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EXISTENCE OF THE CATEGORY $DTC_2(k)$ WHICH IS EQUIVALENT TO THE GIVEN CATEGORY KAC_2

ПРО ІСНУВАННЯ КАТЕГОРІЇ $DTC_2(k)$, ЩО ЕКВІВАЛЕНТНА ЗАДАНІЙ КАТЕГОРІЇ KAC_2

For a given category KAC_2, the present paper deals with an existence problem of the category $DTC_2(k)$ which is equivalent to KAC_2, where $DTC_2(k)$ is the category whose objects are simple closed k-curves with even number l of elements in $\mathbf{Z}^n$, $l \neq 6$ and morphisms are (digitally) k-continuous maps, and KAC_2 is the category whose objects are simple closed A-curves and morphisms are A-maps. To address this issue, the paper starts with the category, denoted by KAC_1, whose objects are connected nD Khalimsky topological subspaces with Khalimsky adjacency and morphisms are A-maps in [*Han S. E., Sostak A.* A compression of digital images derived from a Khalimsky topological structure // Comput. and Appl. Math. – 2013. – **32**. – P. 521 – 536]. Based on this approach, in KAC_1 the paper proposes the notions of an A-homotopy and an A-homotopy equivalence, and classifies spaces in KAC_1 or KAC_2 in terms of an A-homotopy equivalence. Finally, the paper proves that for a given category KAC_2 there is $DTC_2(k)$ which is equivalent to KAC_2.

Для заданої категорії KAC_2 вивчено проблему існування категорії $DTC_2(k)$, що еквівалентна KAC_2, де $DTC_2(k)$ — категорія, об'єктами якої є прості замкнені k-криві з парним числом l, $l \neq 6$, елементів в $\mathbf{Z}^n$, а морфізмами — (цифрово) k-неперервні відображення, тоді як KAC_2 — категорія, об'єктами якої є прості замкнені A-криві, а морфізми є A-відображеннями. Наш виклад ми починаємо з категорії, що позначена KAC_1, об'єктами якої є nD зв'язні топологічні підпростори Халімського з суміжністю Халімського, а морфізми є A-відображеннями, що визначені в [*Han S. E., Sostak A.* A compression of digital images derived from a Khalimsky topological structure // Comput. and Appl. Math. – 2013. – **32**. – P. 521 – 536]. На основі запропонованого підходу в категорії KAC_1 введено поняття A-гомотопії та A-гомотопічної еквівалентності, а простори з KAC_1 або KAC_2 класифіковано в термінах A-гомотопічної еквівалентності. Насамкінець доведено, що для заданої категорії KAC_2 існує $DTC_2(k)$, еквівалентна KAC_2.

1. Introduction. Let $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{Z}^n$ represent the sets of integers, natural numbers and points in the Euclidean nD space with integer coordinates, respectively. To recognize a set $X \subset \mathbf{Z}^n$ with graph theoretical structures, A. Rosenfeld introduced digital topology [19]. Furthermore, many researchers have developed several tools such as a Marcuse Wyse topological structure [20], a graph theoretical method [4 – 7, 16, 19], a Khalimsky topological structure [3, 10, 11, 14, 15, 18], a locally finite topological approach [17] and so forth. Nowadays, digital topology plays an important role in some areas of computer science and applied topology such as image processing, computer graphics, image analysis, mathematical morphology and so forth. It has been used to study topological properties and features, e.g., connectedness and boundaries of two, three or nD digital images. Since the paper will frequently refer a Khalimsky topological structure, hereafter, for convenience we will use the terminology K — instead of "Khalimsky" if there is no danger of ambiguity.

Since continuity of maps between digital spaces is also an important topic in digital topology, many studies have examined various properties of a K-continuous map, connectedness, Khalimsky adjacency, a K-homeomorphism [10, 11, 14, 15]. However, it turns out that a K-continuous map have some limitations [13]: it does not contain some rotations and transformations. Thus the recent paper [13] developed a broader class of maps, called A-maps, which are generalizations of both a K-continuous map and a Khalimsky adjacency (for short, KA-) map. Furthermore, it introduced an A-isomorphism which is a generalization of a K-homeomorphism. But we still need a mathematical structure associated with a K-topological structure which is equivalent to a Rosenfeld's digital image (X, k) in $\mathbf{Z}^n$. Let DTC be a category whose objects are digital images (X, k) in $\mathbf{Z}^n$ and morphisms are (digitally) k-continuous maps [6, 19]. Thus we see that DTC is not a topological category because these digital images in $Ob(DTC)$ have only graph theoretical structures instead of topological ones. Besides, let $DTC_1(k)$ (resp. $DTC_2(k)$) be the category whose objects are k-connected digital images (resp. simple closed k-curves with even number l of elements, $l \neq 6$) in $\mathbf{Z}^n$ and morphisms are k-continuous maps for the categories $DTC_1(k)$ and $DTC_2(k)$. It is obvious that both $DTC_1(k)$ and $DTC_2(k)$ are subcategories of DTC. Let KAC_1 (resp. KAC_2) be the category whose objects are connected K-topological spaces with K-adjacency (resp. simple closed A-curves) and morphisms are A-maps for both KAC_1 and KAC_2. Let us now raise the following question:

Given the category KAC_2, is there a category $DTC_2(k)$ which is equivalent to KAC_2 ?

This problem is strongly related to the development of a homotopy associated with K-topology. Up to now, there is no research into this construction from the viewpoint of K-topology. To address this issue, the paper proposes an equivalence between KAC_2 and $DTC_2(k)$ so that an A-map in KAC_2 is equivalent to a (digitally) k-continuous map in $DTC_2(k)$. Besides, in KAC_1 the paper establishes the notions of an A-homotopy and an A-homotopy equivalence (see Sections 5 and 6).

2. Preliminaries. Let us recall some basic facts and terminology from digital topology such as $\mathbf{Z}^n$ with digital k-connectivity and Khalimsky (for brevity, K-) topology. For two distinct points a and b in $\mathbf{Z}$ let $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \mid a \leq n \leq b\}$ [16]. The *Khalimsky line topology* on $\mathbf{Z}$ is generated by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ as a subbase [1] (see also [14, 15]). Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, T)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or *Khalimsky nD space*) which is denoted by $(\mathbf{Z}^n, T^n)$ [14]. In the present paper each space $X \subset \mathbf{Z}^n$ related to K-topology is considered to be a subspace (X, T^n_X) induced by $(\mathbf{Z}^n, T^n)$. It is well known that $(\mathbf{Z}^n, T^n)$ is a T_0-Alexandroff space [15] (cf. [1]).

Let us now recall the structure of $(\mathbf{Z}^n, T^n)$. A point $x = (x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd; and it is *pure closed* if each of the coordinates is even [15]. The other points in $\mathbf{Z}^n$ are called *mixed* [15]. In each of the spaces of Figures 1, 2 and 3, a black jumbo dot means a pure open point and further, the symbols $\blacksquare$ and $\bullet$ mean a pure closed point and a mixed point, respectively. In relation to the further statement of a pure point and a mixed point, we say that a point x is open if $SN(x) = \{x\}$, where $SN(x)$ means the smallest neighborhood of $x \in \mathbf{Z}^n$. The point $x \in \mathbf{Z}^n$ is called closed if $C(x) = \{x\}$, where $C(x)$ stands for a closure of the singleton $\{x\}$.

Many studies have investigated various properties of a K-continuous map, connectedness, Khalimsky adjacency, a K-homeomorphism [6, 8, 14, 15, 18]. It is important and well known that in $(\mathbf{Z}^n, T^n)$ for two distinct points an equivalence exists between connectedness and Khalimsky adjacency [15]. In $(\mathbf{Z}^n, T^n)$, a Khalimsky adjacency relation is symmetric [15]. Besides, for any Khalimsky adjacent set $A \subset X$, the image by a K-continuous map $f : (X, T^{n_0}_X) := X \to (Y, T^{n_1}_Y) := Y$, $f(A)$, is also a Khalimsky adjacent subset of Y, which is very useful properties for studying Khalimsky topological spaces. However, both a K-continuous map and a K-homeomorphism are very rigid so that their contributions can be so limited. For instance, not every rotation and an odd vector translation are K-continuous maps (see Remark 3.2). In addition, a map preserving Khalimsky adjacency (or a KA-map) does not allow a constant map because the KA-relation deals with only distinct points (see (3.1)). To overcome this difficulty, the recent paper [13] develops two maps called an *A-map* and an *A-isomorphism* which are expansions of a K-continuous map and a K-homeomorphism, respectively.

To study several types of digital neighborhoods and their properties, we need to recall the *digital k-adjacency relation* of $\mathbf{Z}^n$ and *a digital k-neighborhood*. As a generalization of the k-adjacency relations of 2D and 3D digital spaces [16, 19], the k-adjacency relations (or digital k-connectivity) of $\mathbf{Z}^n$ have been established [5] (see also [9, 10]).

For the natural numbers m, n with $1 \leq m \leq n$, two distinct points $p = (p_i)_{i \in [1,n]_{\mathbf{Z}}}$ and $q = (q_i)_{i \in [1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ are called $k(m,n)$- (for brevity, k-) adjacent if there are at most m indices i such that $|p_i - q_i| = 1$ and for all other indices i, $p_i = q_i$.

Concretely, the $k(m,n)$-adjacency relations of $\mathbf{Z}^n$ are determined according to the numbers $m, n \in \mathbf{N}$ [5] (see also [9, 10]) as follows:

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \quad \text{where} \quad C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.1}$$

A pair (X, k) (or digital image) is assumed to be a (binary) set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations (see (2.1)) in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $(k, \bar{k}) \in \{(k, 2n), (2n,\ 3^n - 1)\}$ with $k \neq \bar{k}$, k represents an adjacency relation for X and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n \setminus X$ [16].

Using the adjacency relations of (2.1), in $\mathbf{Z}^n$ we say that a *digital k-neighborhood* of p is the set [19] $N_k(p) := \{q \mid p \text{ is } k\text{-adjacent to } q\}$. Furthermore, we often use the notation [16] $N_k^*(p) := N_k(p) \cup \{p\}$.

Given a k-adjacency relation of $\mathbf{Z}^n$, a simple k-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [16]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x,y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [5], is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i + 1 (\operatorname{mod} l)$ or $i = j + 1 (\operatorname{mod} l)$ [16].

For a digital image (X, k) let us recall a digital k-neighborhood with radius 1 which is a generalization of $N_k^*(p)$ [5].

$$N_k(x, 1) = N_k^*(x) \cap X. \tag{2.2}$$

3. Some properties of maps between digital topological spaces. To map every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [19] established the notion of digital continuity. Motivated by this continuity, we represent the digital continuity of maps between digital images, which can be efficiently used for studying spaces $X \subset \mathbf{Z}^n$, $n \in \mathbf{N}$.

Proposition 3.1 [5, 8]. *Let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x,1)) \subset N_{k_1}(f(x), 1)$.*

Based on these concepts, let us consider a digital topological category, denoted by *DTC*, consisting of two things [5]:

the set of (X, k) in $\mathbf{Z}^n$ as objects;

the set of (k_0, k_1)-continuous maps as morphisms.

In *DTC*, in case $n_0 = n_1$ and $k_0 = k_1 := k$, we will particularly use the notation $DTC(k)$.

Since the digital image (X, k) is considered to be a set $X \subset \mathbf{Z}^n$ with one of the adjacency relation of (2.1), we use the term a (k_0, k_1)-isomorphism as in [6] (see also [12]) rather than a (k_0, k_1)-homeomorphism as in [2].

Definition 3.1 [12] (see also [6, 8]). *For two digital images (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h\colon X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1}\colon Y \to X$ is (k_1, k_0)-continuous.*

In Definition 3.1, in case $n_0 = n_1$ and $k_0 = k_1$, we call it a k_0-*isomorphism*.

Let us now recall some properties of Khalimsky adjacency induced by $(\mathbf{Z}^n, T^n)$.

Definition 3.2 [15]. *In $(\mathbf{Z}^2, T^2)$, we say that two distinct points x, y in $\mathbf{Z}^n$ are Khalimsky adjacent if $y \in SN(x)$ or $x \in SN(y)$, $SN(q)$ means the smallest open set containing the point $q \in \mathbf{Z}^n$, $q \in \{x, y\}$.*

In Definition 3.2, we can extend the notion into the case $(\mathbf{Z}^n, T^n)$ [13].

For a point $p \in \mathbf{Z}^n$ the Khalimsky adjacency neighbor of p, denoted by $A(p)$, is defined [13, 18] as follows (for the case $(\mathbf{Z}^2, T^2)$, see [15]):

$$A(p) = \{q \in N_{3^n-1}(p) \mid \{p, q\} \text{ is connected under } (\mathbf{Z}^n, T^n)\}. \tag{3.1}$$

For a point $p := (\overbrace{0, \ldots, 0}^{\alpha}, \overbrace{1, \ldots, 1}^{\beta}) := \langle \alpha, \beta \rangle \in \mathbf{Z}^n$, the Khalimsky adjacency neighbor of p is defined as follows:

$$A(p) := ([-1, 1]^{\alpha}_{\mathbf{Z}} \times \{1\}^{\beta} \cup \{0\}^{\alpha} \times [0, 2]^{\beta}_{\mathbf{Z}}) \setminus (\{0\}^{\alpha} \times \{1\}^{\beta}). \tag{3.2}$$

According to (3.2), since $A(p)$ does not contain the point $p := \langle \alpha, \beta \rangle$, we obtain that cardinality of $A(p)$ is $(3^{\alpha} - 1) + (3^{\beta} - 1) = 3^{\alpha} + 3^{\beta} - 2$ [18].

***Example* 3.1** [13]. Let $(\mathbf{Z}^n, T^n)$ be the Khalimsky nD space. Then we obtain the Khalimsky adjacency neighbor of a point $p := \langle \alpha, \beta \rangle \in \mathbf{Z}^n$ (for short, $A(p) := A\langle \alpha, \beta \rangle$) as follows:

If $n = 2$, then for a point $p \in \mathbf{Z}^2$ we obtain $A(p) = N_8(p)$ if p is a pure point, and $A(p) = N_4(p)$ if p is a mixed point. If $n \geq 3$, then for a point $p \in \mathbf{Z}^n$ $A(p) = N_{3^n-1}(p)$ if p is a pure point, and if given a point $p := (p_i)_{i \in [1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ is a mixed point, then according to the component of the given coordinates p_i, $A(p)$ is determined by the method suggested in (3.2).

To investigate some properties of maps between K-topological spaces, for a space $(X, T^n_X) := X$ let us recall the *Khalimsky adjacency relation* for any two points in X as follows:

Definition 3.3 [18]. *For $(X, T^n_X) := X$ assume that $A_X(p) := A(p) \cap X$. We say that for two distinct points $p, q \in X$ they are Khalimsky adjacent to each other if $q \in A_X(p)$ or $p \in A_X(q)$.*

In view of Definition 3.3, we observe that Khalimsky adjacency holds only the *symmetric relation* without the reflexive relation. In $(\mathbf{Z}^n, T^n)$, since every point $x \in \mathbf{Z}^n$ has its smallest neighborhood $SN(x)$, we say that a map $f\colon (X, T^{n_0}) := X \to (Y, T^{n_1}) := Y$ is K-continuous [13] if for every point $x \in X$ we have

$$f(SN(x)) \subset SN(f(x)). \tag{3.3}$$

Besides, we say that the map f is a KA-map [13] if for two Khalimsky adjacent points $x_1, x_2 \in X$ the images $f(x_1), f(x_2)$ are Khalimsky adjacent, which we define the following terminology [13].

Definition 3.4. *For a space* $(X, T_X^n) := X$ *we define the following*:

(1) *Two distinct points* $x, y \in X$ *are called Khalimsky adjacency* (*for brevity, A-*) *connected if there is a sequence* (*or path*) $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ *on* X *with* $\{x_0 = x, x_1, \ldots, x_l = y\}$ *such that* x_i *and* x_{i+1} *are Khalimsky adjacent,* $i \in [0, l-1]_{\mathbf{Z}}$, $l \geq 1$. *This sequence is called an A-path. Furthermore, the number* l *is called the length of this A-path. Furthermore, an A-path is called a closed A-curve if* $x_0 = x_l$ [13].

(2) *A simple A-path in* X *is the sequence* $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ *such that* x_i *and* x_j *are Khalimsky adjacent if and only if either* $j = i + 1$ *or* $i = j + 1$.

Furthermore, we say that a simple closed A-curve with l *elements* $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ *is a simple A-path with* $x_0 = x_l$ *such that* x_i *and* x_j *are Khalimsky adjacent if and only if either* $j = i + 1(\mathrm{mod}\, l)$ *or* $i = j + 1(\mathrm{mod}\, l)$, $l \geq 4$ [13].

Hereafter, we denote $SC_A^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ *by a simple closed A-curve with* l *elements in* $\mathbf{Z}^n$, $n \in \mathbf{N} - \{1\}$, $l \geq 4$ [13].

(3) *A* K*-continuous map* $h\colon (X, T_X^{n_0}) \to (Y, T_Y^{n_1})$ *is called a local* K*-homeomorphism if for any* $x \in X$, h *maps* $SN(x)$ K*-homeomorphically onto* $SN(h(x)) \subset Y$.

In view of Definition 3.4(3), it is clear that while a K-homeomorphism implies a local K-homeomorphism, the converse does not hold.

In $(\mathbf{Z}^n, T^n)$ we say that a simple closed K-curve with l elements in $\mathbf{Z}^n$ is a path $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ that is K-homeomorphic to a quotient space of a Khalimsky line interval $[a, b]_{\mathbf{Z}}$ in terms of the identification of the only two end points a and b. We denote it by $SC_K^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, $l \geq 4$ [13].

***Remark* 3.1.** In view of the property of $SC_K^{n,l}$, we say that it is a finite subspace in $(\mathbf{Z}^n, T^n)$ which is locally K-homeomorphic to the Khalimsky line $(\mathbf{Z}, T)$.

Now we refer a difference between a K-continuous map and a KA-map.

***Example* 3.2.** (1) Consider two spaces X, Y in Fig. 1(1). While X is considered to be both $SC_K^{2,4}$ and $SC_A^{2,4}$, Y is neither $SC_K^{2,4}$ nor $SC_A^{2,4}$ because Y cannot be connected under K-topology.

(2) Consider the following five spaces X, Y, Z, V and W in Fig. 1(2). While each of X, Y, V and W is both $SC_K^{2,8}$ and $SC_A^{2,8}$, Z is neither $SC_K^{2,8}$ (see the points z_2, z_6) nor $SC_A^{2,8}$ (see the points z_i, $i \in \{0, 2, 4, 6\}$).

(3) Consider the spaces X in Fig. 1(2) and D in Fig. 1(3). While these consist of eight elements, the former contains mixed points and the latter contains only pure points. However, they have the same structure as $SC_K^{2,8}$ as well as $SC_A^{2,8}$.

Although both a K-continuous map and a KA-map play important roles in studying spaces (X, T_X^n), they have some limitations as follows:

***Remark* 3.2** [13]**.** As said in the previous part, we observe that not every one click rotation of a set $X \subset \mathbf{Z}^n$ and an odd vector translation are K-continuous maps. To be specific,

(1) Consider the self-map $f : SC_K^{n,l} := (c_i)_{i\in[0,l-1]} \to SC_K^{n,l}$ given by $f(c_i) = c_{i+1(\mathrm{mod}\, l)}$. Then f cannot be a K-continuous map (see the space X in Fig. 1(1)).

(2) Let us consider the map $f : (\mathbf{Z}, T) \to (\mathbf{Z}, T)$ given by $f(t) = t + (2n + 1)$, $n \in \mathbf{Z}$, which is a parallel translation with an odd vector. Then we clearly observe that f cannot be a K-continuous map.

(3) In addition, a KA-map also has the following limitation: a Khalimsky adjacency map does not allow a constant map. When we consider a KA-relation of two points p and q in $\mathbf{Z}^n$, we always assume that the given two points are distinct.

In view of Remark 3.2, we strongly need to develop another map overcoming the limitations (see Definition 4.2). Besides, $SC_K^{n,l}$ has the following property.

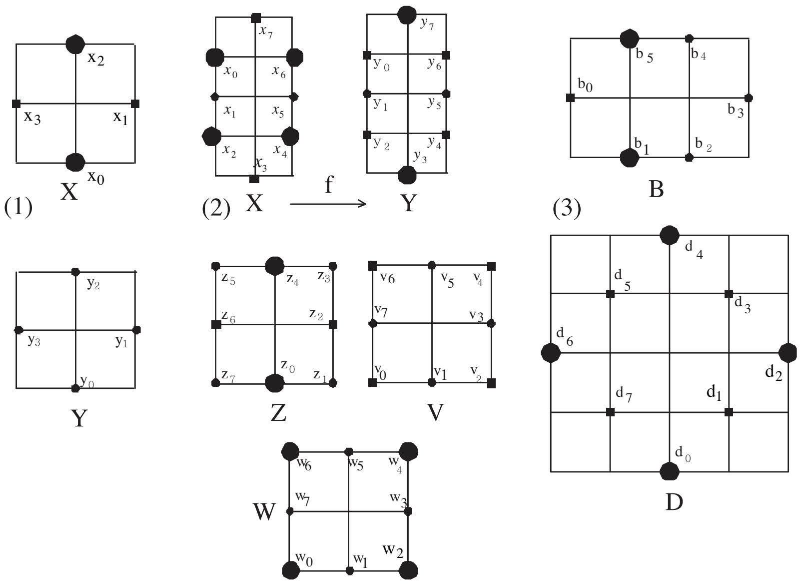

Fig. 1. Explanation of $SC_K^{n,l}$ and $SC_A^{n,l}$.

Lemma 3.1. *For* $SC_K^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, *we obtain that*

(1) $l \neq 6$,

(2) l *is an even number such that* $l \in 2\mathbf{N} \setminus \{6\}$ *and* $l \geq 4$.

Proof. (1) Let us take any point $x_i \in SC_K^{n,l}$. Then, according to the structure of $SC_K^{n,l}$, we may assume the following six cases:

(1) assume x_i is pure closed and $x_{i+1(\mathrm{mod}\, l)}$ is pure open;

(2) assume x_i is pure closed and $x_{i+1(\mathrm{mod}\, l)}$ is mixed;

(3) assume x_i is pure open and $x_{i+1(\mathrm{mod}\, l)}$ is pure closed;

(4) assume x_i is pure open and $x_{i+1(\mathrm{mod}\, l)}$ is mixed;

(5) assume x_i is mixed and $x_{i+1(\mathrm{mod}\, l)}$ is pure closed;

(6) assume x_i is mixed and $x_{i+1(\mathrm{mod}\, l)}$ is pure open.

Depending on the point $x_i \in SC_K^{n,l}$ above, each point x_i has its smallest open neighborhood as follows:

$$SN(x_i) = (x_{i-1(\mathrm{mod}\, l)}, x_i, x_{i+1(\mathrm{mod}\, l)}) \quad \text{if} \quad x_i \quad \text{is pure closed or mixed, and}$$

$$SN(x_i) = \{x_i\} \quad \text{if} \quad x_i \quad \text{is pure open.}$$

To establish $SC_K^{n,6} := (x_i)_{i\in[0,5]_{\mathbf{Z}}}$, we may take $x_0 \in SC_K^{n,l}$ as a pure open, a pure closed or a mixed point. Firstly, in case the given point $x_0 \in SC_K^{n,l}$ is a pure open point, we can consider Case (3) or (4) above. Let us take (Case 3). Then the point x_1 can be pure closed (see Fig. 2(1-1)) and further, according to (Case 2), we can take x_2 as a mixed point. Then we fail to make $SC_K^{n,6}$. If we take x_2 as an pure open point via (Case 1), then we cannot have $SC_K^{n,6}$ either.

As the other case, let us follow (Case 4) to take point x_1 as a mixed point (see Fig. 2(1-2)). Then we should take x_2 as a pure open point in $N_8(x_1)$ (see Cases (5), (6)), which we fail to have $SC_K^{n,6}$.

Secondly, in case the given point $x_0 \in SC_K^{n,l}$ is a pure closed point, the proof is completed by using the method similar to the first case (see Fig. 2(2-1) and (2-2)).

Finally, in case the given point $x_0 \in SC_K^{n,l}$ is a mixed point, the proof is also completed by using the method similar to the first and second cases (see Fig. 2(3-1) and (3-2)).

(2) According to the properties of Cases (1) – (6) above, to establish $SC_K^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, when starting at a point $x_0 \in \mathbf{Z}^n$ as an pure open, pure closed or mixed point, we cannot have the consecutive points $x_i, x_{i+1\,(\mathrm{mod}\,l)}$ such that they are all pure open, pure closed or mixed points. Thus, by the properties of Cases (1) – (6) above, the number l of $SC_K^{n,l}$ should be even. By the property of (1) of this theorem, we obtain $l \neq 6$. However, we have $SC_K^{n,4}$ (see Fig. 1(1)).

Lemma 3.1 is proved.

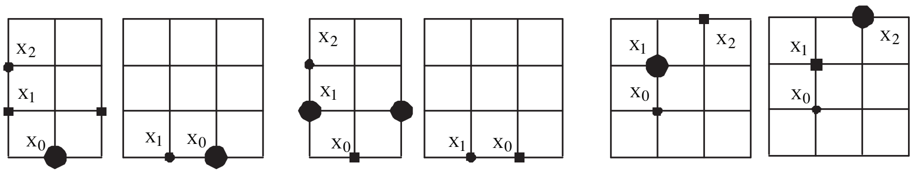

Fig. 2. Several cases of the points related to $SC_K^{2,6}$.

Theorem 3.1. *Consider two spaces $SC_K^{n_1,l_1}$ and $SC_K^{n_2,l_2}$. They are K-homeomorphic to each other if and only if $l_0 = l_1$.*

Proof. Owing to the topological structure of $SC_K^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, for $i \in [0, l-1]_{\mathbf{Z}}$ the subspace $\{x_i, x_{i+1(\mathrm{mod}\,l)}\}$ is a connected set and we need to recall that the number l should be even (see Lemma 3.1). If not, it cannot be locally K-homeomorphic to $(\mathbf{Z}, T)$. Besides, each point $x_i \in SC_K^{n,l}$, $i \in [0, l-1]_{\mathbf{Z}}$, has the smallest open neighborhood satisfying

$$\left.\begin{cases} \text{in case} \quad SN(x_i) = \{x_{i-1(\mathrm{mod}\,l)}, x_i, x_{i+1(\mathrm{mod}\,l)}\}, \\ \text{we obtain} \quad SN(x_{i+1(\mathrm{mod}\,l)}) = \{x_{i+1(\mathrm{mod}\,l)}\}; \\ \text{and in case} \quad SN(x_i) = \{x_i\}, \\ \text{we have} \quad SN(x_{i+1(\mathrm{mod}\,l)}) = \{x_{i(\mathrm{mod}\,l)}, x_{i+1(\mathrm{mod}\,l)}, x_{i+2(\mathrm{mod}\,l)}\}. \end{cases}\right\} \tag{3.4}$$

Consequently, for two spaces $SC_K^{n_1,l_1} := (x_i)_{i\in[0,l_1-1]_{\mathbf{Z}}}$ and $SC_K^{n_2,l_2} := (y_j)_{j\in[0,l_2-1]_{\mathbf{Z}}}$ if $l_1 = l_2$, then we have a map $f\colon SC_K^{n_1,l_1} \to SC_K^{n_2,l_2}$ satisfying that the restriction map on $SN(x_i)$, denoted by $f|_{SN(x_i)}$, is (locally) K-homeomorphic to $SN(y_j)$ for each $i \in [0, l_1 - 1]_{\mathbf{Z}}$. Thus the map f is a K-homeomorphism.

Conversely, if $SC_K^{n_1,l_1}$ is K-homeomorphic to $SC_K^{n_2,l_2}$, then it is clearly $l_1 = l_2$.

Theorem 3.1 is proved.

In relation to the geometric transformation of $SC_K^{n,l}$, we obtain the following property of a rotation of $SC_K^{n,l}$. In view of the structure of $SC_K^{n,l}$ and Lemma 3.1, since the number l should be even, we obtain the following theorem.

Theorem 3.2. *Let $h\colon SC_K^{n,2l} := (x_i)_{i\in[0,2l-1]_{\mathbf{Z}}} \to SC_K^{n,2l}$ be the self-map given by $h(x_i) = x_{i+l(\mathrm{mod}\,2l)}$, $l \in \mathbf{N}\setminus\{1,3\}$. Then h is a K-homeomorphism.*

Proof. For a given $SC_K^{n,2l}$, the point $x_i \in SC_K^{n,2l}$ has the property (3.4). Furthermore, by Lemma 3.1 and further, according to the Cases (1)–(6) in the proof of Lemma 3.1, we can define the map $h\colon SC_K^{n,2l} \to SC_K^{n,2l}$ given by $h(x_i) = x_{i+l(\mathrm{mod}\,2l)}$, which is a K-homeomorphism.

Theorem 3.2 is proved.

4. Establishment of two categories of K-topological spaces with Khalimsky adjacency. To overcome the limitations discussed in Remark 3.2, in this section we formulate two categories associated with K-topology which can be substantially helpful to study a space (X, T_X^n). In relation to the establishment, we will use the following Khalimsky adjacency neighborhood of a point $p \in X$.

Definition 4.1 [13]. *For a space $(X, T_X^n) := X$ and a point $p \in X$ we define a Khalimsky adjacency neighborhood of p to be the set $A_X(p) \cup \{p\} := AN_X(p)$.*

Hereafter, in (X, T_X^n) we will for brevity use $AN(p)$ instead of $AN_X(p)$ if there is no danger of ambiguity. Indeed, using $AN(p)$, we develop an A-map and an A-isomorphism (see Definitions 4.2 and 4.3).

Definition 4.2 [13]. *For two spaces $(X, T_X^{n_0}) := X$ and $(Y, T_Y^{n_1}) := Y$, we say that a function $f\colon X \to Y$ is an A-map at a point $x \in X$ if*

$$f(AN(x)) \subset AN(f(x)).$$

Furthermore, we say that a map $f\colon X \to Y$ is an A-map if the map f is an A-map at every point $x \in X$.

Indeed, an A-map can be also represented by using a smallest neighborhood [13]:

$f\colon (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ is an A-map

$$\left\{\begin{array}{l} \text{if for every} \quad x,\, x' \quad \text{in} \quad X \quad \text{such that} \quad x' \in SN(x) \quad \text{or} \quad x \in SN(x') \\ \text{it holds that} \quad f(x') \in SN(f(x)) \quad \text{or} \quad f(x) \in SN(f(x')). \end{array}\right\} \tag{4.1}$$

According to (4.1), we observe that an A-map implies a map preserving connected subsets of X into connected ones and further, it generalizes both a K-continuous map and a KA-map (see Example 4.1). Thus an A-map can be useful for studying K-topological spaces with K-adjacency.

***Example* 4.1.** Consider the map $f : X \to Y$ in Fig. 1(2) given by $f(x_i) = y_i$, $i \in [0, 7]_{\mathbf{Z}}$. While the map f cannot be a K-continuous map, it is an A-map.

Using spaces $(X, T_X^n) := X$ and A-maps, we establish a category of K-topological spaces with Khalimsky adjacency denoted by *KAC* in terms of the following two sets [13].

(1) The set of spaces $(X, T_X^n) := X$ with K-adjacency as objects denoted by $Ob(KAC)$;

(2) The set of A-maps between all pairs of elements in $Ob(KAC)$ as morphisms.

Since $(\mathbf{Z}^n, T^n)$ is an Alexandroff T_0-space, we obtain that in $(\mathbf{Z}^n, T^n)$, for two distinct points p and q in $\mathbf{Z}^n$ the point p is Khalimsky adjacent to q if and only if the set $\{p, q\}$ is connected [13, 18], in case $(\mathbf{Z}^2, T^2)$ the property was studied in the paper [15]. Thus, in view of (3.1), (3.2) and Definition 4.1, since for each point $p \in \mathbf{Z}^n$ $SN(p) \subset AN(p)$, we obtain the following corollary.

***Corollary* 4.1.** *For a point $p \in X := (X, T_X^n)$ we obtain $SN(p) \subset AN(p) \subset N_{3^n-1}(p, 1)$.*

By Example 4.1, Corollary 4.1 and Definition 4.2, we obtain the following theorem.

Theorem 4.1 [13]. *Let $f\colon (X, T_X^{n_0}) := X \to (Y, T_Y^{n_1}) := Y$ be a map.*

Every K-continuous map is an A-map. But the converse does not hold.

Definition 4.3 [13]. *For two spaces $(X, T_X^{n_0}) := X$ and $(Y, T_Y^{n_1}) := Y$ in KAC, a map $h\colon X \to Y$ is called an A-isomorphism if h is a bijective A-map (for short, A-bijection) and if $h^{-1}\colon Y \to X$ is an A-map.*

Definition 4.4. *We say that an A-map $h\colon X \to Y$ is a local A-isomorphism if for any $x \in X$, h maps $AN(x)$ A-isomorphically onto $AN(h(x)) \subset Y$.*

In view of Definitions 4.3 and 4.4, it is clear that while an A-isomorphism implies a local A-isomorphism, the converse does not hold.

Using the method similar to the establishment of $SC_K^{n,l}$, we can rewrite $SC_A^{n,l}$ in such a way: $SC_A^{n,l}$ is a finite space in $\mathbf{Z}^n$ that is locally A-isomorphic to the Khalimsky line with Khalimsky adjacency, $l \geq 4$ because in $(\mathbf{Z}, T)$ we obtain that for each point $p \in \mathbf{Z}$ $AN(p) = \{p-1, p, p+1\}$. However, $SC_A^{n,l}$ and $SC_K^{n,l}$ have different structures [13]. While each point $c_i \in SC_K^{n,l} := (c_i)_{i\in[0,l-1]_\mathbf{Z}}$ has either $SN(c_i) = \{c_i\}$ or $SN(c_i) = \{c_{i-1(\mathrm{mod}\, l)}, c_i, c_{i+1(\mathrm{mod}\, l)}\}$, each point $x_i \in SC_A^{n,l} := := (x_i)_{i\in[0,l-1]_\mathbf{Z}}$ has $AN(x_i) = \{x_{i-1(\mathrm{mod}\, l)}, x_i, x_{i+1(\mathrm{mod}\, l)}\}$.

Example **4.2.** Consider the spaces B and D in Fig. 1(3). While D is a kind of $SC_A^{2,8}$, B is neither $SC_A^{2,6}$ nor $SC_K^{2,6}$ (see the point b_3 and further, Lemma 3.1).

5. Development of an A-homotopy in KAC_1 and an A-homotopy equivalence. In KAC_1 the paper proposes the notions of an A-homotopy and an A-homotopy equivalence, and classifies spaces in KAC_1 or KAC_2 in terms of an A-homotopy equivalence. Finally, the paper proves that KAC_2 is equivalent to $DTC_2(k)$ for any k-adjacency of $\mathbf{Z}^n$ in (2.1). For a space X in KAC_1, let B be a subset of X in KAC_1. Then (X, B) is called a space pair in KAC_1. Furthermore, if B is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed space in KAC_1. Motivated by the pointed digital homotopy in [2] and the digital relative homotopy in [7], we will establish the notions of an A-homotopy relative to a subset $B \subset X$, an A-homotopy equivalence, which will be helpful to study spaces in KAC_1.

Definition 5.1. *Let (X, B) and Y be a space pair and a space in KAC_1, respectively. Let $f, g\colon X \to Y$ be A-maps. Suppose there exist $m \in \mathbf{N}$ and a function $F\colon X \times [0, m]_\mathbf{Z} \to Y$ such that*

for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;

for all $x \in X$, the induced function $F_x\colon [0, m]_\mathbf{Z} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_\mathbf{Z}$ is an A-map;

for all $t \in [0, m]_\mathbf{Z}$, the induced function $F_t\colon X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is an A-map.

Then we say that F is an A-homotopy between f and g.

Furthermore, for all $t \in [0, m]_\mathbf{Z}$, assume that the induced map F_t on B is a constant which follows the prescribed function from B to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in B$ and for all $t \in [0, m]_\mathbf{Z}$.

Then we call F an A-homotopy relative to B between f and g, and we say that f and g are A-homotopic relative to B in Y, $f \simeq_{A\,\mathrm{rel}\,B} g$ in symbols.

In Definition 5.1, if $B = \{x_0\} \subset X$, then we say that F is a pointed A-homotopy at $\{x_0\}$. In case f and g are pointed A-homotopic in Y, we use the notation $f \simeq_A g$. In addition, if $n_0 = n_1$, then we say that f and g are pointed A-homotopic in Y and we use the notation that $f \simeq_A g$ and $f \in [g]$ which denotes the A-homotopy class of g. If, for some $x_0 \in X$, 1_X is A-homotopic to the constant map in the space x_0 relative to $\{x_0\}$, then we say that (X, x_0) pointed A-contractible.

Motivated by the notion of a digital homotopy equivalence [4], we propose the following definition.

Definition 5.2. *For two spaces $(X, T_X^{n_1})$ and $(Y, T_Y^{n_2})$ in KAC_1, if there is an A-map $h\colon X \to Y$ and an A-map $l : Y \to X$ such that $l \circ h$ is A-homotopic to 1_X and $h \circ l$ is A-homotopic to 1_Y, then the map $h\colon X \to Y$ is called an A-homotopy equivalence and denote it by $X \simeq_{A \cdot h \cdot e} Y$.*

***Example* 5.1.** Consider the spaces X and Y in KAC_1 (see Fig. 3). Using the processes presented by the arrows in Fig. 3, we observe that they are A-homotopy equivalent to each other because the spaces X' and Y' in Fig. 3 are A-isomorphic to each other.

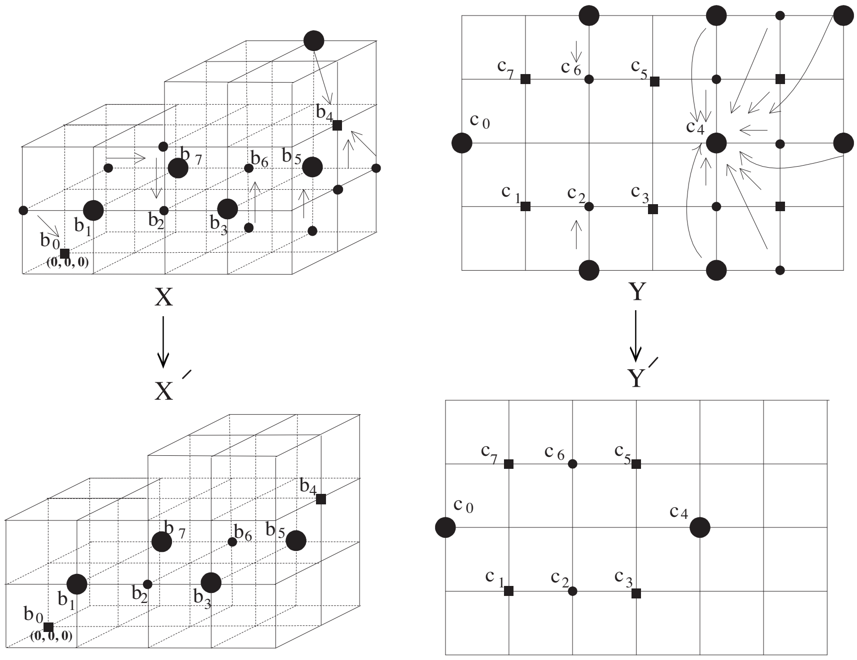

Fig. 3. Explanation of an A-homotopy equivalence.

Let us recall the category KAC_2 whose objects are simple closed A-curves and morphisms are A-maps so that KAC_2 is a subcategory of KAC_1. In view of the property of $SC_A^{n,l}$, we obtain the following theorem.

Theorem 5.1. *In KAC_2, consider two spaces $SC_A^{n_i,l_i}$, $i \in \{1, 2\}$. They are A-homotopy equivalent to each other if and only if $l_1 = l_2$.*

Proof. Firstly, in case all $SC_A^{n_i,l_i}$, $i \in \{1, 2\}$, are A-contractible, we easily see that $l_1 = 4 = l_2$.

Secondly, one of these is A-contractible. Consider $SC_A^{n_1,l_1}$ with $l_1 = 4$ and $SC_A^{n_2,l_2}$ which is not A-contractible. Then l_2 is greater than 4, the proof is completed.

Finally, as the other cases, without loss of generality, assume $l_1 \lneq l_2$ with $l_1 \gneq 4$. Let $f\colon SC_A^{n_1,l_1} \to SC_A^{n_2,l_2}$ be any A-map. From the hypothesis $l_1 \lneq l_2$, it follows that $f(SC_A^{n_1,l_1})$ is a proper and A-connected subset of $SC_A^{n_2,l_2}$. Then $f(SC_A^{n_1,l_1})$ is A-contractible in $SC_A^{n_2,l_2}$. It implies that if $g\colon SC_A^{n_2,l_2} \to SC_A^{n_1,l_1}$ is any A-map, then $g \circ f(SC_A^{n_1,l_1})$ is A-contractible in $SC_A^{n_1,l_1}$. Since $SC_A^{n_2,l_2}$ is not A-contractible, $1_{SC_A^{n_1,l_1}}$ and $g \circ f$ cannot be A-homotopic in $SC_A^{n_1,l_1}$.

Conversely, if $l_1 = l_2$, then $SC_A^{n_1,l_1}$ is A-isomorphic to $SC_A^{n_2,l_2}$. Hence we obtain $SC_A^{n_1,l_1} \simeq_{A \cdot h \cdot e}$ $\simeq_{A \cdot h \cdot e} SC_A^{n_2,l_2}$.

Theorem 5.1 is proved.

6. Existence of the category $DTC_2(k)$ which is equivalent to the given category KAC_2. This section proves that given the category KAC_2 there is the category $DTC_2(k)$ which is equivalent to KAC_2, so that for any $SC_A^{n,l}$ we prove that there is $SC_k^{n,l}$ which is equivalent to $SC_A^{n,l}$, where the k-adjacency is that of the digital connectivity of $\mathbf{Z}^n$ in (2.1). Unlike Remark 3.2(1), we have the following property supporting rotations of the space $SC_A^{n,l}$ under an A-isomorphism.

Theorem 6.1. *Let $f\colon SC_A^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}} \to SC_A^{n,l}$ be the self-map given by $f(x_i) = x_{i+m(\mathrm{mod}\, l)}, m \in \mathbf{N}$. Then f is an A-isomorphism.*

Proof. Let $f\colon SC_A^{n,l} \to SC_A^{n,l}$ be the map given by $f(x_i) = x_{i+m(\mathrm{mod}\, l)}, m \in \mathbf{N}$. Since each point $x_i \in SC_A^{n,l}$ has $AN(x_i) = \{x_{i-1(\mathrm{mod}\, l)}, x_i, x_{i+1(\mathrm{mod}\, l)}\}$, the map f is an A-bijection and its inverse f^{-1} is also an A-map.

Theorem 6.1 is proved.

Motivated by Theorem 6.1, we obtain the following (indeed, a small part of Theorem 6.4 in [13] was missing):

Theorem 6.2 (correcting Theorem 6.4 in [13]). *SC_A^{n,l_1} is A-isomorphic to SC_A^{n,l_2} if and only if $l_1 = l_2$.*

Although two spaces $SC_A^{n,l}$ and $SC_k^{n,l}$ have different features because the former is considered in KAC_2 and the latter is considered in $DTC_2(k)$, they are equivalent to each other.

Theorem 6.3. *For any $SC_A^{n,l} \in Ob(KAC_2)$, there is both $SC_k^{n,l} \in Ob(DTC_2(k))$ and a bijection $h\colon SC_A^{n,l} \to SC_k^{n,l}$ preserving the KA-relation of $SC_A^{n,l}$ into the k-connectivity of $SC_k^{n,l}$ and its inverse $h^{-1}\colon SC_k^{n,l} \to SC_A^{n,l}$ preserving the k-connectivity of $SC_k^{n,l}$ into the KA-relation of $SC_A^{n,l}$, where the k-adjacency is any k-adjacency relation of $\mathbf{Z}^n$ in* (2.1).

Proof. Consider $SC_A^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ and put $SC_k^{n,l} := (d_i)_{i\in[0,l-1]_{\mathbf{Z}}}$. Owing to the properties of $SC_A^{n,l}$ and $SC_k^{n,l}$, each point $c_i \in SC_A^{n,l}$ has an $AN(c_i) = \{c_{i-1(\mathrm{mod}\, l)}, c_i, c_{i+1(\mathrm{mod}\, l)}\}$ and further, each point $d_i \in SC_k^{n,l}$ has $N_k(d_i, 1) = \{d_{i-1(\mathrm{mod}\, l)}, d_i, d_{i+1(\mathrm{mod}\, l)}\} \subset SC_k^{n,l}$. Then, define the map $h\colon SC_A^{n,l} \to SC_k^{n,l}$ given by $h(c_i) = d_{i+m(\mathrm{mod}\, l)}$, $m \in \mathbf{N}$ so that it is a bijection. Besides, if each c_i, $i \in [0, l-1]_{\mathbf{Z}}$, is Khalimsky adjacent to $c_{i+1(\mathrm{mod}\, l)}$, then the images by the map h are also k-adjacent in $SC_k^{n,l}$. This implies that the map h preserves a KA-relation of $SC_A^{n,l}$ into a k-adjacent relation of $SC_k^{n,l}$ like a *homomorphism* in category theory. Similarly, the inverse of the map h, denoted by h^{-1}, preserves a k-adjacent relation of $SC_k^{n,l}$ into a KA-relation of $SC_A^{n,l}$. Besides, the compositions $h \circ h^{-1}$ and $h^{-1} \circ h$ are obviously identities on $SC_k^{n,l}$ and $SC_A^{n,l}$, respectively.

Theorem 6.3 is proved.

Theorem 6.4. *Given the category KAC_2, there is a category $DTC_2(k)$ which is equivalent to KAC_2.*

Proof. (1) Objects: In general, for two categories C and D, an equivalence between C and D requires that the space $X \in Ob(C)$ is equivalent to $Y \in Ob(D)$. Namely, there are equivalent functors $F\colon C \to D$ and $G\colon D \to C$ between C and D, and two natural isomorphisms $G \circ F = 1_F$ and $F \circ G = 1_G$. Thus we obtain that $F|_{Ob(C)}\colon Ob(C) \to Ob(D)$, $G|_{Ob(D)}\colon Ob(D) \to Ob(C)$, and $F|_{Ob(C)}(X) = Y$, $G|_{Ob(D)}(Y) = X$.

To be specific, consider the map $F\colon KAC_2 \to DTC_2(k)$ such that for any $SC_A^{n,l} \in Ob(KAC_2)$ $F(SC_A^{n,l}) = SC_k^{n,l}$ and for any $f \in Mor(KAC_2)$ $F(f) = f$. Thus for any object $X \in KAC_2$ and each point $x \in X := SC_A^{n,l}$, we have $AN(x)$ which is equivalent to $N_k(x, 1) \subset SC_k^{n,l} \in DTC_2(k)$. By Theorem 6.3, we obtain that

$$F|_{Ob(KAC_2)}\colon Ob(KAC_2) \to Ob(DTC_2(k)), \qquad G|_{Ob(DTC_2(k))}\colon Ob(DTC_2(k)) \to Ob(KAC_2),$$

such that

$$F|_{Ob(KAC_2)}(SC_A^{n,l}) = SC_k^{n,l}, \qquad (F|_{Ob(KAC_2)})^{-1} := G|_{Ob(DTC_2(k))}(SC_k^{n,l}) = SC_A^{n,l}.$$

(2) Morphisms: Owing to their own structures of $AN(x)$ and $N_k(x,1)$, an A-map is equivalently considered to be a k-continuous map in $DTC_2(k)$ (see Proposition 3.1 and Definition 4.2).

(3) Since the morphisms in KAC_2 and $DTC_2(k)$ have the transitive property, we consider the following two functors $F\colon KAC_2 \to DTC_2(k)$ satisfying $F(f_1 \circ_1 f_2) = F(f_1) \circ_2 F(f_2)$ and $G\colon DTC_2(k) \to KAC_2$ satisfying $G(g_1 \circ_2 g_2) = G(g_1) \circ_1 G(g_2)$, where "$\circ_1$" and "$\circ_2$" are compositions in the morphisms of KAC_1 and $DTC_2(k)$, respectively. Then we obtain two natural isomorphisms $\epsilon\colon F \circ G \to I_G$ and $\eta\colon I_F \to G \circ F$, where I_F and I_G are identity functors on KAC_2 and $DTC_2(k)$, respectively.

Theorem 6.4 is proved.

***Example* 6.1.** Let us consider $SC_A^{n,l} \in KAC_2$ such as $SC_A^{2,8}$, to be specific, the spaces X, Y, V and W in Fig. 1(2). Then we obtain $SC_k^{n,l} \in DTC_2(k)$ such as $SC_4^{2,8}$ and $SC_8^{2,8}$ such that both X and Y (resp. V and W) are equivalent to $SC_8^{2,8}$ (resp. $SC_4^{2,8}$).

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