UDC 517.54

E. Deniz (Kafkas Univ., Kars, Turkey),
H. Orhan (Ataturk Univ., Erzurum, Turkey),
J. Sokół (Rzeszów Univ. Technology, Poland)

CLASSES OF ANALYTIC FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR RELATED TO CONIC DOMAINS*

КЛАСИ АНАЛІТИЧНИХ ФУНКЦІЙ, ЩО ВИЗНАЧЕНІ ДИФЕРЕНЦІАЛЬНИМ ОПЕРАТОРОМ, ВІДНЕСЕНИМ ДО КОНІЧНИХ ОБЛАСТЕЙ

Let $\mathcal{A}$ be the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disc Δ. We use a generalized linear operator closely related to the multiplier transformation to investigate certain subclasses of $\mathcal{A}$ which map Δ onto the conic domains. Using the principle of the differential subordination and the techniques of convolution, several properties of these classes including some inclusion relations, convolution and coefficient bounds are studied. In particular, we derive many known and new results as special cases.

Нехай $\mathcal{A}$ — клас функцій $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, аналітичних у відкритому одиничному крузі Δ. До вивчення деяких підкласів $\mathcal{A}$, що відображають Δ на конічні області, застосовано узагальнений лінійний оператор, тісно пов'язаний з перетворенням множення. За допомогою принципу диференціального підпорядкування та техніки згорток вивчено деякі властивості цих класів, що включають деякі співвідношення включення та згорток, а також оцінки для коефіцієнтів. Наприклад, низку відомих та нових результатів отримано як частинні випадки.

1. Introduction. Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in Δ. If f and g are analytic in Δ, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, is analytic in Δ (with $w(0) = 0$ and $|w(z)| < 1$ in Δ) such that $f(z) = g(w(z))$, $z \in \Delta$. In particular, if the function $g(z)$ is univalent in Δ, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

A function $f \in \mathcal{A}$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $\beta - \mathcal{UCV}(\gamma)$ [5] if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \quad z \in \Delta, \tag{1.2}$$

where $\beta \geq 0$, $-1 \leq \gamma < 1$, $\beta + \gamma \geq 0$ and it is said to be in the corresponding class denoted by $\beta - \mathcal{SP}(\gamma)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \quad z \in \Delta, \tag{1.3}$$

where $\beta \geq 0$, $-1 \leq \gamma < 1$ and $\beta + \gamma \geq 0$.

* This paper was supported by the Commission for the Scientific Research Projects of Kafkas Univertsity (Project number 2012-FEF-30).

These classes generalize various other classes which are worthy to mention here. For example the class $\beta-\mathcal{UCV}(0)=\beta-\mathcal{UCV}$ is the known class of β-*uniformly convex* functions [11]. Using the Alexander type relation, we can obtain the class $\beta-\mathcal{SP}(\gamma)$ in the following way:

$$f\in\beta-\mathcal{SP}(\gamma)\Leftrightarrow\frac{1}{z}\int\limits_0^z f(t)dt\in\beta-\mathcal{UCV}(\gamma)\ \text{ or }\ f\in\beta-\mathcal{UCV}(\gamma)\Leftrightarrow zf'\in\beta-\mathcal{SP}(\gamma).$$

The class $1-\mathcal{UCV}(0)=\mathcal{UCV}$ of uniformly convex functions was defined by Goodman [9] while the class $1-\mathcal{SP}(0)=\mathcal{SP}$ was considered by Rønning [26].

Geometric interpretation. It is known that $f\in\beta-\mathcal{UCV}(\gamma)$ and $g\in\beta-\mathcal{SP}(\gamma)$ if and only if the quantities $1+zf''(z)/f'(z)$ and $zg'(z)/g(z)$, respectively, takes its all the values in the conic domain $\mathcal{R}_{\beta,\gamma}$ which is included in the right half plane $\Re(w)>(\beta+\gamma)/(1+\beta)$ and is given by

$$\mathcal{R}_{\beta,\gamma}:=\left\{w=u+iv\in\mathbb{C}\colon u>\beta\sqrt{(u-1)^2+v^2}+\gamma,\ \beta\geq 0 \text{ and } \gamma\in[-1,1)\right\}. \tag{1.4}$$

Let $\widehat{P}_{\beta,\gamma}=1+P_1z+\ldots$ denote the function which maps the unit disk conformally onto the domain $\mathcal{R}_{\beta,\gamma}$ given in (1.4). Let $\partial\mathcal{R}_{\beta,\gamma}$ be a curve defined by the equality

$$\partial\mathcal{R}_{\beta,\gamma}:=\left\{w=u+iv\in\ \mathbb{C}\colon u^2=\left(\beta\sqrt{(u-1)^2+v^2}+\gamma\right)^2,\ \beta\geq 0 \text{ and } \gamma\in[-1,1)\right\}. \tag{1.5}$$

After some calculations we can see that for $\beta\neq 0$, $\partial\mathcal{R}_{\beta,\gamma}$ represents conic curves symmetric about the real axis. Thus $\mathcal{R}_{\beta,\gamma}$ is an elliptic domain for $\beta>1$, a parabolic domain for $\beta=1$, a hyperbolic domain for $0<\beta<1$ and the right half plane $\Re(w)>\gamma$, for $\beta=0$.

The functions $\widehat{P}_{\beta,\gamma}$ play the role of extremal functions in the classes $\mathcal{P}(\widehat{P}_{\beta,\gamma})$ and were given in [1] (also see for Taylor series expansion of $\widehat{P}_{\beta,\gamma}$, [14, 16, 26]) as follows:

$$\widehat{P}_{\beta,\gamma}(z)=\begin{cases}\dfrac{1+(1-2\gamma)z}{1-z}, & \beta=0,\\ 1+\dfrac{2(1-\gamma)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta=1,\\ \dfrac{1-\gamma}{1-\beta^2}\cos\left\{\dfrac{2}{\pi}(\arccos\beta)i\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\dfrac{\beta^2-\gamma}{1-\beta^2}, & 0<\beta<1,\\ \dfrac{1-\gamma}{\beta^2-1}\sin\left\{\dfrac{\pi}{2\mathcal{K}(t)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}}\dfrac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right\}+\dfrac{\beta^2-\gamma}{\beta^2-1}, & \beta>1,\end{cases} \tag{1.6}$$

where $u(z)=\dfrac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t\in(0,1)$, $z\in\Delta$ and

$$\mathcal{K}(t)=\int\limits_0^1\frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}} \tag{1.7}$$

is called Legendre's complete elliptic integral of the first kind and $t\in(0,1)$ is such that $\beta=$ $=\cosh\pi\mathcal{K}'(t)/4\ \mathcal{K}(t)$.

For functions $f,g\in\mathcal{A}$, given by $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ and $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$, the Hadamard product (or *convolution*) of f and g is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \Delta.$$

Note that $f * g \in \mathcal{A}$. Let $a \in \mathbb{R}$, $c \in \mathbb{R}$, $c \neq 0, -1, -2, \ldots$ and let

$$\varphi(a, c; z) := z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad z \in \Delta, \tag{1.8}$$

where $(\kappa)_n$ is the Pochhammer symbol (or the *shifted factorial*) in terms of the gamma function, given by

$$(\kappa)_n := \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = \begin{cases} 1, & n = 0, \quad \kappa \in \mathbb{C} \setminus \{0\}, \\ \kappa(\kappa + 1) \ldots (\kappa + n - 1), & n \in \mathbb{N} = \{1, 2, \ldots\}, \quad \kappa \in \mathbb{C}. \end{cases}$$

The Carlson – Shaffer operator [6] $\mathcal{L}(a, c)$ is defined in terms of Hadamard product by

$$\mathcal{L}(a, c) f(z) = \varphi(a, c; z) * f(z), \quad z \in \Delta, \quad f \in \mathcal{A}. \tag{1.9}$$

Note that $\mathcal{L}(a, a)$ is the identity operator and $\mathcal{L}(a, c) = \mathcal{L}(a, b)\mathcal{L}(b, c)$, $(b, c \neq 0, -1, -2, \ldots)$. We also need the following definitions of a *fractional derivative*.

Definition 1.1 [21]. *Let the function f be analytic in a simply-connected region of the z-plane containing the origin. The fractional derivative of f of order α is defined by*

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{\alpha}} d\zeta, \quad 0 \leq \alpha < 1,$$

where the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Using D_z^{α} Owa and Srivastava [22] introduced the operator $\Omega^{\alpha} \colon \mathcal{A} \to \mathcal{A}$, $\alpha \in [0, 1)$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^{\alpha} f(z) = \Gamma(2 - \alpha) z^{\alpha} D_z^{\alpha} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k + 1)\Gamma(2 - \alpha)}{\Gamma(k + 1 - \alpha)} a_k z^k =$$

$$= \varphi(2, 2 - \alpha; z) * f(z) = \mathcal{L}(2, 2 - \alpha) f(z). \tag{1.10}$$

Note that $\Omega_z^0 f(z) = f(z)$.

In [20], Orhan, Deniz and Răducanu introduced the *generalized linear multiplier fractional differential operator* $D_{\lambda,\mu}^{n,\alpha} f \colon \mathcal{A} \to \mathcal{A}$ of functions $f \in \mathcal{A}$ defined by

$$D_{\lambda,\mu}^{0,\alpha} f(z) = f(z),$$

$$D_{\lambda,\mu}^{1,\alpha} f(z) = D_{\lambda,\mu}^{\alpha} f(z) = \lambda\mu z^2 [\Omega^{\alpha} f(z)]'' + (\lambda - \mu) z [\Omega^{\alpha} f(z)]' + (1 - \lambda + \mu)[\Omega^{\alpha} f(z)],$$

$$D_{\lambda,\mu}^{2,\alpha} f(z) = D_{\lambda,\mu}^{\alpha} \left(D_{\lambda,\mu}^{1,\alpha} f(z) \right), \tag{1.11}$$

$$D_{\lambda,\mu}^{n,\alpha} f(z) = D_{\lambda,\mu}^{\alpha} \left(D_{\lambda,\mu}^{n-1,\alpha} f(z) \right),$$

where $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}$.

If f is given by (1.1), then from the definitions of the $D^{n,\alpha}_{\lambda,\mu}$ and Ω^{α} it is easy to see that

$$D^{n,\alpha}_{\lambda,\mu}f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\lambda,\mu,\alpha) a_k z^k, \tag{1.12}$$

where

$$\Psi_{k,n}(\lambda,\mu,\alpha) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left(1+(\lambda\mu k+\lambda-\mu)(k-1)\right)\right]^n. \tag{1.13}$$

From (1.10) and (1.13), $D^{n,\alpha}_{\lambda,\mu}f(z)$ can be written, in terms of convolution as

$$D^{n,\alpha}_{\lambda,\mu}f(z) = \left(\varphi(2,2-\alpha;z) * g_{\lambda,\mu}(z)\right)^{[n]} * f(z), \tag{1.14}$$

where $\underbrace{f * \ldots * f}_{n \text{ times}} = f^{[n]}$ and

$$g_{\lambda,\mu}(z) = \frac{z^3(1-\lambda+\mu) + z^2(\lambda-\mu+2\lambda\mu-2) + z}{(1-z)^3} = z + \sum_{k=2}^{\infty}\left(1+(\lambda\mu k+\lambda-\mu)(k-1)\right)z^k.$$

It should be remarked that the operator $D^{n,\alpha}_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$ we have the following:

(i) $D^{n,0}_{1,0}f(z) \equiv D^n f(z)$, the operator investigated by Salagean [32];

(ii) $D^{n,0}_{\lambda,0}f(z) \equiv D^n_{\lambda}f(z)$, the operator considered by Al-Oboudi [3];

(iii) $D^{1,\alpha}_{0,0}f(z) \equiv \Omega^{\alpha}f(z)$, the fractional derivative operator studied by Owa and Srivastava [22];

(iv) $D^{n,0}_{\lambda,\mu}f(z) \equiv D^n_{\lambda,\mu}f(z)$, the operator investigated by Răducanu and Orhan [28] (also see Deniz and Orhan [7]);

(v) $D^{n,\alpha}_{\lambda,0}f(z) \equiv D^{n,\alpha}_{\lambda}f(z)$, the operator considered by Al-Oboudi and Al-Amoudi [4];

(vi) $D^{1,\alpha}_{\lambda,0}f(z) \equiv D^{\alpha}_{\lambda}f(z)$, the operator studied by Noor, Arif and Ul-Haq [19].

Using the operator $D^{n,\alpha}_{\lambda,\mu}$, authors defined in [20] the classes $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. For a unified class of k-uniformly convex functions defined by the Dziok – Srivastava linear operator [23].

Definition 1.2. *For $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$, $\beta \geq 0$, $-1 \leq \gamma < 1$ and $\beta+\gamma \geq 0$ a function $f \in \mathcal{A}$ is said to be in the class $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$ if it satisfies the following condition*:

$$\Re\left\{1+\frac{z\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)''}{\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)'}\right\} > \beta\left|\frac{z\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)''}{\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)'}\right| + \gamma, \quad z \in \Delta. \tag{1.15}$$

Definition 1.3. *For $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$, $\beta \geq 0$, $-1 \leq \gamma < 1$ and $\beta+\gamma \geq 0$ a function $f \in \mathcal{A}$ is said to be in the class $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ if it satisfies the following condition*:

$$\Re\left\{\frac{z\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)'}{D^{n,\alpha}_{\lambda,\mu}f(z)}\right\} > \beta\left|\frac{z\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)'}{D^{n,\alpha}_{\lambda,\mu}f(z)} - 1\right| + \gamma, \quad z \in \Delta. \tag{1.16}$$

Note that $f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ if and only if $D^{n,\alpha}_{\lambda,\mu} f \in \beta - SP(\gamma)$. Using the Alexander type relation, it is clear that

$$f \in \beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma) \Leftrightarrow zf' \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma). \tag{1.17}$$

Geometric interpretation. From (1.15) and (1.16), $f \in \beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $g \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ if and only if $p(z) = 1 + zD^{n,\alpha}_{\lambda,\mu} f(z))''/(D^{n,\alpha}_{\lambda,\mu} f(z))'$ and $q(z) = z(D^{n,\alpha}_{\lambda,\mu} g(z))'/D^{n,\alpha}_{\lambda,\mu} g(z)$ take all its the values in the domain $\mathcal{R}_{\beta,\gamma}$ given in (1.4) which is included in the half plane $\Re w > (\beta+\gamma)/(1+\beta)$. Thus we may rewrite the conditions (1.15) and (1.16) in the form

$$p \prec \widehat{P}_{\beta,\gamma}, \; q \prec \widehat{P}_{\beta,\gamma}, \quad z \in \Delta, \tag{1.18}$$

where the function $\widehat{P}_{\beta,\gamma}$ given by (1.6).

By virtue of (1.15) and (1.16) and the properties of domain $\mathcal{R}_{\beta,\gamma}$, we have, respectively

$$\Re \left\{ 1 + \frac{z\left(D^{n,\alpha}_{\lambda,\mu} f(z)\right)''}{\left(D^{n,\alpha}_{\lambda,\mu} f(z)\right)'} \right\} > \frac{\beta+\gamma}{1+\beta} > 0, \quad z \in \Delta, \tag{1.19}$$

and

$$\Re \left\{ \frac{z\left(D^{n,\alpha}_{\lambda,\mu} f(z)\right)'}{D^{n,\alpha}_{\lambda,\mu} f(z)} \right\} > \frac{\beta+\gamma}{1+\beta} > 0, \quad z \in \Delta, \tag{1.20}$$

which means that

$$f \in \beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma) \Rightarrow D^{n,\alpha}_{\lambda,\mu} f \in \mathcal{CV}\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq \mathcal{CV} \tag{1.21}$$

and

$$f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \Rightarrow D^{n,\alpha}_{\lambda,\mu} f \in \mathcal{ST}\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq \mathcal{ST}, \tag{1.22}$$

where $\mathcal{CV}(\gamma)$, $\mathcal{ST}(\gamma)$, $\mathcal{CV}$, $\mathcal{ST}$ denote the well-known classes of γ-convex, γ-starlike, convex and starlike functions, respectively.

We note that by specializing the parameters n, α, λ, μ, β and γ, the class $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ reduces to several well-known subclasses of analytic functions. These subclasses are:

(a) $0 - \mathcal{SP}^{1,0}_{0,0}(0) \equiv 0 - \mathcal{SP}^{0,\alpha}_{\lambda,\mu}(0) \equiv \mathcal{ST}$ and $0 - \mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(0) \equiv 0 - \mathcal{UCV}^{1,0}_{0,0}(0) \equiv 0 - \mathcal{SP}^{1,0}_{1,0}(0) \equiv$ $\equiv \mathcal{CV}$ (see [8, p. 40–43]),

(b) $0 - \mathcal{SP}^{1,0}_{0,0}(\gamma) \equiv 0 - \mathcal{SP}^{0,\alpha}_{\lambda,\mu}(\gamma) \equiv \mathcal{ST}(\gamma)$ and $0 - \mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(\gamma) \equiv 0 - \mathcal{UCV}^{1,0}_{0,0}(\gamma) \equiv 0 -$ $- \mathcal{SP}^{1,0}_{1,0}(\gamma) \equiv \mathcal{CV}(\gamma)$ (see [24]),

(c) $1 - \mathcal{SP}^{0,\alpha}_{\lambda,\mu}(0) \equiv 1 - \mathcal{SP}^{1,0}_{0,0}(0) \equiv \mathcal{SP}$ (see [27]),

(d) $1 - \mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(0) \equiv 1 - \mathcal{UCV}^{1,0}_{0,0}(0) \equiv 1 - \mathcal{SP}^{1,0}_{1,0}(0) \equiv \mathcal{UCV}$ (see [9, 16]),

(e) $\beta - \mathcal{SP}^{0,\alpha}_{\lambda,\mu}(0) \equiv \beta - \mathcal{SP}^{1,0}_{0,0}(0) \equiv \beta - \mathcal{SP}$ (see [12]),

(f) $\beta - \mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(0) \equiv \beta - \mathcal{UCV}^{1,0}_{0,0}(0) \equiv \beta - \mathcal{SP}^{1,0}_{1,0}(0) \equiv \beta - \mathcal{UCV}$ (see [11]),

(g) $1-\mathcal{SP}^{0,\alpha}_{\lambda,\mu}(2\rho-1)\equiv 1-\mathcal{SP}^{1,0}_{0,0}(2\rho-1)\equiv PS^*(\rho)$ $(0\le\rho<1)$ and $1-\mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(2\rho-1)\equiv$ $\equiv 1-\mathcal{UCV}^{1,0}_{0,0}(2\rho-1)\equiv 1-\mathcal{SP}^{1,0}_{1,0}(2\rho-1)\equiv\mathcal{UCV}(\rho)$ (see [2]),

(h) $1-\mathcal{SP}^{0,\alpha}_{\lambda,\mu}(\gamma)\equiv 1-\mathcal{SP}^{1,0}_{0,0}(\gamma)\equiv\mathcal{SP}(\gamma)$ and $1-\mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(\gamma)\equiv\ 1-\mathcal{UCV}^{1,0}_{0,0}(\gamma)\equiv 1-$ $-\mathcal{SP}^{1,0}_{1,0}(\gamma)\equiv\mathcal{UCV}(\gamma)$ (see [26]),

(i) $\beta-\mathcal{SP}^{0,\alpha}_{\lambda,\mu}(\gamma)\equiv\beta-\mathcal{SP}^{1,0}_{0,0}(\gamma)\equiv\beta-\mathcal{SP}(\gamma)$ and $\beta-\mathcal{UCV}^{0,\alpha}_{\lambda,\mu}(\gamma)\equiv\beta-\mathcal{UCV}^{1,0}_{0,0}(\gamma)\equiv$ $\equiv\beta-\mathcal{SP}^{1,0}_{1,0}(\gamma)\equiv\beta-\mathcal{UCV}(\gamma)$ (see [5]),

(j) $0-\mathcal{SP}^{n,0}_{1,0}(\gamma)\equiv\mathcal{ST}^n(\gamma)$ (see [32]),

(k) $\beta-\mathcal{SP}^{n,0}_{1,0}(0)\equiv\beta-\mathcal{SP}^n$ (see [14, 17]),

(l) $0-\mathcal{SP}^{1,\alpha}_{0,0}(\gamma)\equiv\mathcal{ST}_\alpha(\gamma)$ (see [33]),

(m) $1-\mathcal{SP}^{1,\alpha}_{0,0}(0)\equiv\mathcal{SP}_\alpha$ (see [34]),

(n) $\beta-\mathcal{SP}^{1,\alpha}_{0,0}(0)\equiv\beta-\mathcal{SP}_\alpha$ (see [18]),

(o) $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,0}(\gamma)\equiv\mathcal{SP}^n_{\alpha,\lambda}(\beta,\gamma)$ and $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,0}(\gamma)\equiv\mathcal{UCV}^n_{\alpha,\lambda}(\beta,\gamma)$ (see [4]).

For special values of parameters $n,\alpha,\lambda,\mu,\beta$ and γ, from the general class $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and the class $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$, the following new classes can be obtained which are open questions:

$$\beta-\mathcal{SP}^{n,0}_{\lambda,\mu}(\gamma)\equiv\beta-\mathcal{SP}^{n}_{\lambda,\mu}(\gamma)\quad\text{and}\quad\beta-\mathcal{UCV}^{n,0}_{\lambda,\mu}(\gamma)\equiv\beta-\mathcal{UCV}^{n}_{\lambda,\mu}(\gamma),$$

$$0-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)\equiv\mathcal{ST}^{n,\alpha}_{\lambda,\mu}(\gamma)\quad\text{and}\quad 0-\mathcal{UCV}y^{n,\alpha}_{\lambda,\mu}(\gamma)\equiv\mathcal{CV}^{n,\alpha}_{\lambda,\mu}(\gamma),$$

$$1-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(0)\equiv\mathcal{SP}^{n,\alpha}_{\lambda,\mu}\quad\text{and}\quad 1-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(0)\equiv\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}.$$

By (1.19) and (1.20), respectively, we note that $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)\subseteq\mathcal{CV}^{n,\alpha}_{\lambda,\mu}\left(\dfrac{\beta+\gamma}{1+\beta}\right)$ and $\beta-$ $-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)\subseteq\mathcal{ST}^{n,\alpha}_{\lambda,\mu}\left(\dfrac{\beta+\gamma}{1+\beta}\right)$.

In the present paper, basic properties of the classes $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ are studied, such as inclusion relations and coefficient bounds. Some interesting consequences of the main results and their relevance to known results are also pointed out.

2. Inclusion relations. In this section, we are going to give several inclusion relationships for the classes $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$, which are associated with the general linear multiplier fractional differential operator $D^{n,\alpha}_{\lambda,\mu}$. To establish our main results, we shall require the following lemmas.

Lemma 2.1 [30]**.** *Let f and g be starlike of order $1/2$. Then so is $f*g$.*

Lemma 2.2 [29, p. 54]**.** *If $f\in\mathcal{CV}$, $g\in\mathcal{ST}$ or $f,g\in\mathcal{ST}(1/2)$, then for each function h analytic in the unit disc Δ we have*

$$\frac{(f*hg)(\Delta)}{(f*g)(\Delta)}\subseteq\overline{\mathrm{co}}\,h(\Delta),$$

where $\overline{\mathrm{co}}\,h(\Delta)$ denotes the closed convex hull of $h(\Delta)$.

Lemma 2.3 [29, p. 60–61]**.** *Suppose that $0<b\le c$. If $c\ge 2$ or $b+c\ge 3$, then*

$$\varphi(b,c;z)=\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k}z^{k+1},\quad z\in\Delta,$$

belongs to the class $\mathcal{CV}$ of convex functions.

Lemma 2.4. *Let $\Omega^\alpha f$ be in the class $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then f is in the class $\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$.*

Proof. Let $\Omega^\alpha f\in\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then from (1.22), $D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f\in\mathcal{ST}$. Using (1.10) and (1.14), we can write $D^{n,\alpha}_{\lambda,\mu}f$ in terms of $D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f$ as follows:

$$D^{n,\alpha}_{\lambda,\mu}f(z)=\varphi(2-\alpha,2;z)*D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f(z).$$

Moreover, $\varphi(2-\alpha,2;z)\in\mathcal{CV}$ by Lemma 2.3 and so $D^{n,\alpha}_{\lambda,\mu}f$ is a starlike function as a convolution of convex and starlike functions (see [29, p. 54]). So $z/D^{n,\alpha}_{\lambda,\mu}f(z)\neq 0$ for $z\in\Delta$ and $z(D^{n,\alpha}_{\lambda,\mu}f(z))'/D^{n,\alpha}_{\lambda,\mu}f(z)$ has no poles in Δ.

Furthermore, by convolution properties, we get

$$z(D^{n,\alpha}_{\lambda,\mu}f(z))'=\varphi(2-\alpha,2;z)*z(D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f(z))'.$$

Since $\varphi(2-\alpha,2;z)\in\mathcal{CV}$ and $D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f\in\mathcal{ST}$, using Lemma 2.2 we have

$$\frac{z\left(D^{n,\alpha}_{\lambda,\mu}f(z)\right)'}{D^{n,\alpha}_{\lambda,\mu}f(z)}=\frac{\varphi(2-\alpha,2;z)*\left[z\left(D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f(z)\right)'/\left(D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f(z)\right)\right]D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f(z)}{\varphi(2-\alpha,2;z)*D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f(z)}\in$$

$$\in\overline{\mathrm{co}}\left(\frac{z\left(D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f\right)'}{\left(D^{n,\alpha}_{\lambda,\mu}\Omega^\alpha f\right)}(\Delta)\right)\subseteq\mathrm{cl}\,\mathcal{R}_{\beta,\gamma}.$$

Therefore, $f\in\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Lemma 2.4 is proved.

***Corollary* 2.1.** *Let $\Omega^\alpha f$ be in the class $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then f is in the class $\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$.*

Proof. By virtue of (1.17) and Lemma 2.4, we obtain

$$\Omega^\alpha f\in\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)\Leftrightarrow z(\Omega^\alpha f)'\in\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)\Leftrightarrow$$

$$\Leftrightarrow\Omega^\alpha zf'\in\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)\Rightarrow zf'\in\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)\Leftrightarrow f\in\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma).$$

Therefore, $f\in\beta-\mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Corollary 2.1 is proved.

Lemma 2.5. *Suppose that $\beta+2\gamma\geq 1$. If $f\in\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$, then $D^{n,\alpha}_{\lambda,\mu}f\in\mathcal{ST}\,(1/2)$.*

Proof. The result follows immediately from (1.20) whenever $(\beta+\gamma)/(1+\beta)\geq 1/2$.

Theorem 2.1. *If*

$$\left[0<\lambda\leq\frac{1+\sqrt{5}}{2}\ \ \textit{and}\ \ 0<\mu\ \ \textit{and}\ \ \lambda-1\leq\mu\leq\frac{\lambda}{1+\lambda}\right]\ \ \textit{or}\ \ [0=\lambda=\mu]\ \ \textit{or}\ \ [0=\mu<\lambda],$$

then

$$\beta-\mathcal{SP}^{n+1,\alpha}_{\lambda,\mu}(\gamma)\subseteq\beta-\mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma).\tag{2.1}$$

Proof. In the proof we will use several convolution results, see for example [29]. In this proof, for simplicity let us denote $\varphi(2,2-\alpha;z)=\varphi$, $g_{\lambda,\mu}=g$. From (1.18) and (1.14) it is easy to see that

$$f \in \beta - \mathcal{SP}_{\lambda,\mu}^{n,\alpha} \Leftrightarrow \frac{z\left[D_{\lambda,\mu}^{n,\alpha} f(z)\right]'}{D_{\lambda,\mu}^{n,\alpha} f(z)} \prec \widehat{P}_{\beta,\gamma}(z) \Leftrightarrow \frac{z\left[(\varphi * g)^{[-1]} * D_{\lambda,\mu}^{n+1,\alpha} f(z)\right]'}{(\varphi * g)^{[-1]} * D_{\lambda,\mu}^{n+1,\alpha} f(z)} \prec \widehat{P}_{\beta,\gamma}(z) \Leftrightarrow$$

$$\Leftrightarrow \frac{(\varphi * g)^{[-1]} * z\left[D_{\lambda,\mu}^{n+1,\alpha} f(z)\right]'}{(\varphi * g)^{[-1]} * D_{\lambda,\mu}^{n+1,\alpha} f(z)} \prec \widehat{P}_{\beta,\gamma}(z), \tag{2.2}$$

where the convex function $\widehat{P}_{\beta,\gamma}(z) = 1 + P_1 z + \ldots$ is given by (1.6) and $(\varphi * g)^{[-1]} = \varphi^{[-1]} * g^{[-1]}$ denotes the convolution inverse with respect to $\varphi * g$. Recall that $f^{[-1]}$ is the convolution inverse to f if $f * f^{[-1]} = z/(1-z)$. Now we will to show that $g^{[-1]}$ is convex. If $\lambda\mu > 0$, then we have

$$(\varphi * g)^{[-1]}(z) = \varphi^{[-1]}(z) * g^{[-1]}(z) = \varphi^{[-1]}(z) * \left[\sum_{k=1}^{\infty} \frac{1}{1 + (k-1)(\lambda\mu k + \lambda - \mu)} z^k\right] =$$

$$= \varphi^{[-1]}(z) * \left[\frac{1}{(1+k_1)\sqrt{\mu\lambda}} \sum_{k=1}^{\infty} \frac{1+k_1}{k+k_1} z^k\right] * \left[\frac{1}{(1+k_2)\sqrt{\mu\lambda}} \sum_{k=1}^{\infty} \frac{1+k_2}{k+k_2} z^k\right] =$$

$$= \varphi^{[-1]}(z) * g_1^{[-1]}(z) * g_2^{[-1]}(z), \tag{2.3}$$

where

$$k_i = \frac{\lambda - \mu - \lambda\mu \pm \sqrt{(\lambda - \mu - \lambda\mu)^2 - 4\lambda\mu(1 + \mu - \lambda)}}{2\lambda\mu}, \quad i = 1, 2.$$

Observe that k_1, k_2 have a positive real under assumptions of Theorem 2.1. For $\Re(x) \geq 0$ or $x = 0$ the function

$$\widetilde{h}(x; z) = \sum_{k=1}^{\infty} \frac{(1+x)}{(k+x)} z^k, \quad z \in \Delta, \tag{2.4}$$

is convex univalent [30]. So $g_1^{[-1]}$ and $g_2^{[-1]}$ in (2.3) are convex when $\lambda\mu > 0$. Otherwise, if $\mu = \lambda = 0$ or if $0 = \mu < \lambda$, then it is easy to see that $g^{[-1]}$ has the form of the type (2.4) so it is convex too. In the famous paper [31] it was proved the Polya – Schoenberg conjecture that the class of convex univalent functions is preserved under convolution. Under our assumptions on α, λ, μ the function $g^{[-1]} = g_1^{[-1]} * g_2^{[-1]}$ is convex as the convolution of two convex functions and by Lemma 2.3

$$\varphi^{[-1]}(z) = (\varphi(2, 2-\alpha; z))^{[-1]} = z + \sum_{k=1}^{\infty} \frac{(2-\alpha)_k}{(2)_k} z^k$$

is a convex function too. Therefore $(\varphi * g)^{[-1]}$ is the convex function. Let $f \in \beta - \mathcal{SP}_{\lambda,\mu}^{n+1,\alpha}(\gamma)$. By Definition 1.3 we have

$$\frac{z(D_{\lambda,\mu}^{n+1,\alpha} f(z))'}{D_{\lambda,\mu}^{n+1,\alpha} f(z)} = \widehat{P}_{\beta,\gamma}(\omega(z)), \tag{2.5}$$

where ω is an analytic function with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$. From (2.5) we have that $D_{\lambda,\mu}^{n+1,\alpha} f$ is a starlike function. Therefore, by (2.3) and by (2.5) we obtain from (2.2)

$$f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \Leftrightarrow \frac{(\varphi * g)^{[-1]} * \left(D^{n+1,\alpha}_{\lambda,\mu} f(z)\right) \widehat{P}_{\beta,\gamma}(\omega(z))}{(\varphi * g)^{[-1]} * D^{n+1,\alpha}_{\lambda,\mu} f(z)} \prec \widehat{P}_{\beta,\gamma}(z). \tag{2.6}$$

The function $\widehat{P}_{\beta,\gamma}$ is univalent so the subordination principle and Lemma 2.2 show that the last subordination is true and so $f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Theorem 2.1 is proved.

***Corollary* 2.2.** *Let* $\alpha \in [0,1)$ *and* $n \in \mathbb{N}$. *Under the conditions stated in Theorem* 2.1 *we have* $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{SP}(\gamma)$.

Proof. Suppose that $f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then as in the proof of Theorem 2.1 we obtain

$$\frac{zf'(z)}{f(z)} = \frac{\left[(\varphi * g)^{[-1]}(z)\right]^{[n]} * \left(D^{n,\alpha}_{\lambda,\mu} f(z)\right) \widehat{P}_{\beta,\gamma}(\omega(z))}{\left[(\varphi * g)^{[-1]}(z)\right]^{[n]} * D^{n,\alpha}_{\lambda,\mu} f(z)} \prec \widehat{P}_{\beta,\gamma}(z).$$

Therefore, $f \in \beta - \mathcal{SP}(\gamma)$.

Corollary 2.2 is proved.

By (1.17) and Theorem 2.1, we deduce the next consequences.

***Corollary* 2.3.** *Let* $\alpha \in [0,1)$ *and* $n \in \mathbb{N}$. *Under the assumptions in Theorem* 2.1 *we have* $\beta - \mathcal{UCV}^{n+1,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Proof. From (1.17) and Theorem 2.1, we get

$$f \in \beta - \mathcal{UCV}^{n+1,\alpha}_{\lambda,\mu}(\gamma) \Leftrightarrow zf' \in \beta - \mathcal{SP}^{n+1,\alpha}_{\lambda,\mu}(\gamma) \Leftrightarrow zf' \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \Leftrightarrow f \in \beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma).$$

Thus $f \in \beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Corollary 2.3 is proved.

***Corollary* 2.4.** *Let* $\alpha \in [0,1)$ *and* $n \in \mathbb{N}$. *Under the conditions stated in Theorem* 2.1 *we have* $\beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{UCV}(\gamma)$.

***Remark* 2.1.** (1) Taking $\gamma = \alpha = \mu = 0$ and $\lambda = 1$ in Theorem 2.1, we get the result by Kanas and Yaguchi [13].

(2) Taking $\mu = 0$ in Theorem 2.1, we get a result due to Al-Oboudi and Al-Amoudi [4].

***Remark* 2.2.** For special values of parameters n, α, β and γ and for λ, μ satisfying the assumptions of Theorem 2.1 we obtain the following new results:

(1) $\beta - \mathcal{SP}^{n+1}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{SP}^{n}_{\lambda,\mu}(\gamma)$ and $\beta - \mathcal{UCV}^{n+1}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{UCV}^{n}_{\lambda,\mu}(\gamma)$,

(2) $\mathcal{ST}^{n+1,\alpha}_{\lambda,\mu}(\gamma) \subseteq \mathcal{ST}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\mathcal{CV}^{n+1,\alpha}_{\lambda,\mu}(\gamma) \subseteq \mathcal{CV}^{n,\alpha}_{\lambda,\mu}(\gamma)$,

(3) $\mathcal{SP}^{n+1,\alpha}_{\lambda,\mu} \subseteq \mathcal{SP}^{n,\alpha}_{\lambda,\mu}$ and $\mathcal{UCV}^{n+1,\alpha}_{\lambda,\mu} \subseteq \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}$.

Theorem 2.2. *Suppose that* $0 \le \delta \le \alpha < 1$. *Then*

$$\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{SP}^{n,\delta}_{\lambda,\mu}(\gamma),$$

whenever $\beta + 2\gamma \ge 1$.

Proof. Let $f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then by (1.14) and convolution properties, we get

$$D^{n,\delta}_{\lambda,\mu} f(z) = [\varphi(2, 2-\delta; z) * g_{\lambda,\mu}(z)]^{[n]} * f(z) = [\varphi(2-\alpha, 2-\delta; z)]^{[n]} * D^{n,\alpha}_{\lambda,\mu} f(z)$$

and

$$z(D^{n,\delta}_{\lambda,\mu} f(z))' = [\varphi(2-\alpha, 2-\delta; z)]^{[n]} * z(D^{n,\alpha}_{\lambda,\mu} f(z))'.$$

Also, it is known [15] that $\varphi(2-\alpha, 2-\delta; z) \in \mathcal{ST}\,(1/2)$. So by applying Lemma 2.1 we have $[\varphi(2-\alpha, 2-\delta; z)]^{[n]} \in \mathcal{ST}\,(1/2)$. For $\beta + 2\gamma \geq 1$, we have by Lemma 2.5 $D^{n,\alpha}_{\lambda,\mu} f \in \mathcal{ST}\,(1/2)$. Using Lemma 2.2, we obtain

$$\frac{z\left(D^{n,\delta}_{\lambda,\mu} f(z)\right)'}{D^{n,\delta}_{\lambda,\mu} f(z)} = \frac{[\varphi(2-\alpha, 2-\delta; z)]^{[n]} * \left[z\left(D^{n,\alpha}_{\lambda,\mu} f(z)\right)' / \left(D^{n,\alpha}_{\lambda,\mu} f(z)\right)\right] D^{n,\alpha}_{\lambda,\mu} f(z)}{[\varphi(2-\alpha, 2-\delta; z)]^{[n]} * D^{n,\alpha}_{\lambda,\mu} f(z)} \in$$

$$\in \overline{\text{co}}\left(\frac{z\left(D^{n,\alpha}_{\lambda,\mu} f\right)'}{\left(D^{n,\alpha}_{\lambda,\mu} f\right)}(\Delta)\right) \subseteq \text{cl}\,\mathcal{R}_{\beta,\gamma}.$$

Therefore, $f \in \beta - \mathcal{SP}^{n,\delta}_{\lambda,\mu}(\gamma)$.

Theorem 2.2 is proved.

***Corollary* 2.5.** *Let* $\beta + 2\gamma \geq 1$. *Then* $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{SP}^{n}_{\lambda,\mu}(\gamma)$.

The proof of the following Corollary 2.6 runs parallel to that of Corollary 2.3, and we omit the details.

***Corollary* 2.6.** *Let* $0 \leq \delta \leq \alpha < 1$. *Then* $\beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{UCV}^{n,\delta}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{UCV}^{n,0}_{\lambda,\mu}(\gamma)$, *where* $\beta + 2\gamma \geq 1$.

***Remark* 2.3.** (1) Taking $\beta = \lambda = \mu = 0$ and $n = 1$ in Theorem 2.2, we get the result of Srivastava, Mishra and Das [33].

(2) Taking $\gamma = \lambda = \mu = 0$ and $\beta = n = 1$ in Theorem 2.1, we get the result of Srivastava and Mishra [34].

(3) Taking $\gamma = \lambda = \mu = 0$ and $n = 1$ in Theorem 2.1 and Corollary 2.6, we get the result by Mishra and Gochhayat [18].

(4) Taking $\mu = 0$ in Theorem 2.1 and Corollary 2.6, we get the result of Al-Oboudi and Al-Amoudi [4].

***Remark* 2.4.** For special values of parameters n, α, β and γ, we arrive the following new results for $0 \leq \delta \leq \alpha < 1$:

(1) $\beta - \mathcal{SP}^{1,\alpha}_{0,0}(\gamma) \subseteq \beta - \mathcal{SP}^{1,\delta}_{0,0}(\gamma)$ and $\beta - \mathcal{UCV}^{1,\alpha}_{0,0}(\gamma) \subseteq \beta - \mathcal{UCV}^{1,\delta}_{0,0}(\gamma)$ for $\beta + 2\gamma \geq 1$,

(2) $\mathcal{ST}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \mathcal{ST}^{n,\delta}_{\lambda,\mu}(\gamma)$ and $\mathcal{CV}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \mathcal{CV}^{n,\delta}_{\lambda,\mu}(\gamma)$ for $1/2 \leq \gamma < 1$,

(3) $\mathcal{SP}^{n,\alpha}_{\lambda,\mu} \subseteq \mathcal{SP}^{n,\delta}_{\lambda,\mu}$ and $\mathcal{UCV}^{n,\alpha}_{\lambda,\mu} \subseteq \mathcal{UCV}^{n,\delta}_{\lambda,\mu}$.

From (1.15) and (1.16) we directly obtain the following useful Theorem 2.3.

Theorem 2.3. *If* $\beta_1 \geq \beta_2$, $\gamma_1 \geq \gamma_2$ *then* $\beta_1 - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma_1) \subseteq \beta_2 - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma_2)$ *and* $\beta_1 - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma_1) \subseteq \beta_2 - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma_2)$.

***Remark* 2.5.** (1) By putting $\alpha = \mu = 0$, $\lambda = 1$ and $\gamma_1 = \gamma_2 = 0$ in Theorem 2.3 for the class $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$, we obtain $\beta_1 - \mathcal{SP}^{n} \subseteq \beta_2 - \mathcal{SP}^{n}$, which was asserted earlier by Kanas and Yaguchi [13].

(2) Taking $\mu = 0$ in Theorem 2.3, we get the result of Al-Oboudi and Al-Amoudi [4].

***Corollary* 2.7.** *Under the conditions stated in Theorem* 2.1 *we have* $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{SP}^{1,\alpha}_{0,0}(\gamma) \subseteq \mathcal{SP}_{\alpha}$ *for* $\beta \geq 1$.

Proof. Let f be in $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then f belongs to $\beta - \mathcal{SP}^{1,\alpha}_{\lambda,\mu}(\gamma)$ by applying Theorem 2.1. By the same steps of the proof of Theorem 2.1 we have $\Omega^{\alpha} f \in \beta - \mathcal{SP}(\gamma)$ and by using Theorem 2.3, $\Omega^{\alpha} f \in \mathcal{SP}$ for $\beta \geq 1$. Thus $f \in \mathcal{SP}_{\alpha}$.

***Corollary* 2.8.** *Under the conditions stated in Theorem* 2.1 *we have* $\beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - \mathcal{UCV}^{1,\alpha}_{0,0}(\gamma) \subseteq 1 - \mathcal{UCV}^{1,\alpha}_{0,0}(0)$ *for* $\beta \geq 1$.

Theorem 2.4. *Let* $f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ *and* $h \in \mathcal{CV}$. *Then*

$$f(z) * h(z) \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma).$$

Proof. Let $f \in \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$. To prove the required result, it is sufficient to prove that

$$\frac{z(h(z) * D^{n,\alpha}_{\lambda,\mu} f(z))'}{h(z) * D^{n,\alpha}_{\lambda,\mu} f(z)} \in \mathcal{R}_{\beta,\gamma}, \quad z \in \Delta.$$

Theorem 2.4 is proved.

The remaining part of the proof of Theorem 2.4 is similar to that of Lemma 2.4 and hence we omit it.

***Remark* 2.6.** Taking $\alpha = \mu = \lambda = 0,\ \gamma = 2\rho - 1,\ 0 \le \rho < 1,\ n = 1$ in Theorem 2.4, we get the result by Ali [2].

3. Coefficient bounds. In the following we give the bounds for the coefficients of series expansion of functions belonging to the classes $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and sufficient for a function to be in these classes.

Let $\widehat{P}_{\beta,\gamma}$ be given in (1.6) and let $f_{\beta,\gamma}$ be defined by

$$\widehat{P}_{\beta,\gamma}(z) = \frac{z(D^{n,\alpha}_{\lambda,\mu} f_{\beta,\gamma}(z))'}{D^{n,\alpha}_{\lambda,\mu} f_{\beta,\gamma}(z)}, \quad z \in \Delta. \tag{3.1}$$

The function $f_{\beta,\gamma}$ is in the class $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ and if we denote

$$\widehat{P}_{\beta,\gamma}(z) = 1 + P_1 z + \ldots, \quad f_{\beta,\gamma}(z) = z + A_2 z^2 + \ldots,$$

then in view of (1.12) and (3.1), we have a coefficient relation

$$(k-1) A_k \Psi_{k,n}(\lambda,\mu,\alpha) = \sum_{j=1}^{k-1} P_{k-j} A_j \Psi_{j,n}(\lambda,\mu,\alpha), \quad A_1 = 1, \quad \Psi_{1,n}(\lambda,\mu,\alpha) = 1. \tag{3.2}$$

In particular, by a straightforward computation we obtain

$$A_2 = \frac{1}{\Psi_{2,n}(\lambda,\mu,\alpha)} P_1, \tag{3.3}$$

observe also, that the coefficients A_k are nonnegative because $\Psi_{k,n}(\lambda,\mu,\alpha) \ge 0$ and P_k are nonnegative (for Taylor series expansion of $\widehat{P}_{\beta,\gamma}$, see [14, 16, 26]).

As simple consequence of the above and the result given in [12], we give sharp bound on the second coefficient for functions of the class $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Theorem 3.1. *If a function* f *of the form* (1.1) *is in* $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$, *then*

$$|a_k| \le \frac{1}{\Psi_{k,n}(\lambda,\mu,\alpha)} \frac{(P_1)_{k-1}}{(k-1)!}, \quad k \ge 2, \tag{3.4}$$

where

$$P_1 := P_1(\beta,\gamma) = \begin{cases} \dfrac{8(1-\gamma)(\arccos\beta)^2}{\pi^2(1-\beta^2)}, & 0 \le \beta < 1, \\ \dfrac{8(1-\gamma)}{\pi^2}, & \beta = 1, \\ \dfrac{\pi^2(1-\gamma)}{4\sqrt{t}(1+t)(\beta^2-1)\mathcal{K}^2(t)}, & \beta > 1. \end{cases} \tag{3.5}$$

The result is sharp for $k = 2$ *or* $\beta = 0$.

For the proof of this theorem, we need the following result by Rogosinski [25].

Rogosinski's theorem [25]. *Let* $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ *be subordinate to* $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$ *in* Δ. *If* $H(z)$ *is univalent in* Δ *and* $H(\Delta)$ *is convex, then* $|c_k| \leq |C_1|$ *for* $k \geq 1$.

***Proof of Theorem* 3.1.** Let $f \in \beta - \mathcal{SP}_{\lambda,\mu}^{n,\alpha}(\gamma)$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. By (1.18), we obtain

$$\frac{z(D_{\lambda,\mu}^{n,\alpha} f(z))'}{D_{\lambda,\mu}^{n,\alpha} f(z)} \prec \widehat{P}_{\beta,\gamma}(z), \quad z \in \Delta.$$

Define $h(z) = \dfrac{z(D_{\lambda,\mu}^{n,\alpha} f(z))'}{D_{\lambda,\mu}^{n,\alpha} f(z)} = 1 + \sum_{k=1}^{\infty} c_k z^k$. The function $\widehat{P}_{\beta,\gamma}$ is univalent in Δ and $\widehat{P}_{\beta,\gamma}(\Delta)$ is the is convex conic domain so Rogosinski's theorem applies. Then we have

$$|c_k| \leq |P_1| = P_1, \quad k \geq 1, \tag{3.6}$$

where $P_1 = P_1(\beta,\gamma)$ is given by (3.5). Now writing $z(D_{\lambda,\mu}^{n,\alpha} f(z))' = h(z) D_{\lambda,\mu}^{n,\alpha} f(z)$ and comparing coefficients of z^k on both sides, we get

$$(k-1) a_k \Psi_{k,n}(\lambda,\mu,\alpha) = \sum_{j=1}^{k-1} c_{k-j} a_j \Psi_{j,n}(\lambda,\mu,\alpha), \quad a_1 = 1, \ \Psi_{1,n}(\lambda,\mu,\alpha) = 1. \tag{3.7}$$

From (3.6) and (3.7) we get $|a_2| = \dfrac{1}{\Psi_{2,n}(\lambda,\mu,\alpha)} |c_1| \leq \dfrac{P_1}{\Psi_{2,n}(\lambda,\mu,\alpha)}$. So the result is true for $k = 2$. Let $k \geq 2$ and assume that the inequality (3.4) is true for all $j \leq k-1$. By using (3.6), (3.7) and applying the induction hypothesis to $|a_j|$, we get

$$|a_k| \leq \frac{1}{(k-1)\Psi_{k,n}(\lambda,\mu,\alpha)} \left[|c_1| + \sum_{j=2}^{k-1} |c_{k-j}| \, |a_j| \, \Psi_{j,n}(\lambda,\mu,\alpha) \right] \leq$$

$$\leq \frac{P_1}{(k-1)\Psi_{k,n}(\lambda,\mu,\alpha)} \left[1 + \sum_{j=2}^{k-1} \frac{(P_1)_{j-1}}{(j-1)!} \right].$$

By applying mathematical induction another time, we find that

$$1 + \sum_{j=2}^{k-1} \frac{(P_1)_{j-1}}{(j-1)!} = \frac{(1+P_1)(2+P_1)\ldots((k-2)+P_1)}{(k-2)!}. \tag{3.8}$$

Thus we get the inequality (3.4). In view of (3.3) the result is sharp for $k = 2$. If $\beta = 0$ then $P_k(0,\gamma) = P_1(0,\gamma) = 2(1-\gamma)$, $k = 1, 2, \ldots$, and in view of (3.2) we have

$$A_k = \frac{1}{\Psi_{k,n}(\lambda,\mu,\alpha)} \frac{(P_1)_{k-1}}{(k-1)!}, \qquad k \geq 2.$$

Applying the relation (1.17), we observe that the extremal function of $\beta - \mathcal{UCV}_{\lambda,\mu}^{n,\alpha}(\gamma)$ denoted by $F_{\beta,\gamma}(z)$, is given by

$$F_{\beta,\gamma}(z) = F_{\beta,\gamma}(z) = z + B_2 z^2 + \ldots = \int\limits_0^z \frac{f_{\beta,\gamma}(\xi)}{\xi} d\xi,$$

where $f_{\beta,\gamma}(z)$ is defined by (3.1). By (3.3) we get

$$B_2 = \frac{1}{2\Psi_{2,n}(\lambda,\mu,\alpha)} P_1.$$

Theorem 3.1 is proved.

Repeating similar consideration as given in the proof of Theorem 3.1 and applying relation (1.17) we can prove the next two results.

***Corollary* 3.1.** *If a function f of the form* (1.1) *is in $\beta - \mathcal{UCV}^{n,\alpha}_{\lambda,\mu}(\gamma)$, then*

$$|a_k| \leq \frac{1}{\Psi_{k,n}(\lambda,\mu,\alpha)} \frac{(P_1)_{k-1}}{(k)!}, \quad k \geq 2,$$

where $P_1 := P_1(\beta,\gamma)$ is given by (3.5). *The result is sharp for $k = 2$ or $\beta = 0$.*

***Corollary* 3.2.** $\bigcap_{n=1}^{\infty} \beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma) = \{z\}$.

Proof. Suppose that there exists $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ for all $n \in \mathbb{N}$. Then Theorem 3.1 gives $|a_k| \leq \dfrac{(2-\alpha)_n}{(2)_n} \dfrac{(P_1)_{k-1}}{(k-1)!}$ for all $n \in \mathbb{N}$. The sequence on the right-hand side is decreasing to 0 with respect to n. Thus $a_k = 0$, for all $k \geq 2$.

***Remark* 3.1.** (1) Putting $\alpha = \mu = \lambda = \beta = 0$, $n = 1$ in Theorem 3.1 and Corollary 3.1 we get the results which, in turn, yields the corresponding results given earlier by Robertson [24].

(2) For special values of the parameters ($\alpha = \mu = \beta = 0$ and$\lambda = 1$) or ($\lambda = \mu = \beta = 0$ and $n = 1$) in Theorem 3.1, we obtain the results of Sălăgean [32] or Srivastava and Mishra [33], respectively, which are sharp results.

(3) Taking $\alpha = \mu = \lambda = \gamma = 0$, $\beta = n = 1$ in Theorem 3.1, we get the result by Rønning [27].

(4) Setting $\alpha = \mu = \lambda = \gamma = 0$, $n = 1$ in Theorem 3.1, we get the result by Kanas and Wiśniowska [12].

(5) Taking $\alpha = \mu = \gamma = 0$, $\lambda = 1$ in Theorem 3.1, we get the result by Kanas and Yaguchi [13].

(6) Upon setting $\alpha = \mu = \lambda = \gamma = 0$, $n = 1$ in Corollary 3.1, we obtain the result which is an improvement of a result due to Kanas and Wiśniowska [10].

(7) For $\mu = 0$ Theorem 3.1 and Corollary 3.1 would lead us, respectively, to the corresponding results obtained by Al-Oboudi and Al-Amoudi [4].

***Remark* 3.2.** For special values of the parameters n, α, μ, λ and β in Theorem 3.1 and Corollary 3.1, we get the coefficients bounds which is a new result for the classes $\beta-\mathcal{SP}^{n}_{\lambda,\mu}(\gamma)$, $\beta-\mathcal{UCV}^{n}_{\lambda,\mu}(\gamma)$, $\mathcal{ST}^{n,\alpha}_{\lambda,\mu}(\gamma)$, $\mathcal{CV}^{n,\alpha}_{\lambda,\mu}(\gamma)$,$\beta - \mathcal{SP}^{1,\alpha}_{0,0}(\gamma)$ and $\beta - \mathcal{UCV}^{1,\alpha}_{0,0}(\gamma)$.

Now for functions in the class $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$, we establish the following result.

Theorem 3.2. *A function f of the form* (1.1) *is in $\beta - \mathcal{SP}^{n,\alpha}_{\lambda,\mu}(\gamma)$ whenever*

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\beta+\gamma)] \, |a_k| \, \Psi_{k,n}(\lambda,\mu,\alpha) \leq 1 - \gamma. \tag{3.9}$$

Proof. From (3.9) we can find that $1-\sum_{k=2}^{\infty}\Psi_{k,n}(\lambda,\mu,\alpha)\,|a_k|>0$. Thus we get

$$\beta\left|\frac{z\left(D_{\lambda,\mu}^{n,\alpha}f(z)\right)'}{D_{\lambda,\mu}^{n,\alpha}f(z)}-1\right|-\Re\left\{\frac{z\left(D_{\lambda,\mu}^{n,\alpha}f(z)\right)'}{D_{\lambda,\mu}^{n,\alpha}f(z)}-1\right\}\le(1+\beta)\left|\frac{z\left(D_{\lambda,\mu}^{n,\alpha}f(z)\right)'}{D_{\lambda,\mu}^{n,\alpha}f(z)}-1\right|\le$$

$$\le\frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)\Psi_{k,n}(\lambda,\mu,\alpha)\,|a_k|\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}\Psi_{k,n}(\lambda,\mu,\alpha)\,|a_k|\,|z|^{k-1}}<$$

$$<\frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)\Psi_{k,n}(\lambda,\mu,\alpha)\,|a_k|}{1-\sum_{k=2}^{\infty}\Psi_{k,n}(\lambda,\mu,\alpha)\,|a_k|}.$$

This last expression is bounded above by $(1-\gamma)$ if (3.9) is satisfied. Therefore $f\in\beta-\mathcal{SP}_{\lambda,\mu}^{n,\alpha}(\gamma)$.

By virtue of (1.17) and Theorem 3.2, we have the following corollary.

***Corollary* 3.3.** *A function f of the form* (1.1) *is in $\beta-\mathcal{UCV}_{\lambda,\mu}^{n,\alpha}(\gamma)$ whenever*

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\beta+\gamma)]k\,|a_k|\,\Psi_{k,n}(\lambda,\mu,\alpha)\le 1-\gamma.$$

***Corollary* 3.4.** *If $|a_2|\le\dfrac{1-\gamma}{(2+\beta-\gamma)\Psi_{k,n}(\lambda,\mu,\alpha)}$, then $f(z)=z+a_2z^2$ belongs to the class $\beta-\mathcal{SP}_{\lambda,\mu}^{n,\alpha}(\gamma)$.*

***Remark* 3.3.** (1) If we consider $\alpha=\mu=\gamma=0$, $\beta=1$, $\lambda=1$ in Theorem 3.2 and Corollary 3.3, we obtain the same result by Bharti, Parvatham and Swaminathan [5].

(2) Taking $\alpha=\mu=\gamma=0$, $\lambda=1$ in Theorem 3.2, we get the result by Kanas and Yaguchi [13].

(3) Taking $\alpha=\mu=\lambda=0$, $\gamma=2\rho-1$, $0\le\rho<1$, $n=1$ in Theorem 3.2, we get the result by Ali [2].

For $\mu=0$, Theorem 3.2 and Corollary 3.3 would lead us, respectively, to the corresponding results obtained by Al-Oboudi and Al-Amoudi [4].

1. *Aghalary R., Azadi G. H.* The Dziok – Srivastava operator and *k*-uniformly starlike functions // J. Inequal. Pure and Appl. Math. – 2005. – **6**, № 2. – P. 1 – 7. – Article 52 (electronic).
2. *Ali R. M.* Starlikeness associated with parabolic regions // Int. J. Math. and Math. Sci. – 2005. – **4**. – P. 561 – 570.
3. *Al-Oboudi F. M.* On univalent functions defined by a generalized Salagean operator // Int. J. Math. and Math. Sci. – 2004. – **27**. – P. 1429 – 1436.
4. *Al-Oboudi F. M., Al-Amoudi K. A.* On classes of analytic functions related to conic domains // J. Math. Anal. and Appl. – 2008. – **339**. – P. 655 – 667.
5. *Bharti R., Parvatham R., Swaminathan A.* On subclasses of uniformly convex functions and corresponding class of starlike functions // Tamkang J. Math. – 1997. – **28**, № 1. – P. 17 – 32.
6. *Carlson B. C., Shaffer D. B.* Starlike and prestarlike hypergeometric functions // SIAM J. Math. Anal. – 1984. – **15**, № 4. – P. 737 – 745.
7. *Deniz E., Orhan H.* The Fekete-Szegö problem for a generalized subclass of analytic functions // Kyungpook Math. J. – 2010. – **50**. – P. 37 – 47.
8. *Duren P. L.* Univalent functions. – New York: Springer-Verlag, 1983.
9. *Goodman A. W.* On uniformly convex functions // Ann. pol. math. – 1991. – **56**. – P. 87 – 92.

10. *Kanas S., Wiśniowska A.* Conic regions and k -uniform convexity, II // Folia Sci. Univ. Techn. Resov. – 1998. – **170**. – P. 65 – 78.
11. *Kanas S., Wiśniowska A.* Conic regions and k-uniform convexity // Comput. and Appl. Math. – 1999. – **105**. – P. 327 – 336.
12. *Kanas S., Wiśniowska A.* Conic domains and starlike functions // Rev. roum. math. pures et appl. – 2000. – **45**, № 3. – P. 647 – 657.
13. *Kanas S., Yaguchi T.* Subclasses of k-uniformly convex and starlike functions defined by generalized derivate, I // Indian J. Pure and Appl. Math. – 2001. – **32**, № 9. – P. 1275 – 1282.
14. *Kanas S., Yaguchi T.* Subclasses of k-uniformly convex and starlike functions defined by generalized derivative, II // Publ. Inst. Math. – 2001. – **69**, № 83. – P. 91 – 100.
15. *Yi Ling, Ding S.* A class of analytic functions defined by fractional derivation // J. Math. Anal. and Appl. – 1994. – **186**. – P. 504 – 513.
16. *Ma W., Minda D.* Uniformly convex functions // Ann. pol. math. – 1992. – **57**. – P. 165 – 175.
17. *Magdas I.* On alpha-type uniformly convex functions // Stud. Univ. Babes-Bolyai Inform. – 1999. – **44**, № 1. – P. 11 – 17.
18. *Mishra A. K., Gochhayat P.* Applications of the Owa – Srivastava operator to the class of k-uniformly convex functions // Fract. Calc. Appl. Anal. – 2006. – **9**, № 4. – P. 323 – 331.
19. *Noor K. I., Arif M., Ul-Haq W.* On k-uniformly close-to-convex functions of complex order // Appl. Math. and Comput. – 2009. – **215**. – P. 629 – 635.
20. *Orhan H., Deniz E., Răducanu D.* The Fekete-Szegö problem for subclasses of analytic functions defined by a differential operator related to conic domains // Comput. and Math. Appl. – 2010. – **59**. – P. 283 – 295.
21. *Owa S.* On the distortion theorems, I // Kyungpook Math. J. – 1978. – **18**, № 1. – P. 53 – 59.
22. *Owa S., Srivastava H. M.* Univalent and starlike generalized hypergeometric functions // Can. J. Math. – 1987. – **39**, № 5. – P. 1057 – 1077.
23. *Ramachandran C., Shanmugam T. N., Srivastava H. M., Swaminathan A.* A unified class of k-uniformly convex functions defined by the Dziok – Srivastava linear operator // Appl. Math. and Comput. – 2007. – **190**, № 2. – P. 1627 – 1636.
24. *Robertson M. S.* On the theory of univalent functions // Ann. Math. – 1936. – **37**, № 2. – P. 374 – 408.
25. *Rogosinski W.* On the coefficients of subordinate functions // Proc. London Math. Soc. – 1943. – **48**. – P. 48 – 82.
26. *Rønning F.* On starlike functions associated with parabolic regions // Ann. Univ. Mariae Curie-Sklodowska Sect. A. – 1991. – **45**, № 14. – P. 117 – 122.
27. *Rønning F.* Uniformly convex functions and a corresponding class of starlike functions // Proc. Amer. Math. Soc. – 1993. – **118**, № 1. – P. 189 – 196.
28. *Răducanu D., Orhan H.* Subclasses of analytic functions defined by a generalized differential operator // Int. J. Math. Anal. – 2010. – **4**, № 1-4. – P. 1 – 15.
29. *Ruscheweyh St.* Convolutions in geometric function theory // Sem. Math. Sup. – Presses Univ. de Montreal, 1982. – Vol. 83.
30. *Rusheweyh St.* New criteria for univalent functions // Proc. Amer. Math. Soc. – 1975. – **49**. – P. 109 – 115.
31. *Ruscheweyh St., Sheil-Small T.* Hadamard products of Schlicht functions and the Pólya – Schöenberg conjecture // Comment. math. helv. – 1973. – **48**. – P. 119 – 135.
32. *Sălăgean G. S.* Subclasses of univalent functions // Lect. Notes Math.: Complex Analysis, Fifth Romanian-Finnish Sem., Bucharest, 1981. – Berlin: Springer, 1983. – **1013**. – P. 362 – 372.
33. *Srivastava H. M., Mishra A. K., Das M. K.* A nested class of analytic functions defined by fractional calculus // Commun. Appl. Anal. – 1998. – **2**, № 3. – P. 321 – 332.
34. *Srivastava H. M., Mishra A. K.* Applications of fractional calculus to parabolic starlike and uniformly convex functions // J. Comput. and Math. Appl. – 2000. – **39**, № 3/4. – P. 57 – 69.

Received 31.12.12,
after revision — 14.05.14