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SOLUTIONS FOR THE QUASILINEAR ELLIPTIC SYSTEMS WITH COMBINED CRITICAL SOBOLEV – HARDY TERMS

РОЗВ'ЯЗКИ ДЛЯ КВАЗІЛІНІЙНИХ ЕЛІПТИЧНИХ СИСТЕМ З КОМБІНОВАНИМИ КРИТИЧНИМИ ЧЛЕНАМИ СОБОЛЄВА – ХАРДІ

We study the existence of multiple solutions for a quasilinear elliptic system. Based upon the Mountain – Pass theorem of Ambrosetti and Rabinowitz and symmetric Mountain – Pass theorem of Rabinowitz, we establish several existence and multiplicity results for the solutions and G-symmetric solutions under certain suitable conditions.

Вивчається задача існування багатьох розв'язків квазілінійної еліптичної системи. На основі теореми перевалу Амброзетті і Рабіновича та симетричної теореми перевалу Рабіновича встановлено кілька результатів про існування та множинність розв'язків та G-симетричних розв'язків за деяких прийнятних умов.

1. Introduction. Our purpose in the first part of this paper is to establish the existence of nontrivial solution to the following quasilinear elliptic system:

$$
\begin{aligned}
&-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \\
&= \frac{|u|^{p^*(a,b_1)-2}u}{|x|^{b_1p^*(a,b_1)}} + \frac{\alpha}{\alpha+\beta}Q(x)\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x-x_0|^{cp^*(a,c)}} + \lambda h(x)\frac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \quad x\in\Omega,\\
&-\operatorname{div}(|x|^{-ap}|\nabla v|^{p-2}\nabla v) - \mu\frac{|v|^{p-2}v}{|x|^{p(a+1)}} = \\
&= \frac{|v|^{p^*(a,b_2)-2}v}{|x|^{b_2p^*(a,b_2)}} + \frac{\beta}{\alpha+\beta}Q(x)\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x-x_0|^{cp^*(a,c)}} + \lambda h(x)\frac{|v|^{q-2}v}{|x|^{dp^*(a,d)}}, \quad x\in\Omega,\\
&u = v = 0, \quad x\in\partial\Omega,
\end{aligned}
\tag{1}
$$

where $0\in\Omega$ is a bounded domain in $\mathbb{R}^N$, $N\geq 3$, with smooth boundary $\partial\Omega$, $\lambda>0$ is a parameter, $1\leq q<p$, $1<p<N$, $0\leq\mu<\overline{\mu}\triangleq\left(\dfrac{N-p}{p-a}\right)^p$, $0\leq a<\dfrac{N-p}{p}$; $Q(x)$ is nonnegative and continuous on $\overline{\Omega}$ satisfying some additional conditions which will be given later, $Q(x_0)=\|Q\|_\infty$ for $0\neq x_0\neq\Omega$, $h(x)\in C(\overline{\Omega})$; $\alpha,\beta>1$, $\alpha+\beta=p^*(a,c)\triangleq\dfrac{pN}{N-p(1+a-c)}$, $p^*(a,b_1)\triangleq$ $\triangleq\dfrac{pN}{N-p(1+a-b_1)}$ $(a\leq b_1,b_2,d\leq c<a+1)$ are critical Sobolev – Hardy exponents. Note that $p^*(0,0)=p^*:=\dfrac{Np}{N-p}$ is the critical Sobolev exponent.

In the second part of this paper, we consider the following quasilinear elliptic system:

$$
\begin{aligned}
&-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \\
&= \frac{\alpha}{\alpha+\beta}Q(x)\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x-x_0|^{cp^*(a,c)}} + \lambda h(x)\frac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \quad x \in \Omega, \\
&-\mathrm{div}(|x|^{-ap}|\nabla v|^{p-2}\nabla v) - \mu\frac{|v|^{p-2}v}{|x|^{p(a+1)}} = \\
&= \frac{\beta}{\alpha+\beta}Q(x)\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x-x_0|^{cp^*(a,c)}} + \lambda h(x)\frac{|v|^{q-2}v}{|x|^{dp^*(a,d)}}, \quad x \in \Omega, \\
&u = v = 0, \quad x \in \partial\Omega,
\end{aligned}
\tag{2}
$$

where $0 \in \Omega$ is a bounded domain, G-symmetric domain (see Section 4 for details) in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $1 < q < p < p^*(a,c)$, $1 < p < N$, $0 \leq \mu < \overline{\mu} \triangleq$ $\triangleq \left(\frac{N-p}{p-a}\right)^p$, $0 \leq a < \frac{N-p}{p}$ and $a \leq d \leq c < a+1$.

The aim this part (second part) is to establish few results on the existence of G-symmetric solutions for (2).

In this paper, if $1 < p < N$ and $-\infty < a < \frac{N-p}{p}$ we denote by $W_a^{1,p}(\Omega, |x|^{-ap})$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$
\|u\| = \left(\int\limits_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{\frac{1}{p}}.
$$

Problem (1) is related to the well known Caffarelli–Kohn–Nirenberg inequality in [1, 2],

$$
\left(\int\limits_{\mathbb{R}^N} |x|^{-bp^*(a,b)}|u|^{p^*(a,b)}dx\right)^{\frac{p}{p^*(a,b)}} \leq C_{a,b}\int\limits_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N), \tag{3}
$$

where $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $a \leq b \leq a+1$, $p^*(a,b) = \frac{Np}{N-p(1+a-b)}$.

If $b = a+1$, then $p^*(a,b) = p$ and the following Hardy inequality holds [1, 3]:

$$
\int\limits_{\mathbb{R}^N} |x|^{-p(a+1)}|u|^p dx \leq \frac{1}{\overline{\mu}}\int\limits_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N), \tag{4}
$$

where $\overline{\mu} \triangleq \left(\frac{N-p}{p-a}\right)^p$ is the best Hardy constant.

In the space $W_a^{1,p}(\Omega, |x|^{-ap})$, we employ the following norm if $\mu < \overline{\mu}$:

$$
\|u\|_\mu = \|u\|_{W_a^{1,p}(\Omega,|x|^{-ap})} = \left(\int\limits_\Omega \left(|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right)dx\right)^{\frac{1}{p}}.
$$

By (4) it is equivalent to the usual norm $\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}$ of the space $W_a^{1,p}(\Omega, |x|^{-ap})$.

Now, we define the space $W = W_a^{1,p}(\Omega, |x|^{-ap}) \times W_a^{1,p}(\Omega, |x|^{-ap})$ with the norm

$$\|(u,v)\|^p = \|u\|_\mu^p + \|v\|_\mu^p.$$

Also, we can define the best Sobolev – Hardy constant

$$S_{\mu,a,b}(\Omega) = \inf_{u \in W_a^{1,p}(\Omega,|x|^{-ap})\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|x|^{-ap}|\nabla u|^p - \mu\frac{|u|^p}{|x|^{p(a+1)}}\right) dx}{\left(\displaystyle\int_\Omega |x|^{-bp^*(a,b)}|u|^{p^*(a,b)}dx\right)^{\frac{p}{p^*(a,b)}}}. \tag{5}$$

From Kang [4] (Lemma 2.2), $S_{\mu,a,b}(\Omega)$ is independent of $\Omega \subset \mathbb{R}^N$. Thus, we will simply denote that $S_{\mu,a,b}(\mathbb{R}^N) = S_{\mu,a,b}(\Omega) = S_{\mu,a,b}$.

For any $0 \le \mu < \overline{\mu}$, $\alpha, \beta > 1$ and $\alpha + \beta = p^*(a,c)$, by (3), (4), $0 \le t < p$ and the Young inequality, the following best constant are well defined:

$$S_{\mu,\alpha,\beta,a,c} := \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\displaystyle\int_\Omega \left(|x|^{-ap}(|\nabla u|^p + |\nabla v|^p) - \mu\frac{|u|^p + |v|^p}{|x|^{p(a+1)}}\right) dx}{\left(\displaystyle\int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^{cp^*(a,c)}}dx\right)^{\frac{p}{p^*(a,b)}}}. \tag{6}$$

Then we have (its proof is the same as that of Theorem 5 in [5])

$$S_{\mu,\alpha,\beta,a,c}(\mu) = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right) S_{\mu,a,c}.$$

Throughout this paper, let R_0 be the positive constant such that $\Omega \subset B(0;R_0)$, where $B(0;R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. By Hölder and Sobolev – Hardy inequalities, for all $u \in W_0^{1,p}(\Omega)$, we obtain

$$\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}} \le \left(\int_{B(0;R_0)} |x|^{-dp^*(a,d)}\right)^{\frac{p^*(a,d)-q}{p^*(a,d)}} \left(\int_\Omega \frac{|u|^{p^*(a,d)}}{|x|^{dp^*(a,d)}}\right)^{\frac{q}{p^*(a,d)}} \le$$

$$\le \left(N\omega_N \int_0^{R_0} r^{-dp^*(a,d)+N-1}dr\right)^{\frac{p^*(a,d)-q}{p^*(a,d)}} (S_{\mu,a,d})^{-\frac{q}{p}}\|u\|^q \le$$

$$\le \mathcal{D}_0 (S_{\mu,a,d})^{-\frac{q}{p}}\|u\|^q, \tag{7}$$

where $\omega_N = \dfrac{2\pi^{N/2}}{N\Gamma\left(\dfrac{N}{2}\right)}$ is the volume of the unit ball in $\mathbb{R}^N$ and $\mathcal{D}_0 := \left(\dfrac{N\omega_N R_0^{N-dp^*(a,d)}}{N - dp^*(a,d)}\right)^{\frac{p^*(a,d)-q}{p^*(a,d)}}$.

Existence of nontrivial nonnegative solutions for elliptic equations with singular potentials were recently studied by several authors, but, essentially, only with a solely critical exponent. We refer to [6 – 13] and the references therein. For example, in [12] the author studied the following equation via the Mountain – Pass theorem:

$$-\mathrm{div}\left(\frac{|Du|^{p-2}Du}{|x|^{ap}}\right)-\mu\frac{|u|^{p-2}u}{|x|^{(a+1)p}}=\frac{|u|^{p^*(b)-2}u}{|x|^{bp^*}}+\frac{|u|^{p^*(c)-2}u}{|x|^{cp^*}}\quad \text{in}\quad \mathbb{R}^N,$$

where $1<p<N$, $0\le\mu<\overline{\mu}\triangleq\left(\frac{N-(a+1)p}{p}\right)^p$, $0\le a<\dfrac{N-p}{p}$, $a\le b,c<a+1$, $p^*(b)= =\dfrac{Np}{N-(a+1-b)p}$ and $p^*(c)=\dfrac{Np}{N-(a+1-c)p}$.

In [6], Deng and Huang studied the following quasilinear elliptic problem:

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right)-\mu\frac{|u|^{p-2}u}{|x|^{(a+1)p}}=Q(x)\frac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}}+h(x,u),\quad x\in\Omega,$$
$$u=0,\quad x\in\partial\Omega, \tag{8}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, $0\in\Omega$ and Ω is G-symmetric with respect to a subgroup G of $O(\mathbb{N})$, Q, h satisfying some suitable conditions and obtained the existence of solutions via variational methods.

In this work, motivated by the above works we are interested to study the problems (1) and (2) by using the Mountain – Pass theorem of Ambrosetti and Rabinowitz and symmetric Mountain – Pass theorem of Rabinowitz [14], respectively. We shall show that the system (1) has at least two positive weak solutions and the system (2) has infinitely many G-symmetric solutions.

Throughout this paper, we assume that $a\le b_1$, b_2, $d\le c<a+1$, $\alpha,\beta>1$ and $\alpha+\beta=p^*(a,c)$. For $0\le\mu<\overline{\mu}$, we set

$$\theta(\mu,a,b_1):=\frac{p^*(a,b_1)-p}{pp^*(a,b_1)}(S_{\mu,a,b_1})^{\frac{p^*(a,b_1)}{p^*(a,b_1)-p}},$$

$$\Upsilon(\mu,\alpha,\beta,a,c):=\frac{p^*(a,c)-p}{pp^*(a,c)}\frac{1}{\|Q\|_\infty^{\frac{N-p(a+1-c)}{p(a+1-c)}}}(S_{\mu,\alpha,\beta,a,c})^{\frac{p^*(a,c)}{p^*(a,c)-p}},$$

$$\theta^*:=\big\{\theta(\mu,a,b_1),\theta(\mu,a,b_2),\Upsilon(\mu,\alpha,\beta,a,c)\big\}.$$

Moreover, assume that $Q(x)$ satisfies some of the following assumptions:

(H_1) $Q\in C(\overline{\Omega})$, $Q(x)\ge 0$ and meas$(\{x\in\Omega,\ Q(x)>0\})>0$.

(H_2) There exist $\vartheta>0$ such that $Q(x_0)=\|Q\|_\infty>0$ and $Q(x)=Q(x_0)+O(|x-x_0|^{\varrho})$, as $x\to x_0$.

(H_3) There exist β_0 and $\rho>0$ such that $B_{2\rho_0}(x_0)\subset\Omega$ and $h(x)\ge\beta_0$ for all $x\in B_{2\rho_0}(x_0)$.

Set $h_+:=\max\{h,0\}$ and $h_-:=\max\{-h,0\}$.

The main results of this paper can be included in the following three theorems.

Theorem 1. *Assume that* $N\ge 3$, $\mu\in[0,\overline{\mu})$, $1<q<p$ *and* (H_1). *Then there exists* $\Lambda_{11}^*>0$, *such that for* $0<\lambda<\Lambda_{11}^*$ *problem* (1) *has at lest one positive solutions.*

Theorem 2. *Assume that*

$$N \geq p^2, \quad 0 \leq \mu < \overline{\mu}, \quad \theta^* = \frac{p^*(a,c) - p}{pp^*(a,c)} \frac{1}{\|Q\|_\infty^{\frac{N-p(a+1-c)}{N}}} (S_{\mu,\alpha,\beta,a,c})^{\frac{p^*(a,c)}{p^*(a,c)-p}},$$

(H$_1$)–(H$_3$), $Q(0) = 0$, $\varrho > b(\mu)p + p - N + t$ *and* $\dfrac{N - dp^*(a,d)}{\beta(\mu)} < q < p$ *hold, and* $b(\mu)$ *is the constant defined as in Lemma* 4. *Then there exists* $\Lambda^{**} > 0$, *such that for* $0 < \lambda < \Lambda^{**}$, *problem* (1) *has at least two positive solutions.*

Theorem 3. *Suppose that* $|G| = +\infty$ *and* $Q, h \in C(\overline{\Omega}) \bigcap L^\infty(\overline{\Omega})$ *is G-symmetric. Then for* $\lambda > 0$ *the problem* (2) *has infinitely many G-symmetric solutions.*

This paper is divided into four sections, organized as follows. In Section 2, we establish some elementary results. In Section 3, we prove our main results (Theorems 1 and 2). In Section 4, we prove another our main result (Theorem 3).

2. Preliminary lemmas. The corresponding energy functional of problem (1) is defined by

$$J(u,v) = \frac{1}{p} \int\limits_\Omega \left(|x|^{-ap} |\nabla u|^p - \mu \frac{|u|^p}{|x|^{p(a+1)}} + |x|^{-ap} |\nabla v|^p - \mu \frac{|v|^p}{|x|^{p(a+1)}} \right) dx -$$

$$-\frac{\lambda}{q} \int\limits_\Omega h(x) \left(\frac{|u|^q}{|x|^{dp^*(a,d)}} + \frac{|v|^q}{|x|^{dp^*(a,d)}} \right) dx - \frac{1}{p^*(a,b_1)} \int\limits_\Omega \frac{|u|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}} dx -$$

$$-\frac{1}{p^*(a,b_2)} \int\limits_\Omega \frac{|v|^{p^*(a,b_2)}}{|x|^{b_2 p^*(a,b_2)}} dx - \frac{1}{\alpha+\beta} \int\limits_\Omega Q(x) \frac{|u|^\alpha |v|^\beta}{|x - x_0|^{cp^*(a,c)}} dx,$$

for each $(u,v) \in W$. Then $J \in C^1(W, \mathbb{R})$.

Lemma 1. *Assume that* $N \geq 3$, $0 \leq \mu < \overline{\mu}$, (H$_1$), $h_+ \neq 0$ *and* (u,v) *is a weak solution of problem* (1). *Then there exists a positive constant* d *depending on* N, $|\Omega|$, $|h_+|_\infty$, $A_{\mu,s}$, s_1, s_2 *and* q *such that*

$$J(u,v) \geq -d\lambda^{\frac{p}{p-q}}.$$

Proof. Since (u,v) is a weak solution of problem (1). Then, note that $\langle J'(u,v),(u,v)\rangle = 0$, we have

$$\langle J'(u,v),(u,v)\rangle = \int\limits_\Omega \left(|x|^{-ap} |\nabla u|^p - \mu \frac{|u|^p}{|x|^{p(a+1)}} + |x|^{-ap} |\nabla v|^p - \mu \frac{|v|^p}{|x|^{p(a+1)}} \right) dx -$$

$$-\lambda \int\limits_\Omega h(x) \left(\frac{|u|^q}{|x|^{dp^*(a,d)}} + \frac{|v|^q}{|x|^{dp^*(a,d)}} \right) dx - \int\limits_\Omega \frac{|u|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}} dx -$$

$$-\int\limits_\Omega \frac{|v|^{p^*(a,b_2)}}{|x|^{b_2 p^*(a,b_2)}} dx - \int\limits_\Omega Q(x) \frac{|u|^\alpha |v|^\beta}{|x - x_0|^{cp^*(a,c)}} dx = 0. \tag{9}$$

Now, by using $h_+ \neq 0$, (9), (7), the Hölder inequality and the Sobolev – Hardy inequality, by a direct calculation, one can get

$$J(u,v) \geq 2 \inf_{t \geq 0} \left[\left(\frac{1}{p} - \frac{1}{p^*(a,c)} \right) t^p - \lambda \left(\frac{1}{q} - \frac{1}{p^*(a,c)} \right) \mathcal{D}_0 (S_{\mu,a,d})^{-\frac{q}{p}} |h_+|_\infty t^q \right] \geq -d\lambda^{\frac{p}{p-q}}.$$

Here $d_\Omega := \sup_{x,y \in \Omega} |x - y|$ is the diameter of Ω and d is a positive constant depending on N, $|\Omega|$, $|h_+|_\infty$, $S_{\mu,a,d}$, b_1, b_2 and q.

Lemma 1 is proved.

Recall that a sequence $(u_n, v_n)_{n \in \mathbb{N}}$ is a $(PS)_c$ sequence for the functional J if $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0$. If any $(PS)_c$ sequence $(u_n, v_n)_{n \in \mathbb{N}}$ has a convergent subsequence, we say that J satisfies the $(PS)_c$ condition.

Lemma 2. *Assume that* $N \geq 3$, $0 \leq \mu < \overline{\mu}$, (H$_1$), $h_+ \neq 0$ *and* $Q(0) = 0$. *Then* $J(u,v)$ *satisfies the* $(PS)_c$ *condition with c satisfying*

$$c < c_* := \min \theta^* - d\lambda^{\frac{p}{p-q}}. \tag{10}$$

Proof. It is easy to see that the $(PS)_c$ sequence $(u_n, v_n)_{n \in \mathbb{N}}$ of $J(u,v)$ is bounded in W. Then $(u_n, v_n) \rightharpoonup (u,v)$ weakly in W as $n \to \infty$, which implies $u_n \rightharpoonup u$ weakly and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Passing to a subsequence we may assume that

$$|x|^{-ap} |\nabla u_n|^p dx \rightharpoonup \overline{\alpha}, \qquad |x|^{-ap} |\nabla v_n|^p dx \rightharpoonup \widetilde{\alpha}, \qquad \frac{|u|^p}{|x|^{p(a+1)}} dx \rightharpoonup \overline{\beta},$$

$$\frac{|v|^p}{|x|^{p(a+1)}} dx \rightharpoonup \widetilde{\beta}, \qquad \frac{|u_n|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}} dx \rightharpoonup \overline{\gamma}, \qquad \frac{|v_n|^{p^*(a,b_2)}}{|x|^{b_2 p^*(a,b_2)}} dx \rightharpoonup \widetilde{\gamma},$$

$$Q(x) \frac{|u_n|^\alpha |v_n|^\beta}{|x - x_0|^{cp^*(a,c)}} dx \rightharpoonup \nu$$

weakly in the sense of measures. Using the concentration-compactness principle in [15], there exist an at most countable set I, a set of points $\{x_i\}_{i \in I} \in \Omega \setminus \{0\}$, real numbers $\overline{a}_{x_i}$, $\widetilde{a}_{x_i}$, d_{x_i}, $i \in I$, $\overline{a}_0$, $\widetilde{a}_0$, $\overline{b}_0$, $\widetilde{b}_0$, $\overline{c}_0$, $\widetilde{c}_0$ and d_0, such that

$$\overline{\alpha} \geq |x|^{-ap} |\nabla u|^p dx + \sum_{i \in I} \overline{a}_{x_i} \delta_{x_i} + \overline{a}_0 \delta_0, \qquad \widetilde{\alpha} \geq |x|^{-ap} |\nabla v|^p dx + \sum_{i \in I} \widetilde{a}_{x_i} \delta_{x_i} + \widetilde{a}_0 \delta_0, \tag{11}$$

$$\overline{\beta} = \frac{|u|^p}{|x|^{p(a+1)}} dx + \overline{b}_0 \delta_0, \qquad \widetilde{\beta} = \frac{|v|^p}{|x|^{p(a+1)}} dx + \widetilde{b}_0 \delta_0, \tag{12}$$

$$\overline{\gamma} = \frac{|u|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}} + \overline{c}_0 \delta_0, \qquad \widetilde{\gamma} = \frac{|v|^{p^*(a,b_2)}}{|x|^{b_2 p^*(a,b_2)}} dx + \widetilde{c}_0 \delta_0, \tag{13}$$

$$\nu = Q(x) \frac{|u|^\alpha |v|^\beta}{|x - x_0|^{cp^*(a,c)}} dx + \sum_{i \in I} Q(x_i) d_{x_i} \delta_{x_i} + Q(0) d_0 \delta_0, \tag{14}$$

where δ_x is the Dirac-mass of mass 1 concentrated at the point x.

First, we consider the possibility of the concentration at $\{x_i\}_{i\in I} \in \Omega \setminus \{0\}$.

Let $\epsilon > 0$ be small enough, take $\eta_{x_i} \in C_c^\infty(B_{2\varepsilon}(x_i))$, such that $\eta_{x_i}|_{B_\varepsilon(x_i)} = 1$, $0 \le \eta_{x_i} \le 1$ and $|\nabla \eta_{x_i}(x)| \le \dfrac{C}{\varepsilon}$. Then

$$o(1) = \langle J'(u_n, v_n), (\eta_{x_i}^p u_n, \eta_{x_i}^p v_n)\rangle =$$

$$= \int\limits_\Omega \left(|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(\eta_{x_i}^p u_n) + |x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n \nabla(\eta_{x_i}^p v_n)\right) dx -$$

$$- \int\limits_\Omega Q(x)\frac{|u_n|^\alpha |v_n|^\beta}{|x-x_0|^{cp^*(a,c)}}\eta_{x_i}^p dx - \underbrace{\mu \int\limits_\Omega \left(\frac{|u_n|^p}{|x|^{p(a+1)}}\eta_{x_i}^p + \frac{|v_n|^p}{|x|^{p(a+1)}}\eta_{x_i}^p\right) dx}_{(I)} -$$

$$\underbrace{-\lambda \int\limits_\Omega h(x)\left(\frac{|u_n|^q}{|x|^{dp^*(a,d)}}\eta_{x_i}^p + \frac{|v_n|^q}{|x|^{dp^*(a,d)}}\eta_{x_i}^p\right) dx}_{(II)} - \underbrace{\left(\int\limits_\Omega \frac{|u_n|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}}\eta_{x_i}^p dx + \int\limits_\Omega \frac{|v_n|^{p^*(a,b_2)}}{|x|^{b_2 p^*(a,b_2)}}\eta_{x_i}^p dx\right)}_{(III)}.$$

From (12) – (14), one can get

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}(I) = \lim_{\varepsilon\to0}\left(\int\limits_\Omega \eta_{x_i}^p d\overline{\beta} + \int\limits_\Omega \eta_{x_i}^p d\widetilde{\beta}\right) = 0,$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}(III) = \lim_{\varepsilon\to0}\left(\int\limits_\Omega \eta_{x_i}^p d\overline{\gamma} + \int\limits_\Omega \eta_{x_i}^p d\widetilde{\gamma}\right) = 0, \qquad \lim_{\varepsilon\to0}\lim_{n\to\infty}(II) = 0 \tag{15}$$

and

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int\limits_\Omega Q(x)\frac{|u_n|^\alpha|v_n|^\beta}{|x-x_0|^{cp^*(a,c)}}\eta_{x_i}^p dx = \lim_{\varepsilon\to0}\int\limits_\Omega \eta_{x_i}^p d\nu = Q(x_i)dx_i.$$

Thus,

$$0 = \lim_{\varepsilon\to0}\lim_{n\to\infty}\int\limits_\Omega \left(|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla(\eta_{x_i}^p u_n) + |x|^{-ap}|\nabla v_n|^{p-2}\nabla v_n \nabla(\eta_{x_i}^p v_n)\right) dx - Q(x_i)dx_i. \tag{16}$$

Moreover, by a direct calculation, we have

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\left|\int\limits_\Omega |x|^{-ap}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\eta_{x_i}^p dx\right| =$$

$$= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left| \int_{\Omega} |x|^{-a(p-1)} |x|^{-a} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \eta_{x_i}^p dx \right| \leq$$

$$\leq C \lim_{\varepsilon \to 0} \left(\int_{B_\varepsilon(x_i)} |x|^{-ap^*} |u|^{p^*} dx \right)^{\frac{p}{p^*}} = 0. \tag{17}$$

Similarly,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \left| \int_{\Omega} |x|^{-ap} v_n |\nabla v_n|^{p-2} \nabla v_n \nabla \eta_{x_i}^p dx \right| = 0. \tag{18}$$

Combining (16) – (18), there holds

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} (|x|^{-ap} |\eta_{x_i} \nabla u_n|^p + |x|^{-ap} |\eta_{x_i} \nabla v_n|^p) dx - Q(x_i) d_{x_i} =$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega} \left(\eta_{x_i}^p d\overline{\alpha} + \eta_{x_i} d\widetilde{\alpha} \right) - Q(x_i) d_{x_i}. \tag{19}$$

On the other hand, (6) implies that

$$\frac{1}{\|Q\|_\infty^{\frac{p}{p^*(a,c)}}} S_{\mu,\alpha,\beta,a,c} \left(\int_{\Omega} Q(x) \frac{|\eta_{x_i} u_n|^\alpha |\eta_{x_i} v_n|^\beta}{|x - x_0|^{cp^*(a,c)}} dx \right)^{\frac{p}{p^*(a,c)}} \leq$$

$$\leq \int_{\Omega} \left(|x|^{-ap} |\nabla(\eta_{x_i} u_n)|^p + |x|^{-ap} |\nabla(\eta_{x_i} v_n)|^p - \mu \frac{|\eta_{x_i} u_n|^p + |\eta_{x_i} v_n|^p}{|x|^{p(a+1)}} \right) dx. \tag{20}$$

Note that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |x|^{-ap} |\nabla \eta_{x_i}|^p |u_n|^p dx = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |x|^{-ap} |\nabla \eta_{x_i}|^p |v_n|^p dx = 0,$$

together with (17) and (18), we obtain that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |x|^{-ap} |\eta_{x_i} \nabla u_n|^p dx = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |x|^{-ap} |\nabla(\eta_{x_i} u_n)|^p dx, \tag{21}$$

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |x|^{-ap} |\eta_{x_i} \nabla v_n|^p dx = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |x|^{-ap} |\nabla(\eta_{x_i} v_n)|^p dx. \tag{22}$$

The relations (14), (15) and (20) – (22) imply that

$$\frac{1}{\|Q\|_{\infty}^{\frac{p}{p^*(a,c)}}} S_{\mu,\alpha,\beta,a,c} \left(Q(x_i)d_{x_i}\right)^{\frac{p}{p^*(a,c)}} \leq \lim_{\varepsilon\to 0} \int\limits_{\Omega} |\eta_{x_i}|^p d\overline{\alpha} + \lim_{\varepsilon\to 0} \int\limits_{\Omega} |\eta_{x_i}|^p d\widetilde{\alpha}. \tag{23}$$

Combining (19) and (23),

$$\frac{1}{\|Q\|_{\infty}^{\frac{p}{p^*(a,c)}}} S_{\mu,\alpha,\beta,a,c} \left(Q(x_i)d_{x_i}\right)^{\frac{p}{p^*(a,c)}} \leq Q(x_i)d_{x_i}, \tag{24}$$

which implies that

$$\text{either } \; Q(x_i)d_{x_i} = 0, \quad \text{or} \quad Q(x_i)d_{x_i} \geq \frac{1}{\|Q\|_{\infty}^{\frac{N-p(a+1-c)}{p(a+1-c)}}} \left(S_{\mu,\alpha,\beta,a,c}\right)^{\frac{N}{p(a+1-c)}}. \tag{25}$$

Now, we consider the possibility of the concentration at 0.

For $\epsilon > 0$ be small enough, take $\eta_0 \in C_c^{\infty}(B_{2\varepsilon}(0))$, such that $\eta_0|_{B_\varepsilon(0)} = 1$, $0 \leq \eta_0 \leq 1$ and $|\nabla \eta_0(x)| \leq \dfrac{C}{\varepsilon}$. Then

$$o(1) = \langle J'(u_n, v_n), (\eta_0^p u_n, 0)\rangle =$$

$$= \int\limits_{\Omega} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla(\eta_0^p u_n) dx - \mu \int\limits_{\Omega} \frac{|u_n|^p}{|x|^{p(a+1)}} \eta_0^p dx - \lambda \int\limits_{\Omega} h(x) \frac{|u_n|^q}{|x|^{dp^*(a,d)}} \eta_0^p dx -$$

$$- \int\limits_{\Omega} \frac{|u_n|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}} \eta_0^p dx - \frac{\alpha}{\alpha+\beta} \int\limits_{\Omega} Q(x) \frac{|u_n|^{\alpha}|v_n|^{\beta}}{|x-x_0|^{cp^*(a,c)}} \eta_0^p dx.$$

From (12) – (14) and $Q(0) = 0$, we obtain that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} \frac{|u_n|^p}{|x|^{p(a+1)}} \eta_0^p dx = \overline{b}_0, \qquad \lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} \frac{|u_n|^{p^*(a,b_1)}}{|x|^{b_1 p^*(a,b_1)}} \eta_0^p dx = \overline{c}_0$$

and

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} Q(x) \frac{|u_n|^{\alpha}|v_n|^{\beta}}{|x-x_0|^{cp^*(a,c)}} \eta_0^p dx = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} h(x) \frac{|u_n|^q}{|x|^{dp^*(a,d)}} \eta_0^p dx = 0.$$

Thus,

$$0 = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla(\eta_0^p u_n) dx - \mu \overline{b}_0 - \overline{c}_0. \tag{26}$$

Note that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} |x|^{-ap} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \eta_0^p dx = 0,$$

together with (26), there holds

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega}\eta_0^p d\overline{\alpha}-\mu\overline{b}_0=\overline{c}_0. \tag{27}$$

On the other hand, (5) implies that

$$S_{\mu,a,b_1}\left(\int\limits_{\Omega}\frac{|\eta_0 u_n|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx\right)^{\frac{p}{p^*(a,b_1)}}\le\int\limits_{\Omega}\left(|x|^{-ap}|\nabla(\eta_0 u_n)|^p-\mu\frac{|\eta_0 u_n|^p}{|x|^{p(a+1)}}\right)dx.$$

Thus

$$S_{\mu,a,b_1}\overline{c}_0^{\frac{p}{p^*(a,b_1)}}\le\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int\limits_{\Omega}|x|^{-ap}|\nabla(\eta_0 u_n)|^p dx-\mu\overline{b}_0. \tag{28}$$

Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int\limits_{\Omega}|x|^{-ap}|\eta_0\nabla u_n|^p dx=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int\limits_{\Omega}|x|^{-ap}|\nabla(\eta_0 u_n)|^p dx,$$

together with (28), there holds

$$S_{\mu,a,b_1}\overline{c}_0^{\frac{p}{p^*(a,b_1)}}\le\lim_{\varepsilon\to 0}\int\limits_{\Omega}|\eta_0|^p d\overline{\alpha}-\mu\overline{b}_0. \tag{29}$$

Therefore, from (27) and (29),

$$S_{\mu,a,b_1}\overline{c}_0^{\frac{p}{p^*(a,b_1)}}\le\overline{c}_0, \tag{30}$$

which implies that

$$\text{either }\ \overline{c}_0=0,\quad\text{or}\quad\overline{c}_0\ge S_{\mu,a,b_1}^{\frac{N}{p(a+1-b_1)}}, \tag{31}$$

similarly,

$$\text{either }\ \overline{c}_0=0,\quad\text{or}\quad\overline{c}_0\ge S_{\mu,a,b_2}^{\frac{N}{p(a+1-b_2)}}. \tag{32}$$

Recall that $u_n\rightharpoonup u$ weakly and $v_n\rightharpoonup v$ weakly in $W_a^{1,p}(\Omega,|x|^{-ap})$, we have

$$c+o(1)=J(u_n,v_n)=$$

$$=\frac{1}{p}\int\limits_{\Omega}\left(|x|^{-ap}|\nabla u_n-\nabla u|^p-\mu\frac{|u_n-u|^p}{|x|^{p(a+1)}}+|x|^{-ap}|\nabla v_n-\nabla v|^p-\mu\frac{|v_n-v|^p}{|x|^{p(a+1)}}\right)dx-$$

$$-\frac{1}{p^*(a,b_1)}\int\limits_{\Omega}\frac{|u_n-u|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx-\frac{1}{p^*(a,b_2)}\int\limits_{\Omega}\frac{|v_n-v|^{p^*(a,b_2)}}{|x|^{b_2p^*(a,b_2)}}dx-$$

$$-\frac{1}{p^*(a,c)}\int\limits_{\Omega} Q(x)\frac{|u_n-u|^{\alpha}|v_n-v|^{\beta}}{|x-x_0|^{cp^*(a,c)}}dx + J(u,v). \tag{33}$$

On the other hand, from $o(1) = J'(u_n, v_n)$, we obtain that

$$J'(u_n, v_n) = 0. \tag{34}$$

Thus, $0 = \langle J'(u,v),(u,v)\rangle$. Together with $o(1) = \langle J'(u_n,v_n),(u_n,v_n)\rangle$, there holds

$$o(1) = \int\limits_{\Omega}\left(|x|^{-ap}|\nabla u_n - \nabla u|^p - \mu\frac{|u_n-u|^p}{|x|^{p(a+1)}} + |x|^{-ap}|\nabla v_n - \nabla v|^p - \mu\frac{|v_n-v|^p}{|x|^{p(a+1)}}\right)dx -$$

$$-\int\limits_{\Omega}\frac{|u_n-u|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx - \int\limits_{\Omega}\frac{|v_n-v|^{p^*(a,b_2)}}{|x|^{b_2p^*(a,b_2)}}dx - \int\limits_{\Omega} Q(x)\frac{|u_n-u|^{\alpha}|v_n-v|^{\beta}}{|x-x_0|^{cp^*(a,c)}}dx. \tag{35}$$

From (33)–(35) and Lemma 1,

$$c + o(1) \geq \left(\frac{1}{p} - \frac{1}{p^*(a,b_1)}\right)\int\limits_{\Omega}\frac{|u_n-u|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx + \left(\frac{1}{p} - \frac{1}{p^*(a,b_2)}\right)\int\limits_{\Omega}\frac{|v_n-v|^{p^*(a,b_2)}}{|x|^{b_2p^*(a,b_2)}}dx +$$

$$+\left(\frac{1}{p} - \frac{1}{p^*(a,c)}\right)\int\limits_{\Omega} Q(x)\frac{|u_n-u|^{\alpha}|v_n-v|^{\beta}}{|x-x_0|^{cp^*(a,c)}}dx - d\lambda^{\frac{p}{p-q}}. \tag{36}$$

Passing to the limit in (36) as $n\to\infty$, we have

$$c \geq \left(\frac{1}{p} - \frac{1}{p^*(a,b_1)}\right)\overline{c}_0 + \left(\frac{1}{p} - \frac{1}{p^*(a,b_2)}\right)\widetilde{c}_0 + \left(\frac{1}{p} - \frac{1}{p^*(a,c)}\right)\sum_{i\in I} Q(x_i)d_{x_i} - d\lambda^{\frac{p}{p-q}}. \tag{37}$$

By the assumption $c < c_*$ and in view of (25), (31) and (32), there holds $\overline{c}_0 = \widetilde{c}_0 = 0$, $Q(x_i)d_{x_i} = 0$, $i \in I$. Up to a subsequence, $(u_n, v_n) \to (u, v)$ strongly in W as $n\to\infty$.

Lemma 2 is proved.

If the restriction $Q(0) = 0$ is removed, we establish the following version of Lemma 2.

Lemma 3. *Assume that* $N \geq 3$, $0 \leq \mu < \overline{\mu}$, (H_1) *and* $h_+ \neq 0$. *Then* $J(u,v)$ *satisfies the* $(PS)_c$ *condition with* c *satisfying*

$$c < c_0 := \min\left\{\frac{p^*(a,b_1)-p}{pp^*(a,b_1)}\left(\frac{1}{p}S_{\mu,a,b_1}\right)^{\frac{p^*(a,b_1)}{p^*(a,b_1)-p}}, \frac{p^*(a,b_2)-p}{pp^*(a,b_2)}\left(\frac{1}{p}S_{\mu,a,b_2}\right)^{\frac{p^*(a,b_2)}{p^*(a,b_2)-p}},\right.$$

$$\left.\frac{p^*(a,c)-p}{pp^*(a,c)}\frac{1}{\|Q\|_\infty^{\frac{N-p(a+1-c)}{N}}}\left(\frac{1}{p}S_{\mu,\alpha,\beta,a,c}\right)^{\frac{p^*(a,c)}{p^*(a,c)-p}}\right\} - d\lambda^{\frac{p}{p-q}}. \tag{38}$$

Proof. The proof is similar to Lemma 2 and is omitted.

Here, we recall a recent result on the extremal functions of $S_{\mu,a,b}$ [4].

Lemma 4 [4]. *Assume that* $0 \le a < \dfrac{N-p}{p}$, $a \le b < a+1$ *and* $0 \le \mu < \overline{\mu}$. *Then* $S_{\mu,a,b}$ *is attained when* $\Omega = \mathbb{R}^N$ *by the radial functions*

$$V_\epsilon(x) \triangleq \epsilon^{-(\frac{N-p}{p}-a)} U_{p,\mu}\left(\frac{|x|}{\epsilon}\right) \quad \forall \epsilon > 0, \tag{39}$$

that satisfy

$$\int\limits_\Omega \left(|x|^{-ap} |\nabla V_\epsilon(x)|^p - \mu \frac{|V_\epsilon(x)|^p}{|x|^{p(a+1)}} \right) dx = \int\limits_\Omega \frac{|V_\epsilon(x)|^{p^*(a,b)}}{|x|^{bp^*(a,c)}} dx = (S_{\mu,a,c})^{\frac{p^*(a,c)}{p^*(a,c)-p}},$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ *is the unique radial solution of the following problem*:

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu \frac{|u|^{p-2}u}{|x|^{p(a+1)}} = \frac{|u|^{p^*(a,c)-2}u}{p^*(a,c)|x|^{cp^*(a,c)}}, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

$$u \in W_a^{1,p}(\mathbb{R}^N), \quad u > 0, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

with

$$U_{p,\mu}(1) = \left(\frac{p^*(a,c)(\overline{\mu}-\mu)}{p} \right)^{\frac{1}{p^*(a,c)-p}}.$$

Furthermore, $U_{p,\mu}$ *have the following properties*:

$$\lim_{r\to 0} r^{\alpha(\mu)} U_{p,\mu}(r) = C_1 > 0, \qquad \lim_{r\to+\infty} r^{\beta(\mu)} U_{p,\mu}(r) = C_2 > 0,$$

$$\lim_{r\to 0} r^{\alpha(\mu)+1} |U'_{p,\mu}(r)| = C_1\alpha(\mu) \ge 0, \qquad \lim_{r\to+\infty} r^{\beta(\mu)+1} |U'_{p,\mu}(r)| = C_2\beta(\mu) > 0,$$

where C_i, $i = 1, 2$, *are positive constants and* $\alpha(\mu)$ *and* $\beta(\mu)$ *are zeros of the function*

$$f(\zeta) = (p-1)\zeta^p - (N - p(a+1))\zeta^{p-1} + \mu, \qquad \zeta \ge 0, \quad 0 \le \mu < \overline{\mu},$$

that satisfy

$$0 \le \alpha(\mu) < \frac{N - p(a+1)}{p} < \beta(\mu) \le \frac{N - p(a+1)}{p-1}.$$

Furthermore, there exist the positive constants $C_3 = C_3(\mu, p, a, c)$ *and* $C_4 = C_4(\mu, p, a, c)$ *such that*

$$C_3 \le U_{p,\mu}(x) \left(|x|^{\frac{\alpha(\mu)}{\delta}} - |x|^{\frac{\beta(\mu)}{\delta}} \right)^\delta < C_4, \qquad \delta = \frac{N - p(a+1)}{p}.$$

Lemma 5. *Under the assumptions of Theorem* 2, *there exists* $(u_1, v_1) \in W \setminus \{(0,0)\}$ *and* $\Lambda_1 > 0$, *such that for* $0 < \lambda < \Lambda_1$, *there holds*

$$\sup_{t\ge 0} J(tu_1, tv_1) < \Upsilon(\mu, \alpha, \beta, a, c) - d\lambda^{\frac{p}{p-q}}. \tag{40}$$

Proof. First, we will give some estimates on the extremal function $V_\epsilon(x)$ defined in (39). Let $V_\epsilon(x)$ be the function in Lemma 4, $\rho > 0$ small enough such that $B_\rho(0) \subset \Omega$, $\psi \in C_0^\infty(B_\rho(0))$ with $0 \le \psi \le 1$ in $B_\rho(0)$ and $\psi = 1$ in $B_{\rho/2}(0)$, then the function given by [4]:

$$u_\epsilon(x) := \psi(x) V_\epsilon(x),$$

satisfies

$$\|u_\epsilon\|_p = (S_{\mu,a,c})^{\frac{p^*(a,c)}{p^*(a,c)-p}} + O\left(\epsilon^{\beta(\mu)p+p(a+1)-N}\right), \tag{41}$$

$$\int\limits_\Omega \frac{|u_\epsilon|^{p^*(a,c)}}{|x|^{cp^*(a,c)}} dx = (S_{\mu,a,c})^{\frac{p^*(a,c)}{p^*(a,c)-p}} + O\left(\epsilon^{(\beta(\mu)+c)p^*(a,c)-N}\right), \tag{42}$$

$$\int\limits_\Omega \frac{|u_\epsilon|^q}{|x|^{dp^*(a,d)}} dx \ge \begin{cases} C\epsilon^{N-dp^*(a,d)-q\delta}, & \text{if } \dfrac{N-dp^*(a,d)}{\beta(\mu)} < q < p^*(a,d), \\ C\epsilon^{q(\beta(\mu)-\delta)} |\ln(\epsilon)|), & \text{if } q = \dfrac{N-dp^*(a,d)}{\beta(\mu)}, \\ C\epsilon^{q(\beta(\mu)-\delta)}, & \text{if } 1 \le q < \dfrac{N-dp^*(a,d)}{\beta(\mu)}, \end{cases} \tag{43}$$

where $\delta = \dfrac{N-p(a+1)}{p} < \beta(\mu) \le \dfrac{N-p(a+1)}{p-1}$.

Now, we consider the functional $I\colon W \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{p} \int\limits_\Omega \left(|x|^{-ap} |\nabla u|^p - \mu \frac{|u|^p}{|x|^{p(a+1)}} + |x|^{-ap} |\nabla v|^p - \mu \frac{|v|^p}{|x|^{p(a+1)}} \right) dx -$$

$$- \frac{1}{p^*(a,c)} \int\limits_\Omega Q(x) \frac{|u|^\alpha |v|^\beta}{|x-x_0|^{cp^*(a,c)}} dx.$$

Let $u_1 = \alpha^{1/p} u_\epsilon$, $v_1 = \beta^{1/p} u_\epsilon$ and define the function $g_1(t) := J(tu_1, tv_1)$, $t \ge 0$. Note that $\lim_{t\to+\infty} g_1(t) = -\infty$ and $g_1(t) > 0$ as t is close to 0. Thus $\sup_{t\ge0} g_1(t)$ is attained at some finite $t_\epsilon > 0$ with $g_1'(t_\epsilon) = 0$. Furthermore, $C' < t_\epsilon < C''$; where C' and C'' are the positive constants independent of ϵ. We have

$$I(tu_1, tv_1) = \left[\frac{t^p}{p} (\alpha+\beta) \int\limits_\Omega \left(|x|^{-ap} |\nabla u_\epsilon|^p - \mu \frac{|u_\epsilon|^p}{|x|^{p(a+1)}} \right) dx - \right.$$

$$\left. - \frac{t^{p^*(a,c)}}{p^*(a,c)} (\alpha^{\alpha/p} \beta^{\beta/p}) \int\limits_\Omega Q(y_0) \frac{|u_\epsilon|^{p^*(a,c)}}{|x-x_0|^{p^*(a,c)c}} dx \right] -$$

$$- \frac{t^{p^*(a,c)}}{p^*(a,c)} (\alpha^{\alpha/p} \beta^{\beta/p}) \int\limits_\Omega (Q(x) - Q(x_0)) \frac{|u_\epsilon|^{p^*(a,c)}}{|x-x_0|^{cp^*(a,c)}} dx :=$$

$$:= y(tu_1, tv_1) - \frac{t^{p^*(a,c)}}{p^*(a,c)}(\alpha^{\alpha/p}\beta^{\beta/p})\int\limits_\Omega (Q(x) - Q(x_0))\frac{|u_\epsilon|^{p^*(a,c)}}{|x - x_0|^{cp^*(a,c)}}dx. \tag{44}$$

Note that

$$\sup_{t\geq 0}\left(\frac{t^p}{p}A - \frac{t^{p^*(a,c)}}{p^*(a,c)}B\right) = \left(\frac{1}{p} - \frac{1}{p^*(a,c)}\right)\left(\frac{A}{B^{\frac{p}{p^*(a,c)}}}\right)^{\frac{p^*(a,c)}{p^*(a,c)-p}}, \quad A, B > 0. \tag{45}$$

From (H_2), (41), (42) and (45) it follows that straightforward

$$\sup_{t\geq 0} y(tu_1, tv_1) \leq \left(\frac{1}{p} - \frac{1}{p^*(a,c)}\right)\frac{1}{\|Q\|_\infty^{\frac{N-p(a+1-c)}{p(a+1-c)}}}\left(S_{\mu,\alpha,\beta,a,c}\right)^{\frac{p^*(a,c)}{p^*(a,c)-p}} + O\left(\epsilon^{\beta(\mu)p+p(a+1)-N}\right). \tag{46}$$

On the other hand, (H_2) implies that there exists $r_1 < r$, such that for $x \in B_{r_1}(y_0)$, $|Q(x) - Q(x_0)| \leq$ $\leq C|x - x_0|^\vartheta$. Thus

$$\left|\int\limits_\Omega (Q(x) - Q(x_0))\frac{|u_\epsilon|^{p^*(a,c)}}{|x - x_0|^{cp^*(a,c)}}dx\right| \leq C\int\limits_\Omega |Q(x) - Q(x_0)|\frac{|u_\epsilon|^{p^*(a,c)}}{|x - x_0|^{cp^*(a,c)}}dx =$$

$$= C\int\limits_{B_{2r}(x_0)}\frac{|x - x_0|^\vartheta|u_\epsilon|^{p^*(a,c)}}{|x - x_0|^{cp^*(a,c)}}dx =$$

$$= O(\epsilon^{\vartheta - cp^*(a,c)}). \tag{47}$$

From (44), (46), one can get

$$\sup_{t\geq 0} I(tu_1, tv_1) = I(t_\epsilon u_1, t_\epsilon v_1) \leq \Upsilon(\mu, \alpha, \beta, a, c) + O\left(\epsilon^{\beta(\mu)p+p(a+1)-N}\right). \tag{48}$$

Observe that there exists $\Lambda_1^* > 0$, such that for $0 < \lambda < \Lambda_1^*$ and

$$\Upsilon(\mu, \alpha, \beta, a, c) - d\lambda^{\frac{p}{p-q}} > 0,$$

Then for $0 < \lambda < \Lambda_1^*$, there exists $t_1 \in (0, 1)$, such that

$$\sup_{0\leq t\leq t_1} J(tu_1, tv_1) \leq \sup_{0\leq t\leq t_1}\frac{1}{p}t^p\int\limits_\Omega\left(|x|^{-ap}|\nabla u_1|^p + |x|^{-ap}|\nabla v_1|^p\right)dx < \Upsilon(\mu, \alpha, \beta, a, c) - d\lambda^{\frac{p}{p-q}}. \tag{49}$$

On the other hand, we have

$$\sup_{t\geq t_1} J(tu_1, tv_1) \leq \sup_{t\geq t_1}\left[I(tu_1, tv_1) - \frac{\lambda}{q}t^q\int\limits_\Omega h(x)\frac{|u_1|^q}{|x|^{dp^*(a,d)}}dx - \frac{t^{p^*(a,b_1)}}{p^*(a,b_1)}\int\limits_\Omega\frac{|u_1|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx\right] \leq$$

$$\leq \sup_{t\geq t_1}\left[I(tu_1,tv_1)-\frac{\lambda}{q}t_1^q\int\limits_\Omega h(x)\frac{|u_1|^q}{|x|^{dp^*(a,d)}}dx-\frac{t_1^{p^*(a,b_1)}}{p^*(a,b_1)}\int\limits_\Omega \frac{|u_1|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx\right]\leq$$

$$\leq \Upsilon(\mu,\alpha,\beta,a,c)+O\left(\epsilon^{\beta(\mu)p+p(a+1)-N}\right)-$$

$$-C\int\limits_\Omega \frac{|u_\epsilon|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx-\lambda C\int\limits_\Omega h(x)\frac{|u_\epsilon|^q}{|x|^{dp^*(a,d)}}dx. \tag{50}$$

From (42),

$$\int\limits_\Omega \frac{|u_\epsilon|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx\geq O\left(\epsilon^{(\beta(\mu)+b_1)p^*(a,b_1)-N}\right). \tag{51}$$

Also, from (43), it follows that

$$\int\limits_\Omega h(x)\frac{|u_\epsilon|^q}{|x|^{dp^*(a,d)}}dx\geq \beta_0\int\limits_\Omega \frac{|u_\epsilon|^q}{|x|^{dp^*(a,d)}}dx\geq$$

$$\geq\begin{cases} C\epsilon^{N-dp^*(a,d)-q\delta}, & \text{if } \dfrac{N-dp^*(a,d)}{\beta(\mu)}<q<p^*(a,d),\\ C\epsilon^{q(\beta(\mu)-\delta)}|\ln(\epsilon)|), & \text{if } q=\dfrac{N-dp^*(a,d)}{\beta(\mu)},\\ C\epsilon^{q(\beta(\mu)-\delta)}, & \text{if } 1\leq q<\dfrac{N-dp^*(a,d)}{\beta(\mu)}.\end{cases} \tag{52}$$

Since $q\geq \dfrac{N-dp^*(a,d)}{\beta(\mu)}$, by (50)–(52) we have

$$\sup_{t\geq t_1}J(tu_1,tv_1)\leq \Upsilon(\mu,\alpha,\beta,a,c)+O\left(\epsilon^{\beta(\mu)p+p(a+1)-N}\right)+O\left(\epsilon^{(\beta(\mu)+b_1)p^*(a,b_1)-N}\right)-$$

$$-\lambda\begin{cases} C\epsilon^{N-dp^*(a,d)-q\delta}, & \text{if } \dfrac{N-dp^*(a,d)}{\beta(\mu)}<q<p^*(a,d),\\ C\epsilon^{q(\beta(\mu)-\delta)}|\ln(\epsilon)|), & \text{if } q=\dfrac{N-dp^*(a,d)}{\beta(\mu)}.\end{cases}$$

Note that $\beta(\mu)p+p(a+1)-N<(\beta(\mu)+b_1)p^*(a,b_1)-N$, then we have

$$\sup_{t\geq t_1}J(tu_1,tv_1)\leq \Upsilon(\mu,\alpha,\beta,a,c)+O\left(\epsilon^{(\beta(\mu)+b_1)p^*(a,b_1)-N}\right)-$$

$$-\lambda\begin{cases} C\epsilon^{N-dp^*(a,d)-q\delta}, & \text{if } \dfrac{N-dp^*(a,d)}{\beta(\mu)}<q<p^*(a,d),\\ C\epsilon^{q(\beta(\mu)-\delta)}|\ln(\epsilon)|), & \text{if } q=\dfrac{N-dp^*(a,d)}{\beta(\mu)}.\end{cases} \tag{53}$$

Note that $N > p^2$, $\beta(\mu) \geq \dfrac{N - dp^*(a,d)}{q}$. Thus

$$\big[N - dp^*(a,d) - q\delta\big]\frac{p-q}{q} < \beta(\mu)p + p(a+1) - N - \big[N - dp^*(a,d) - q\delta\big].$$

Choose $\lambda = \epsilon^{\tau}$, where $\big[N - dp^*(a,d) - q\delta\big]\dfrac{p-q}{q} < \tau < \beta(\mu)p + p(a+1) - N - \big[N - dp^*(a,d) - q\delta\big]$. Then

$$\lambda O(\epsilon^{N-dp^*(a,d)-q\delta}) = O(\epsilon^{\tau+N-dp^*(a,d)-q\delta}) \quad \text{and} \quad d\lambda^{\frac{p}{p-q}} = O(\epsilon^{\frac{p\tau}{p-q}}).$$

Since $\tau + N - dp^*(a,d) - q\delta < \dfrac{p\tau}{p-q}$, $\tau + N - dp^*(a,d) - q\delta < \beta(\mu)p + p(a+1) - N$, taking ϵ small enough we deduce that there exists $\delta > 0$, such that

$$O\left(\epsilon^{(\beta(\mu)+b_1)p^*(a,b_1)-N}\right) - \lambda O(\epsilon^{N-dp^*(a,d)-q\delta}) < -d\lambda^{\frac{p}{p-q}} \quad \forall\lambda\colon\ 0 < \lambda^{\frac{p}{p-q}} < \delta. \tag{54}$$

Choose $\Lambda_1 = \min\left\{\Lambda_1^*, \dfrac{p-q}{p}\delta\right\}$. Then for all $\lambda \in (0, \Lambda_1)$ we have

$$\sup_{t \geq t_1} J(tu_1, tv_1) \leq \Upsilon(\mu, \alpha, \beta, a, c) - d\lambda^{\frac{p}{p-q}}.$$

Together with (49), we get the conclusion of Lemma 5.

3. Proof of the main results. ***Proof of Theorem* 1.** Let

$$r := \|(u,v)\|,$$

$$f(r) := \frac{1}{p}r^p - \frac{1}{p^*(a,b_1)}S_{\mu,a,b_1}^{-\frac{p^*(a,b_1)}{p}} r^{p^*(a,b_1)} -$$

$$-\frac{1}{p^*(a,b_2)}S_{\mu,a,b_2}^{-\frac{p^*(a,b_2)}{p}} r^{p^*(a,b_2)} - \frac{1}{p^*(a,c)}S_{\mu,\alpha,\beta,a,c}^{-\frac{p^*(a,c)}{p}}\|Q\|_\infty,$$

$$h(r) := \mathcal{D}_0(S_{\mu,a,d})^{-\frac{q}{p}} r^q.$$

From (6) and (7),

$$J(u,v) \geq f(r) - h(r),$$

Note that $p < p^*(a,b_1),\ p^*(a,b_2),\ p^*(a,c)$, it is easy to see that there exists $\varrho > 0$ such that $f(r)$ achieves its maximum at ϱ and $f(\varrho) > 0$. Therefore, there exists $\Lambda_{11} > 0$, such that for $0 < \lambda < \Lambda_{11}$,

$$\inf_{\|(u,v)\|=\varrho} I(u,v) \geq f(\varrho) - h(\varrho) > 0. \tag{55}$$

On the other hand, set $B_\varrho = \{(u,v);\ \|(u,v)\| \leq \varrho\}$. For $(u,v) \neq (0,0)$, we can choose $d > 0$ small enough, such that $(du, dv) \in B_\varrho$ and

$$I(du, dv) < 0. \tag{56}$$

Thus,

$$-\infty < \inf_{(u,v)\in B_\varrho} I(u,v) < 0. \tag{57}$$

Now we can apply the Ekeland variational principle in [16] and obtain $\{(u_n,v_n)\}\subset B_\varrho$, such that

$$I(u_n,v_n)\le \inf_{(u,v)\in B_\varrho} I(u,v)+\frac{1}{n}, \tag{58}$$

$$I(u_n,v_n)\le I(u,v)+\frac{1}{n}\|(u_n-u,v_n-v)\|, \tag{59}$$

for all $(u,v)\in B_R$. Define

$$J_1(u,v):=J(u,v)+\frac{1}{n}\|(u_n-u,v_n-v)\|. \tag{60}$$

By (59), we have (u_n,v_n) is the minimizer of $J_1(u,v)$ on B_ϱ. (55), (57) and (58) imply that there exists $\epsilon>0$ and $k\in N$, such that for $n\ge k$, $\{(u,v),\ \|(u,v)\|\le\varrho-\epsilon\}$. Therefore, for $n\ge k$ and $(\phi,\varphi)\in W$, we can choose $t>0$ small enough, such that $(u_n+t\phi,v_n+t\varphi)\in B_\varrho$ and

$$\frac{J_1(u_n+t\phi,v_n+t\varphi)-J_1(u_n,v_n)}{t}\ge 0.$$

That is,

$$\frac{J(u_n+t\phi,v_n+t\varphi)-J(u_n,v_n)}{t}+\frac{1}{n}\|(\phi,\varphi)\|\ge 0. \tag{61}$$

Passing to the limit in (61) as $n\to 0$, one can get

$$\langle J'(u_n,v_n),(\phi,\varphi)\rangle\ge -\frac{1}{n}\|(\phi,\varphi)\|,$$

which implies that

$$\|J'(u_n,v_n)\|\le\frac{1}{n}. \tag{62}$$

Combining (58) and (62), there holds

$$\lim_{n\to\infty} J(u_n,v_n)=\inf_{(u,v)\in B_\varrho} J(u,v)<0, \tag{63}$$

$$\lim_{n\to\infty} J'(u_n,v_n)=0. \tag{64}$$

We note that there exists $\Lambda_{11}^*\in(0,\Lambda_{11})$, such that for $0<\lambda<\Lambda_{11}^*$, and $c_0>\inf_{(u,v)\in B_\varrho} I(u,v)$, where c_0 is defined in Lemma 3. Thus, (63) and (64) and Lemma 5 imply that for $0<\lambda<$ $<\Lambda_{11}^*$, $(u_n,v_n)\to(u,v)$ strongly in W. Therefore, (u,v) is a nontrivial solution of problem (1) satisfying $J(u,v)=\inf_{(u,v)\in B_\varrho} J(u,v)<0$. Note that $J(u,v)=J(|u|,|v|)$ and

$$(|u|,|v|)\in\{(u,v),\ \|(u,v)\|\le\varrho-\epsilon\},$$

we have $I(|u|,|v|) = \inf_{(u,v)\in B_\varrho} J(u,v)$ and $J'(|u|,|v|) = 0$. Then problem (1) has a nontrivial nonnegative solution. By the strongly maximum principle, we get the conclusion of Theorem 1.

***Proof of Theorem* 2.** In view of the proof of Theorem 1, we know that for $0 < \lambda < \Lambda_{11}$, there exists $\varrho > 0$, such that $\inf_{\|(u,v)\|=\varrho} I(u,v) \geq \vartheta^* > 0$. Moreover, (63) and (64) hold. We note that there exists $\Lambda_{12} \in (0,\Lambda_{11})$, such that for $0 < \lambda < \Lambda_{12}$, $c_* > \inf_{(u,v)\in B_\varrho} J(u,v)$, where c_* is defined in Lemma 2. Thus (63) and (64) and Lemma 2 imply that $(u_n,v_n) \to (u,v)$ strongly in W. Standard argument shows that for $0 < \lambda < \Lambda_{12}$, problem (1) has at least one positive solution satisfying $J(u,v) < 0$ and $J'(u,v) = 0$.

Now we prove a second positive solution. It is easy to see $J(0,0) = 0$. Set $\Lambda^{**} = \min\{\Lambda_{12},\Lambda_1\}$, where Λ_1 is given in Lemma 5. Then it follows from Lemma 5 there exists $(u',v') \in W \setminus \{0\}$, such that for $0 < \lambda < \Lambda^{**}$,

$$\sup_{t\geq 0} J(tu',tv') < c_*.$$

On the other hand we obtain that $\lim_{l\to\infty} J(lu',lv') = -\infty$. Thus there exists $l' > 0$ such that $\|(l'u',l'v')\| > \varrho$ and $J(l'u',l'v') < 0$. Let

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma := \left\{\gamma \in C^0([0,1],W) \mid \gamma(0) = (0,0),\ \gamma(1) = (l'u',l'v')\right\}.$$

Thus, it follows from the Mountain pass theorem in [14] that there exists a sequence $(u_n,v_n) \in W$ such that

$$\lim_{n\to\infty} J(u_n,v_n) = c$$

and

$$\lim_{n\to\infty} J'(u_n,v_n) = 0.$$

Moreover, $c \in (0,c_*)$. From Lemma 2, $(u_n,v_n) \to (\overline{u},\overline{v})$ strongly in W, which implies that $J(\overline{u},\overline{v}) = c$ and $J'(\overline{u},\overline{v}) = 0$, Therefore, $(\overline{u},\overline{v})$ is a second nontrivial solution of problem(1). Set $u^+ = \max\{u,0\}, v^+ = \max\{v,0\}$. Replacing

$$\int\limits_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}dx, \qquad \int\limits_\Omega \frac{|v|^q}{|x|^{dp^*(a,d)}}dx, \qquad \int\limits_\Omega \frac{|u|^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx,$$

$$\int\limits_\Omega \frac{|v|^{p^*(a,b_2)}}{|x|^{b_2p^*(a,b_2)}}dx, \qquad \int\limits_\Omega Q(x)\frac{|u|^\alpha|v|^\beta}{|x-x_0|^{cp^*(a,c)}}dx$$

by

$$\int\limits_\Omega \frac{(u^+)^q}{|x|^{dp^*(a,d)}}dx, \qquad \int\limits_\Omega \frac{(v^+)^q}{|x|^{dp^*(a,d)}}dx, \qquad \int\limits_\Omega \frac{(u^+)^{p^*(a,b_1)}}{|x|^{b_1p^*(a,b_1)}}dx,$$

$$\int\limits_\Omega \frac{(v^+)^{p^*(b_2)}}{|x|^{b_2p^*(a,b_2)}}dx, \qquad \int\limits_\Omega Q(x)\frac{(u^+)^\alpha(v^+)^\beta}{|x-x_0|^{cp^*(a,c)}}dx$$

and repeating the above process, we have a nonnegative solution $(\widetilde{u},\widetilde{v})$ of problem (1) satisfying $J(\widetilde{u},\widetilde{v})>0$. Then by the strongly maximum principle, we have a second positive solution.

Theorem 2 is proved.

4. Symmetric solution. In this section, similar to the method in [6], we prove the existence of infinitely many G-symmetric solutions of problem (2).

The corresponding energy functional of problem (2) is defined by

$$\widetilde{J}(u,v)=\frac{1}{p}\int\limits_{\Omega}\left(|x|^{-ap}|\nabla u|^p-\mu\frac{|u|^p}{|x|^{p(a+1)}}+|x|^{-ap}|\nabla v|^p-\mu\frac{|v|^p}{|x|^{p(a+1)}}\right)dx-$$

$$-\frac{\lambda}{q}\int\limits_{\Omega}h(x)\left(\frac{|u|^q}{|x|^{dp^*(a,d)}}+\frac{|v|^q}{|x|^{dp^*(a,d)}}\right)dx-\frac{1}{\alpha+\beta}\int\limits_{\Omega}Q(x)\frac{|u|^\alpha|v|^\beta}{|x-x_0|^{cp^*(a,c)}}dx. \qquad (65)$$

First, we present some notations and definitions that will be used in this section. Let $O(\mathbb{N})$ be the group of orthogonal linear transformations of $\mathbb{R}^N$ with natural action and let $G\subset O(\mathbb{N})$ be a subgroup with the property that $\mathrm{Fix}\{G\}=\{0\}$, where $\mathrm{Fix}\{G\}=\{x\in\mathbb{R}^N\colon gx=x\ \forall g\in G\}$ is the fixed point set of the action of G on $\mathbb{R}^N$. For $x\neq 0$ we denote the cardinality of $G_x=\{gx\colon g\in G\}$ by $|Gx|$ and set $|G|=\inf_{0\neq x\in\mathbb{R}^N}|Gx|$. Note that, here, $|G|$ may be $+\infty$. We call Ω a G-symmetric subset of $\mathbb{R}^N$, if $x\in\Omega$, then $gx\in\Omega$ for all $g\in G$. For any function $f(x)$ defining on $\mathbb{R}^N$, We call $f(x)$ a G-symmetric function if for all $g\in G$ and $x\in\mathbb{R}^N$, $f(gx)=f(x)$ holds. In particular, if f is radially symmetric, then the corresponding group G is $O(\mathbb{N})$ and $|G|=+\infty$. Other further examples of G-symmetric functions can be found in [17].

For a bounded and G-symmetric domain $\Omega\subset\mathbb{R}^N$, $0\in\Omega$, the natural functional space to study problem (2) is the Banach space $W_G(\Omega)=W^{1,p}_{a,G}(\Omega,|x|^{-ap})\times W^{1,p}_{a,G}(\Omega,|x|^{-ap})$ which $W^{1,p}_{a,G}(\Omega,|x|^{-ap})$ is the subspace of $W^{1,p}_a(\Omega,|x|^{-ap})$ consisting of all G-symmetric functions.

Lemma 6. *Assume that* $N\geq 3$, $0\leq\mu<\overline{\mu}$, $h_+\neq 0$ *and* $Q\in C(\overline{\Omega})\bigcap L^\infty(\overline{\Omega})$ *is G-symmetric. Then* $\widetilde{J}(u,v)$ *satisfies the* $(PS)_c$ *condition in* W_G *with c satisfying*

$$c<c^*:=\frac{p^*(a,c)-p}{pp^*(a,c)}\frac{|G|}{\|Q\|_\infty^{\frac{N-p(a+1-c)}{N}}}\left(\frac{1}{p}S_{\mu,\alpha,\beta,a,c}\right)^{\frac{p^*(a,c)}{p^*(a,c)-p}}. \qquad (66)$$

Proof. The proof is similar to the Lemma 3.3 of [6] and the Lemma 3 and is omitted.

***Corollary* 1.** *If* $|G|=+\infty$, *then the functional* $\widetilde{J}$ *satisfies* $(PS)_c$ *condition for every* $c\in\mathbb{R}$.

To prove Theorem 3 we need the following version of symmetric mountain pass theorem (see [14], Theorem 9.12).

Theorem 4. *Let E be an infinite dimensional Banach space and* $\mathcal{F}\in C^1(E,\mathbb{R})$ *be an even functional satisfying* $(PS)_c$ *condition for each c and* $\mathcal{F}=0$. *Further, we suppose that*:

(i) *there exist constants* $\widetilde{\alpha}>0$ *and* $\rho>0$ *such that* $\mathcal{F}\geq\widetilde{\alpha}$ *for all* $\|u\|=\rho$;

(ii) *there exist an increasing sequence of subspaces* $\{E_m\}$ *of* E, *with* $\dim E_m=m$, *such that for every* m *one can find a constant* $R_m>0$ *such that* $\mathcal{F}\leq 0$ *for all* $u\in E_m$ *with* $\|u\|\geq R_m$.

Then $\mathcal{F}$ *possesses a sequence of critical values* $\{c_m\}$ *tending to* ∞ *as* $m\to\infty$.

***Proof of Theorem* 3**. The proof is similar to that of Theorem 2.2 in [6]. From $\alpha+\beta=p^*(a,c)$, the Young inequality and (5) it follows that

$$\int\limits_\Omega Q(x)\frac{|u|^\alpha|v|^\beta}{|x-x_0|^{cp^*(a,c)}}dx \le \frac{\alpha\|Q\|_\infty}{\alpha+\beta}\int\limits_\Omega \frac{|u|^{\alpha+\beta}}{|x-x_0|^{cp^*(a,c)}}dx+\frac{\beta\|Q\|_\infty}{\alpha+\beta}\int\limits_\Omega \frac{|v|^{\alpha+\beta}}{|x-x_0|^{cp^*(a,c)}}dx\le$$

$$\le \frac{\alpha\|Q\|_\infty}{p^*(a,c)}(S_{\mu,a,c})^{-\frac{p^*(a,c)}{p}}\|u\|_\mu^{p^*(a,c)}+\frac{\beta\|Q\|_\infty}{p^*(a,c)}(S_{\mu,a,c})^{-\frac{p^*(a,c)}{p}}\|v\|_\mu^{p^*(a,c)}. \tag{67}$$

So, by (7), (65) and (67), one can get

$$\widetilde{J}(z)=\widetilde{J}(u,v)\ge$$

$$\ge \frac{1}{p}\|u\|_\mu^p+\frac{1}{p}\|v\|_\mu^p-\lambda\mathcal{D}_0(S_{\mu,a,d})^{-\frac{q}{p}}(\|u\|^q+\|v\|^q)-$$

$$-\frac{\alpha\|Q\|_\infty}{p^*(a,c)}(S_{\mu,a,c})^{-\frac{p^*(a,c)}{p}}\|u\|_\mu^{p^*(a,c)}-\frac{\beta\|Q\|_\infty}{p^*(a,c)}(S_{\mu,a,c})^{-\frac{p^*(a,c)}{p}}\|v\|_\mu^{p^*(a,c)}.$$

Since $1<q<p<p^*(a,c)$, we see that

$$\widetilde{J}(z)\ge\|z\|^q\left[\frac{1}{p}\|z\|^{p-q}-\lambda\mathcal{D}_0(S_{\mu,a,d})^{-\frac{q}{p}}\right]-$$

$$-\frac{\alpha\|Q\|_\infty}{p^*(a,c)}(S_{\mu,a,c})^{-\frac{p^*(a,c)}{p}}\|z\|^{p^*(a,c)}-\frac{\beta\|Q\|_\infty}{p^*(a,c)}(S_{\mu,a,c})^{-\frac{p^*(a,c)}{p}}\|z\|^{p^*(a,c)}.$$

Now, taking $\|z\|=\rho$ such that $\rho^{p-q}=2p\lambda\left(\dfrac{N\omega_N R_0^{N-dp^*(a,d)}}{N-dp^*(a,d)}\right)^{\frac{p^*(a,d)-q}{p^*(a,d)}}(S_{\mu,a,d})^{-q/p}>0$ with $\lambda>0$. Finally, we take $\widehat{\beta}>0$ such that

$$\widetilde{J}(z)\ge\lambda\mathcal{D}_0(S_{\mu,a,d})^{-\frac{q}{p}}\rho^q-\widehat{\beta}\rho^{p^*(a,c)}-\widehat{\beta}\rho^{p^*(a,c)}>0,$$

for every $z=(u,v)\in W_G(\Omega)$ and $\|z\|=\rho$. Therefore, there exist $\widehat{\alpha}>0$ and $\rho>0$ such that $\widetilde{J}(z)\ge\widehat{\alpha}$ for every z with $\|z\|=\rho$.

On the other hand, to find a suitable sequence of finite dimensional subspaces of $W_G(\Omega)$, we set $\omega=\{x\in\Omega; Q(x)>0\}$. Since the set ω is G-symmetric, we can define $W_G(\omega)$, which is the subspace of G-symmetric functions of $W|_{\Omega=\omega}$. By extending functions in $W_G(\omega)$ to 0 outside ω we can assume $W_G(\omega)\subset W_G(\Omega)$. Let $\{E_m\}$ be an increasing sequence of subspaces of $W_G(\omega)$ with $\dim E_m=m$ for each m. Now, we take $\varphi_{1,m},\ldots,\varphi_{m,m}\in C_0^\infty(\Omega)$ such that $0\le\varphi_{i,m}\le1$, $\mathrm{supp}(\varphi_{i,m})\bigcap\mathrm{supp}(\varphi_{j,m})=\varnothing$, $i\ne j$, and $|\mathrm{supp}(\varphi_{i,m})\bigcap\omega|>0$, for all $i,j\in\{1,\ldots,m\}$. Let $e_{i,m}=(a\varphi_{i,m},b\varphi_{i,m})\in E_m$, $i=1,\ldots,m$, and $E_m=\mathrm{span}\{e_{1,m},\ldots,e_{m,m}\}$, where a,b are tow positive constants. By construction, $\dim E_m=m$. Now, for $z=(u,v)=\sum_{i=1}^m t_{i,m}e_{i,m}\in E_m$, one can get

$$\frac{1}{\alpha+\beta}\int\limits_\omega Q(x)\frac{|u|^\alpha|v|^\beta}{|x-x_0|^{cp^*(a,c)}}dx=\frac{1}{\alpha+\beta}\int\limits_\omega Q(x)\frac{\left|\sum_{i=1}^m at_{i,m}\varphi_{i,m}\right|^\alpha\left|\sum_{i=1}^m bt_{i,m}\varphi_{i,m}\right|^\beta}{|x-x_0|^{cp^*(a,c)}}dx,$$

then there exists a constant $C(m) > 0$ such that

$$\frac{1}{\alpha+\beta}\int\limits_{\omega} Q(x)\frac{|u|^{\alpha}|v|^{\beta}}{|x-x_0|^{cp^*(a,c)}}dx \geq c(m), \quad \text{for all} \quad (u,v)\in E_m, \quad \text{with} \quad \|z\| = \|(u,v)\| = 1.$$

Consequently, if $0 \neq u \in E_m$, then we write $z = (u,v) = (tu_1, tv_1)$, with $t = \|z\|$ and $\|(u_1,v_1)\| = 1$. Thus we have

$$\widetilde{J}(z) \leq \widetilde{J}_0(u,v) = \frac{1}{p}t^p - \frac{t^{p^*(a,c)}}{\alpha+\beta}\int\limits_{\omega} Q(x)\frac{|u_1|^{\alpha}|v_1|^{\beta}}{|x-x_0|^{cp^*(a,c)}}dx \leq \frac{1}{p}t^p - C(m)t^{p^*(a,c)} \leq 0$$

for t large enough. By Corollary 1 and Theorem 4 we conclude that there exists a sequence of critical values $c_m \to \infty$ as $m \to \infty$.

Theorem 3 is proved.

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