. Introduction. In [1] (see also [2]) it was proposed a regularization with the help of quasi–derivatives of the two-term formal differential expression

$$l(y) = i^m y^{(m)} + qy, \quad m \geq 3, \tag{1}$$

with distributional potential $q = Q^{([m/2])}$, where $Q \in L_2[a,b]$ if m is even and $Q \in L_1[a,b]$ otherwise. Note that the case $m = 2$ was considered earlier in [3] which started a new development in the theory of Schrödinger operators with distributional potentials. We mention here only papers [4, 5] and references therein. In particular in [5] (see also [6]) spectral properties of Schrödinger operators with matrix distributional potentials were studied.

The main purpose of this paper is the extension of the results of [1] to the case of matrix differential operators of the form (1), acting in the Hilbert space $L_2([a,b],\mathbb{C}^s) \equiv (L_2([a,b]))^s$, $s \in \mathbb{N}$. In the case of formally self-adjoint quasidifferential expression we apply the boundary triple technique to give the explicit description of the various classes of extensions of the corresponding minimal operator.

The paper is organized as follows. In Section 1 we recall basic definitions and known facts concerning the matrix quasidifferential operators. Section 2 presents the regularization of the formal differential expression (1) using the quasiderivatives. In Section 3 the boundary triplets for the minimal symmetric operators are constructed and maximal dissipative, maximal accumulative and self-adjoint extensions of these operators are explicitly described in terms of boundary conditions. Section 4 deals with the formally self-adjoint quasidifferential operators with real-valued coefficients. In this case we prove that every maximal dissipative (or accumulative) extension of the minimal operator is self-adjoint and describe all such extensions. In Section 5 the extensions with separated boundary conditions are considered. Section 6 deals with generalized resolvents of the minimal operator.

2. Matrix quasidifferential expressions. In this section we recall some basic facts concerning the matrix quasidifferential operators on a finite interval. For a more detailed discussion of quasi-differential equations the reader is referred to [7, 8] in the scalar coefficients case and to [9, 10] for general case with matrix coefficients.

Let $m, s \in \mathbb{N}$ and a finite (closed) interval $[a,b]$ be given. For a given set T, $M_s(T)$ denotes the set of $(s\times s)$-matrices with entries in T. Denote by $Z_{m,s}([a,b])$ the set of the $(m\times m)$-matrix-valued functions A with entries $a_{k,j}$ satisfy

1) $a_{k,j} \in M_s(L_1[a,b])$, $k,j = 1,2,\ldots,m$,

2) $a_{k,j} = 0$, $j \geq k+2$, $a_{k,k+1}$ is invertible a. e. on $\mathcal{J}$ for $k = 1,2,\ldots,m-1$.

Such matrices will be referred to as Shin – Zettl matrices of order m. Define inductively the associated quasiderivatives of orders $k \leq m$ of a (vector) function $y \in \mathrm{Dom}\,(A)$ in the following way:

$$D^{[0]}y := y,$$

$$D^{[k]}y := a_{k,k+1}^{-1}(t)\left((D^{[k-1]}y)' - \sum_{j=1}^{k} a_{k,j}(t)D^{[j-1]}y\right), \quad k = 1,2,\ldots,m-1,$$

where $a_{m,m+1} := I_s$, the identity $(s\times s)$-matrix, and the associated domain $\mathrm{Dom}\,(A)$ is defined by

$$\mathrm{Dom}(A) := \left\{y \,\middle|\, D^{[k]}y \in AC([a,b],\mathbb{C}^s),\ k = \overline{0,m-1}\right\}.$$

The above yields $D^{[m]}y \in L_1([a,b],\mathbb{C}^s)$. The quasidifferential expression $l(y)$ of order m associated with A is defined by

$$l(y) := i^m D^{[m]}y. \tag{2}$$

It gives rise to the associated *maximal* quasidifferential operator

$$L_{\max}\colon y \mapsto l(y),$$

$$\mathrm{Dom}(L_{\max}) = \left\{y \in \mathrm{Dom}(A) \,\middle|\, D^{[m]}y \in L_2([a,b],\mathbb{C}^s)\right\}$$

in the Hilbert space $L_2([a,b],\mathbb{C}^s)$, and the associated *minimal* quasidifferential operator is defined as the restriction of $L_{\max}$ onto the set

$$\mathrm{Dom}(L_{\min}) := \left\{y \in \mathrm{Dom}(L_{\max}) \,\middle|\, D^{[k]}y(a) = D^{[k]}y(b) = 0,\ k = \overline{0,m-1}\right\}.$$

If the matrix functions $a_{k,s}$ are sufficiently smooth, then all the brackets in the definition of the quasiderivatives can be expanded, and we come to the usual ordinary differential operators.

Let us recall the definition of the formally adjoint quasidifferential expression $l^+(y)$. The formally adjoint (also called the Lagrange adjoint) matrix A^+ for $A \in Z_{m,s}([a,b])$ is defined by

$$A^+ := -\Lambda_m^{-1}\overline{A^T}\Lambda_m,$$

where $\overline{A^T}$ is the conjugate transposed matrix to A and

$$\Lambda_m := \begin{pmatrix} 0 & 0 & \dots & 0 & -I_s \\ 0 & 0 & \dots & I_s & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^{m-1}I_s & \dots & 0 & 0 \\ (-1)^m I_s & 0 & \dots & 0 & 0 \end{pmatrix}.$$

One can easily see that $\Lambda_m^{-1} = (-1)^{m-1}\Lambda_m$. In the similar way one can define the Shin – Zettl quasiderivatives associated with A^+ which will be denoted by $D^{\{0\}}y, D^{\{1\}}y, \dots, D^{\{m\}}y$, acting on the domain

$$\mathrm{Dom}(A^+) := \left\{ y \,\middle|\, D^{\{k\}}y \in AC([a,b],\mathbb{C}^s),\ k = \overline{0, m-1} \right\}.$$

The formally adjoint quasidifferential expression is defined as $l^+(y) := i^m D^{\{m\}}y$. Denote the associated maximal and minimal operators by $L^+_{\max}$ and $L^+_{\min}$ respectively. The following results are proved in [9] (see also [10]).

Lemma 1 (Green's formula)**.** *For any* $y \in \mathrm{Dom}(L_{\max})$, $z \in \mathrm{Dom}(L^+_{\max})$ *there holds*

$$\int\limits_a^b \left(D^{[m]}y \cdot \overline{z} - y \cdot \overline{D^{\{m\}}z} \right) dt = \sum_{k=1}^m (-1)^{k-1} D^{[m-k]}y \cdot \overline{D^{\{k-1\}}z} \,\Big|_{t=a}^{t=b}.$$

Lemma 2. *For any* $(\alpha_0, \alpha_1, \dots, \alpha_{m-1}), (\beta_0, \beta_1, \dots, \beta_{m-1}) \in \mathbb{C}^{ms}$ *there exists a function* $y \in$ $\in \mathrm{Dom}(L_{\max})$ *such that*

$$D^{[k]}y(a) = \alpha_k, \quad D^{[k]}y(b) = \beta_k, \quad k = 0, 1, \dots, m-1.$$

Theorem 1. *The operators* $L_{\min}$, $L^+_{\min}$, $L_{\max}$, $L^+_{\max}$ *are closed and densely defined in* $L_2([a,b],\mathbb{C}^s)$, *and satisfy*

$$L^*_{\min} = L^+_{\max}, \quad L^*_{\max} = L^+_{\min}.$$

If $l(y) = l^+(y)$, *then the operator* $L_{\min} = L^+_{\min}$ *is symmetric with the deficiency indices* (ms, ms), *and*

$$L^*_{\min} = L_{\max}, \quad L^*_{\max} = L_{\min}.$$

3. Regularizations by quasiderivatives. Consider the formal matrix differential expression

$$l(y) = i^m y^{(m)}(t) + q(t)y(t), \quad m \geq 2,$$

assuming that

$$q = Q^{(n)}, \quad n = \left[\frac{m}{2}\right],$$

$$Q \in \begin{cases} M_s(L_2[a,b]), & m = 2n, \\ M_s(L_1[a,b]), & m = 2n+1, \end{cases} \tag{3}$$

where the derivatives of Q are understood in the sense of distributions. Introduce the quasiderivatives as follows:

$$\begin{aligned} &D^{[r]}y = y^{(r)}, \quad r = 0, 1, \ldots, m-n-1, \\ &D^{[m-n+k]}y = (D^{[m-n+k-1]}y)' + i^{-m}(-1)^k \binom{n}{k} QD^{[k]}y, \quad k = 0, 1, \ldots, n-1, \\ &D^{[m]}y = \begin{cases} (D^{[m-1]}y)' + i^{-m}(-1)^n \dbinom{n}{n} QD^{[n]}y, & m = 2n+1, \\ (D^{[m-1]}y)' + QD^{[n]}y + (-1)^{n+1}Q^2 y, & m = 2n, \end{cases} \end{aligned} \tag{4}$$

where $\binom{k}{j}$ are the binomial coefficients. It is easy to verify that for sufficiently smooth matrix functions Q we have $l(y) = i^m D^{[m]}y$. The Shin – Zettl matrix corresponding to (4) has the form

$$A := \begin{pmatrix} 0 & I_s & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_s & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -i^{-m}\dbinom{n}{0} Q & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & i^{-m}\dbinom{n}{1} Q & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & I_s \\ (-1)^{\frac{m}{2}} \delta_{2n,m} Q^2 & 0 & 0 & \ldots & i^{-m}(-1)^{n+1}\dbinom{n}{n} Q & \ldots & 0 & 0 \end{pmatrix}, \tag{5}$$

where δ_{ij} is the Kronecker symbol. Note that under assumptions (3), all the coefficients of the Shin – Zettl matrix (5) are integrable matrix functions. The regularization of the initial formal differential expression (1) is defined by (2) and generates the corresponding quasidifferential operators $L_{\min}$ and $L_{\max}$.

***Remark* 1.** For $m - n \leq r < m$ the quasiderivatives $D^{[r]}$ depend on the choice of the antiderivative Q of order n of the (matrix) distribution q which is defined up to a a polynomial of order $\leq n-1$. Nevertheless in the sense of distributions $i^m D^{[m]}y = l[y]$ does not depend on this polynomial. Morever, it is easy to see that the corresponding maximal and minimal operators also do not depend on the choice of antiderivative (cf. [2]).

In the case $s = 1, m = 2$ the above regularization was proposed in [3]. The extension to arbitrary even m was announced in [11]. The case of general $m \geq 3$ was considered in [1, 2]. Here we extend this approach on the arbitrary $s \geq 1$.

4. Extensions of symmetric quasidifferential operators. Throughout the rest of the paper we assume that the matrix distribution q is Hermitian (due to Remark 1 one can suppose that Q is Hermitian). It follows from Theorem 1 that the minimal quasidifferential operator $L_{\min}$ is symmetric with deficiency indices (ms, ms). Therefore it is interesting to describe various classes of extensions of $L_{\min}$ in $L_2([a,b], \mathbb{C}^s)$. For this purpose we will exploit the theory of boundary triplets [12].

Definition 1. *Let T be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficiency indices. The triplet (H, Γ_1, Γ_2), where H is an auxiliary Hilbert* *space and* Γ_1, Γ_2 *are the linear maps from* $\mathrm{Dom}\,(T^*)$ *to* H, *is called a boundary triplet for* T, *if the following two conditions are satisfied*:

(1) *for any* $f, g \in \mathrm{Dom}\,(L^*)$ *there holds*

$$(T^* f, g)_{\mathcal{H}} - (f, T^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_2 g)_H - (\Gamma_2 f, \Gamma_1 g)_H\,,$$

(2) *for any* $g_1, g_2 \in H$ *there is a vector* $f \in \mathrm{Dom}\,(T^*)$ *such that* $\Gamma_1 f = g_1$ *and* $\Gamma_2 f = g_2$.

The above definition implies that $f \in \mathrm{Dom}\,(T)$ if and only if $\Gamma_1 f = \Gamma_2 f = 0$. A boundary triplet (H, Γ_1, Γ_2) with $\dim H = k$ exists for any symmetric operator T with equal non-zero deficiency indices (k, k) $(k \leq \infty)$, but it is not unique [12–14].

The following result is crucial for the rest of the paper.

Lemma 3. *Let* n *be a positive integer. Define linear maps* $\Gamma_{[1]}$, $\Gamma_{[2]}$ *from* $\mathrm{Dom}(L_{\max})$ *to* $\mathbb{C}^{ms}$ *as follows: for* $m = 2n$ *we set*

$$\Gamma_{[1]} y := i^{2n} \begin{pmatrix} -D^{[2n-1]}y(a) \\ \ldots \\ (-1)^n D^{[n]}y(a) \\ D^{[2n-1]}y(b) \\ \ldots \\ (-1)^{n-1} D^{[n]}y(b) \end{pmatrix}, \quad \Gamma_{[2]} y := \begin{pmatrix} D^{[0]}y(a) \\ \ldots \\ D^{[n-1]}y(a) \\ D^{[0]}y(b) \\ \ldots \\ D^{[n-1]}y(b) \end{pmatrix} \tag{6}$$

and for $m = 2n + 1$ *and we set*

$$\Gamma_{[1]} y := i^{2n+1} \begin{pmatrix} -D^{[2n]}y(a) \\ \ldots \\ (-1)^n D^{[n+1]}y(a) \\ D^{[2n]}y(b) \\ \ldots \\ (-1)^{n-1} D^{[n+1]}y(b) \\ \alpha D^{[n]}y(b) + \beta D^{[n]}y(a) \end{pmatrix}, \quad \Gamma_{[2]} y := \begin{pmatrix} D^{[0]}y(a) \\ \ldots \\ D^{[n-1]}y(a) \\ D^{[0]}y(b) \\ \ldots \\ D^{[n-1]}y(b) \\ \gamma D^{[n]}y(b) + \delta D^{[n]}y(a) \end{pmatrix},$$

where

$$\alpha = 1, \quad \beta = 1, \quad \gamma = \frac{(-1)^n}{2} + i, \quad \delta = \frac{(-1)^{n+1}}{2} + i.$$

Then $(\mathbb{C}^{ms}, \Gamma_{[1]}, \Gamma_{[2]})$ *is a boundary triplet for* $L_{\min}$.

***Remark* 2.** The values of the coefficients α, β, γ, δ may be replaced by an arbitrary quadruple of numbers satisfying the conditions

$$\alpha\overline{\gamma} + \overline{\alpha}\gamma = (-1)^n, \quad \beta\overline{\delta} + \overline{\beta}\delta = (-1)^{n+1}, \quad \alpha\overline{\delta} + \overline{\beta}\gamma = 0,$$

$$\beta\overline{\gamma} + \overline{\alpha}\delta = 0, \quad \alpha\delta - \beta\gamma \neq 0.$$

***Proof of Lemma* 3.** The proof follows from Lemmas 1 and 2. It repeats the arguments of [1, 2] in the case of scalar differential operators $(s = 1)$.

For any bounded operator K in $\mathbb{C}^{ms}$ denote by L_K the restriction of $L_{\max}$ onto the set of the functions $y \in \mathrm{Dom}\,(L_{\max})$ satisfying the homogeneous boundary condition in the canonical form (see [12])

$$(K-I)\,\Gamma_{[1]}y+i\,(K+I)\,\Gamma_{[2]}y=0. \tag{7}$$

Similarly, denote by L^K the restriction of $L_{\max}$ onto the set of the functions $y\in\operatorname{Dom}(L_{\max})$ satisfying the boundary condition

$$(K-I)\,\Gamma_{[1]}y-i\,(K+I)\,\Gamma_{[2]}y=0. \tag{8}$$

Recall that a densely defined linear operator T on a complex Hilbert space $\mathcal{H}$ is called *dissipative* (resp. *accumulative*) if

$$\Im\,(Tx,x)_{\mathcal{H}}\geq 0 \quad (\text{resp.}\leq 0), \quad \text{for all} \quad x\in \mathrm{Dom}(T)$$

and it is called *maximal dissipative* (resp. *maximal accumulative*) if, in addition, T has no nontrivial dissipative (resp. accumulative) extensions in $\mathcal{H}$. Every symmetric operator is both dissipative and accumulative, and every self-adjoint operator is a maximal dissipative and maximal accumulative one. According to Phillips' theorem (see [12, p. 154]) every maximal dissipative or accumulative extension of a symmetric operator is a restriction of its adjoint operator. Abstract results of [12] and Lemma 3 lead to the following description of dissipative, accumulative and self-adjoint extensions of $L_{\min}$.

Theorem 2. *Every L_K with K being a contracting operator in $\mathbb{C}^{ms}$, is a maximal dissipative extension of $L_{\min}$. Similarly every L^K with K being a contracting operator in $\mathbb{C}^{ms}$, is a maximal accumulative extension of the operator $L_{\min}$. Conversely, for any maximal dissipative (respectively, maximal accumulative) extension $\widetilde{L}$ of the operator $L_{\min}$ there exists a contracting operator K such that $\widetilde{L}=L_K$ (respectively, $\widetilde{L}=L^K$). The extensions L_K and L^K are self-adjoint if and only if K is a unitary operator on $\mathbb{C}^{ms}$. These correspondences between operators $\{K\}$ and the extensions $\{\widetilde{L}\}$ are all bijective.*

***Remark* 3.** It follows from [10] (Theorem 7.2) that in the case of even m $L_{\min}$ and therefore all its extensions are bounded below. Otherwise, for odd m the operator $L_{\min}$ is unbounded below and above (see, e.g., [10], Theorem 10.3).

***Remark* 4.** Analogously to [2] one can prove that the mapping $K\to L_K$ is not only bijective but also continuous. More accurately, if unitary operators K_n strongly converge to an operator K, then

$$\left\|(L_K-\lambda)^{-1}-(L_{K_n}-\lambda)^{-1}\right\|\to 0, \qquad n\to\infty, \quad \operatorname{Im}\lambda\neq 0.$$

The converse is also true, because the set of unitary operators in the space $\mathbb{C}^{ms}$ is a compact set. This means that the mapping

$$K\to (L_K-\lambda)^{-1}, \quad \operatorname{Im}\lambda\neq 0,$$

is a homeomorphism for any fixed $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

5. Real extensions. Recall that a linear operator L acting in $L_2([a,b],\mathbb{C}^s)$ is called *real* if for every function $y\in\mathrm{Dom}(L)$ the complex conjugate function $\overline{y}$ also lies in $\mathrm{Dom}(L)$ and $L(\overline{y})=\overline{L(y)}$.

If the minimal quasidifferential operator is real, one arrives at the natural question on how to describe its real extensions. The following theorem is valid.

Theorem 3. *Let $m=2n$, and let the entries of the Hermitian matrix distribution q be real-valued, then the maximal and minimal quasi-differential operators $L_{\max}$ and $L_{\min}$ generated by Shin – Zettle matrix* (5) *are real. All real maximal dissipative and maximal accumulative extensions of $L_{\min}$ are self-adjoint. The self-adjoint extensions L_K or L^K are real if and only if the unitary matrix K is symmetric.*

Proof. By Remark 1 one can assume that Q is real all the coefficients of the quasi-derivatives are real matrix functions. Therefore

$$D^{[i]}\overline{y} = \overline{D^{[i]}y}, \quad i = 1, 2, \ldots, m,$$

which implies $l(\overline{y}) = \overline{l(y)}$. Thus for any $y \in \mathrm{Dom}\,(L_{\max})$ we have $\overline{y} \in \mathrm{Dom}(L_{\max})$ and $L_{\max}(\overline{y}) = \overline{L_{\max}(y)}$. It follows that the operator $L_{\max}$ is real. Analogously, the operator $L_{\min}$ is a also real. Let L_K be an arbitrary real maximal dissipative extension given by the boundary conditions (7), then for any $y \in \mathrm{Dom}(L_K)$ the complex conjugate $\overline{y}$ satisfies (7) too, that is

$$(K - I)\,\Gamma_{[1]}\overline{y} + i\,(K + I)\,\Gamma_{[2]}\overline{y} = 0.$$

Due to the real-valuedness of the coefficients of the quasiderivatives, the equalities (6) imply

$$\Gamma_{[1]}\overline{y} = \overline{\Gamma_{[1]}y}, \quad \Gamma_{[2]}\overline{y} = \overline{\Gamma_{[2]}y}.$$

By taking the complex conjugates we obtain

$$\left(\overline{K} - I\right)\Gamma_{[1]}y - i\left(\overline{K} + I\right)\Gamma_{[2]}y = 0,$$

and $L_K \subset L^{\overline{K}}$ due to Theorem 2. Thus, the dissipative extension L_K is also accumulative, which means that it is symmetric. As L_K is a maximal dissipative extension of $L_{\min}$ we have that the operator $L_K = L^{\overline{K}}$ is self-adjoint. It follows that K is a unitary operator. In this case the boundary condition (7) is equivalent to

$$\left(K^{-1} - I\right)\Gamma_{[1]}y - i\left(K^{-1} + I\right)\Gamma_{[2]}y = 0.$$

It follows that $L_K = L^{K^{-1}}$. On the other hand $L_K = L^{\overline{K}}$ and therefore $K^{-1} = \overline{K}$. As K is unitary, we have $K^{-1} = \overline{K^T}$, which gives $K = K^T$. Here K^T is the transpose of the matrix K. In a similar way one can show that a maximal accumulative extension L^K is real if and only if it is self-adjoint and $K = K^T$.

Theorem 3 is proved.

6. Separated boundary conditions. In this section we discuss the extensions of $L_{\min}$ defined by the so-called separated boundary conditions. Denote by $\mathbf{f_a}$ the germ of a continuous function f at the point a. We recall that the boundary conditions that define an operator $L \subset L_{\max}$ are called *separated* if for any $y \in \mathrm{Dom}(L)$ and any $g, h \in \mathrm{Dom}(L_{\max})$ with

$$\mathbf{g_a} = \mathbf{y_a}, \qquad \mathbf{g_b} = 0, \qquad \mathbf{h_a} = 0, \qquad \mathbf{h_b} = \mathbf{y_b}$$

we have $g, h \in \mathrm{Dom}(L)$.

The following theorem gives a description of the operators L_K and L^K with separated boundary conditions in the case of an even order $m = 2n$.

Theorem 4. *The boundary conditions* (7) *and* (8) *defining L_K and L^K respectively are separated if and only if the matrix K has the block form*

$$K = \begin{pmatrix} K_a & 0 \\ 0 & K_b \end{pmatrix}, \tag{9}$$

where K_a and K_b are $(ns \times ns)$-matrices.

Proof. We consider the operators L_K only, the case of L^K can be considered in a similar way. Denote

$$\Gamma_{[1]} =: (\Gamma_{1a}, \Gamma_{1b}), \quad \Gamma_{[2]} =: (\Gamma_{2a}, \Gamma_{2b}),$$

where

$$\Gamma_{1a}y = i^{2n}\left(-D^{[2n-1]}y(a), \ldots, (-1)^n D^{[n]}y(a)\right),$$

$$\Gamma_{1b}y = i^{2n}\left(D^{[2n-1]}y(b), \ldots, (-1)^{n-1} D^{[n]}y(b)\right),$$

$$\Gamma_{2a}y = \left(D^{[0]}y(a), \ldots, D^{[n-1]}y(a)\right),$$

$$\Gamma_{2b}y = \left(D^{[0]}y(b), \ldots, D^{[n-1]}y(b)\right).$$

Let $y, g \in \mathrm{Dom}\,(L_{\max})$. Clearly, for any $c \in [a,b]$ the equality $\mathbf{y_c} = \mathbf{g_c}$ implies that $D^{[k]}y_c = D^{[k]}g_c$, $k = 0, 1, \ldots, m-1$. In particular, the equality $\mathbf{y_a} = \mathbf{g_a}$ implies $\Gamma_{1a}y = \Gamma_{1a}g$ and $\Gamma_{2a}y = \Gamma_{2a}g$, and the equality $\mathbf{y_b} = \mathbf{h_b}$ implies $\Gamma_{1b}y = \Gamma_{1b}h$ and $\Gamma_{2b}y = \Gamma_{2b}h$.

If K has the form (9), then the boundary condition (7) can be rewritten as a system

$$(K_a - I)\Gamma_{1a}y + i(K_a + I)\Gamma_{2a}y = 0,$$

$$-(K_b - I)\Gamma_{1b}y + i(K_b + I)\Gamma_{2b}y = 0,$$

and these boundary conditions are obviously separated. Conversely, let the boundary conditions (7) be separated. The matrix $K \in \mathbb{C}^{2ns\times 2ns}$ can be written in the block form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

with $ns \times ns$ blocks K_{jk}. We need to show that $K_{12} = K_{21} = 0$. Let us rewrite boundary conditions (7) in the form of the system

$$(K_{11} - I)\Gamma_{1a}y + K_{12}\Gamma_{1b}y + i(K_{11} + I)\Gamma_{2a}y + iK_{12}\Gamma_{2b}y = 0,$$

$$K_{21}\Gamma_{1a}y + (K_{22} - I)\Gamma_{1b}y + iK_{21}\Gamma_{2a}y + i(K_{22} + I)\Gamma_{2b}y = 0.$$

As the boundary conditions are separated, any function g with $\mathbf{g_a} = \mathbf{y_a}$ and $\mathbf{g_b} = 0$ also satisfies this system, which gives

$$K_{11}\left[\Gamma_{1a}y + i\Gamma_{2a}y\right] = \Gamma_{1a}y - i\Gamma_{2a}y,$$

$$K_{21}\left[\Gamma_{1a}y + i\Gamma_{2a}y\right] = 0.$$

This means that

$$\Gamma_{1a}y + i\Gamma_{2a}y \in \mathrm{Ker}(K_{21}), \quad y \in \mathrm{Dom}(L_K). \tag{10}$$

For any $z = (z_1, z_2) \in \mathbb{C}^{2ns}$ consider the vectors $-i\,(K+I)\,z$ and $(K-I)\,z$. Due to Lemma 3 there is a function $y_z \in \mathrm{Dom}(L_{\max})$ such that

$$\begin{gathered} -i\,(K+I)\,z = \Gamma_{[1]} y_z, \\ (K-I)\,z = \Gamma_{[2]} y_z. \end{gathered} \tag{11}$$

A simple calculation shows that y_z satisfies the boundary conditions (7) and, therefore, $y_z \in \mathrm{Dom}(L_K)$. We can rewrite (11) as a system

$$-i(K_{11}+I)z_1 - iK_{12}z_2 = \Gamma_{1a}y_z,$$

$$-iK_{21}z_1 - i(K_{22}+I)z_2 = \Gamma_{1b}y_z,$$

$$(K_{11}-I)z_1 + K_{12}z_2 = \Gamma_{2a}y_z,$$

$$K_{21}z_1 + (K_{22}-I)z_2 = \Gamma_{2b}y_z.$$

It follows from the first and the third equations of the system above that

$$\Gamma_{1a}y_z + i\Gamma_{2a}y_z = -2iz_1$$

for any $z = (z_1, z_2) \in \mathbb{C}^{2ns}$. By (10) we have that $\mathrm{Ker}(K_{21}) = \mathbb{C}^{ns}$ or equivalently $K_{21} = 0$. Similarly one can prove that $K_{12} = 0$.

Theorem 4 is proved.

7. Generalized resolvents. Let us recall [15] that a *generalized resolvent* of a closed symmetric operator L in a Hilbert space $\mathcal{H}$ is an operator-valued function $\lambda \mapsto R_\lambda$ defined on $\mathbb{C} \setminus \mathbb{R}$ which can be represented as

$$R_\lambda x = P^+ \left(L^+ - \lambda I^+\right)^{-1} x, \quad x \in \mathcal{H},$$

where L^+ is a self-adjoint extension L which acts in a certain Hilbert space $\mathcal{H}^+$ containing $\mathcal{H}$ as a subspace, I^+ is the identity operator on $\mathcal{H}^+$, and P^+ is the orthogonal projection operator from $\mathcal{H}^+$ onto $\mathcal{H}$. It is known [15] that an operator-valued function $R_\lambda \quad (\mathrm{Im}\,\lambda \neq 0)$ is a generalized resolvent of a symmetric operator L if and only if it can be represented as

$$(R_\lambda x, y)_{\mathcal{H}} = \int\limits_{-\infty}^{+\infty} \frac{d\,(F_\mu x, y)}{\mu - \lambda}, \quad x, y \in \mathcal{H},$$

where F_μ is a generalized spectral function of the operator L, i.e., $\mu \mapsto F_\mu$ is an operator-valued function F_μ defined on $\mathbb{R}$ and taking values in the space of continuous linear operators in $\mathcal{H}$ with the following properties:

(1) for $\mu_2 > \mu_1$, the difference $F_{\mu_2} - F_{\mu_1}$ is a bounded nonnegative operator;

(2) $F_{\mu+} = F_\mu$ for any real μ;

(3) for any $x \in \mathcal{H}$ there holds

$$\lim_{\mu \to -\infty} ||F_\mu x||_{\mathcal{H}} = 0, \quad \lim_{\mu \to +\infty} ||F_\mu x - x||_{\mathcal{H}} = 0.$$

The following theorem provides a description of all generalized resolvents of the operator $L_{\min}$.

Theorem 5. 1. *Every generalized resolvent R_λ of the operator $L_{\min}$ in the half-plane $\operatorname{Im}\lambda < 0$ acts by the rule $R_\lambda h = y$, where y is the solution of the boundary-value problem*

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I)\,\Gamma_{[1]}f + i\,(K(\lambda) + I)\,\Gamma_{[2]}f = 0.$$

Here $h(x) \in L_2([a,b],\mathbb{C}^s)$ and $K(\lambda)$ is an $(ms \times ms)$-matrix-valued function which is holomorph in the lower half-plane and satisfy $||K(\lambda)|| \le 1$.

2. *In the half-plane $\operatorname{Im}\lambda > 0$, every generalized resolvent of $L_{\min}$ acts by $R_\lambda h = y$, where y is the solution of the boundary-value problem*

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I)\,\Gamma_{[1]}f - i\,(K(\lambda) + I)\,\Gamma_{[2]}f = 0.$$

Here $h(x) \in L_2([a,b],\mathbb{C}^s)$ and $K(\lambda)$ and $K(\lambda)$ is an $(ms \times ms)$-matrix-valued function which is holomorph in the upper half-plane and satisfy $||K(\lambda)|| \le 1$.

The parametrization of the generalized resolvents by the matrix-valued functions K is bijective.

Proof. The result directly follows from Lemma 3 and [14] (Theorem 1) which prove a description of generalized resolvents in terms of boundary triplets. We need only to take as an auxiliary Hilbert space $\mathbb{C}^{ms}$ and as the operator $\gamma y := \{\Gamma_{[1]}y, \Gamma_{[2]}y\}$.

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