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Sh. Zviadadze (Javakhishvili Tbilisi State Univ., Georgia)

DETERMINATION OF JUMPS IN TERMS OF LINEAR OPERATORS*

ВИЗНАЧЕННЯ СТРИБКІВ У ТЕРМІНАХ ЛІНІЙНИХ ОПЕРАТОРІВ

A theorem of Lukács [J. reine und angew. Math. – 1920. – **150**. – S. 107 – 112] states that the partial sums of conjugate Fourier series of a periodic Lebesgue integrable function f diverge with a logarithmic rate at the points of discontinuity of f of the first kind. Móricz [Acta math. hung. – 2003. – **98**. – P. 259 – 262] proved a similar theorem for the rectangular partial sums of double conjugate trigonometric Fourier series.

We consider analogs of the Móricz theorem for generalized Cesáro means and for positive linear means.

In the present paper we prove a similar theorem in terms of linear operators satisfying certain conditions.

Теорема Лукаша [J. reine und angew. Math. – 1920. – **150**. – S. 107 – 112] стверджує, що частинні суми спряжених рядів Фур'є періодичної функції f, інтегровної за Лебегом, розбігаються з логарифмічною швидкістю в точках розриву першого роду функції f. Моріч [Acta math. hung. – 2003. – **98**. – P. 259 – 262] довів подібну теорему для прямокутних частинних сум двічі спряжених тригонометричних рядів Фур'є.

Розглянуто теореми, що аналогічні теоремі Моріча для узагальнених середніх Чезаро та для позитивних лінійних середніх.

У цій статті доведено аналогічну теорему в термінах лінійних операторів, що задовольняють певні умови.

1. Introduction. Let f be a 2π-periodic Lebesgue integrable function. The Fourier trigonometric series of the function f is defined by

$$\frac{a_0}{2} + \sum_{i=1}^{\infty}(a_i \cos ix + b_i \sin ix), \tag{1.1}$$

where

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos ix dx \qquad \text{and} \qquad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin ix dx$$

are the Fourier coefficients of f. The conjugate series of (1.1) is defined by

$$\sum_{i=1}^{\infty}(a_i \sin ix - b_i \cos ix). \tag{1.2}$$

Let $\tilde{S}_k(f;x)$ be the kth partial sum of series (1.2). Lukács [4] proved the following theorem.

Theorem 1.1. *If $f \in L(-\pi,\pi]$ and for some point $x \in (-\pi,\pi]$, there exists a number $d_x(f)$ such that*

$$\lim_{t\to 0+}\frac{1}{t}\int_0^t |f(x+s) - f(x-s) - d_x(f)|ds = 0,$$

then

$$\lim_{k\to+\infty}\frac{\tilde{S}_k(f;x)}{\ln k} = -\frac{d_x(f)}{\pi}.$$

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R. Riad [10] proved an analogous theorem in terms of the conjugate Walsh series.

F. Móricz [5, 6] generalized Lukács's theorem in terms of the Abel – Poisson means and proved estimate of the partial derivative of the Abel – Poisson mean of an integrable function at those points where it is smooth.

Pinsky [9] generalized Fourier partial sums by using a family of convolusion operators with some classes of kernels.

Q. Shi and X. Shi [11] discuss about the concentration factor methods for determination of jumps in terms of spectral data.

Dansheng Yu, P. Zhou and S. Zhou [14] show how jumps can be determined by the higher order partial derivatives of the of its Abel – Poisson means.

We [17, 18] examine the analogous theorems for the generalized Cesáro means, introduced by Akhobadze [1 – 3], as well as positive regular linear means, and consider [19] Lukás theorem for the functions and series introduced by Taberski [12, 13] as well as generalized Cesáro and positive regular linear means. Some results of this paper were announced in [17, 18].

P. Zhou and S. P. Zhou [15] proved an analogous theorem in terms of the linear operators which satisfy some certain conditions.

F. Móricz [7] examined Lukács theorem for double trigonometric series. F. Móricz and W. R. Wade [8] generalized Lukács theorem for double Walsh series.

We [21] generalized Móricz's theorem and we proved that conditions in Móricz's theorem is the best option for indices not to be dependent on each other. Also we considered analogues of these theorems for generalized Cesáro means and positive linear means matrix of which satisfy necessary conditions of regularity.

2. Determination of jumps in terms of linear operators. Let $(m(k))$, $(n(k))$ be a nonnegative sequences of real numbers such that

$$\lim_{k\to+\infty} m(k) = \lim_{k\to+\infty} n(k) = +\infty.$$

Suppose there are given two sequences of 2π-periodic, odd and Lebesgue integrable functions G_k and H_k for which the following is true

$$|G_k(t)| = O(m(k)), \qquad |H_k(t)| = O(n(k)) \quad \text{for all} \quad t, \tag{2.1}$$

$$|G_k(t)| = O(1/t), \qquad |H_k(t)| = O(1/t) \quad t \in (0;\pi], \tag{2.2}$$

$$\int_0^\pi G_k(t)dt \simeq \ln m(k), \qquad \int_0^\pi H_k(t)dt \simeq \ln n(k). \tag{2.3}$$

Suppose that it is given two variable function f, 2π-periodic in both variables and Lebesgue integrable on $(-\pi;\pi]^2$.

Let's consider the following linear operator:

$$F_{kj}(f;x,y) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)G_k(u-x)H_j(v-y)dudv. \tag{2.4}$$

Let

$$\varphi(x,y,u,v) := f(x+u,y+v) - f(x-u,y+v) - f(x+u,y-v) + f(x-u,y-v) - d_{xy}(f),$$

where $d_{xy}(f)$ is a number and

$$\Psi(x,y,s,t) := \int_0^s \int_0^t |\varphi(x,y,u,v)| du dv.$$

Theorem 2.1. *Let $f \in L(-\pi;\pi]^2$ and suppose that for a point $(x,y) \in (-\pi;\pi]^2$*

$$\lim_{s,t\to 0+} \Psi(x,y,s,t)(ts)^{-1} = 0, \tag{2.5}$$

$$\Psi(x,y,s,t) = O(\min\{s,t\}), \qquad 0 < s, \quad t \le \pi. \tag{2.6}$$

Then for all sequences of convolution type operators F_{kj} which are defined by (2.4) *where kernels satisfy conditions* (2.1) – (2.3) *the following equality is valid*:

$$\lim_{k,j\to+\infty} \frac{F_{kj}(f;x,y)}{\ln m(k) \ln n(j)} = \frac{d_{xy}(f)}{\pi^2}. \tag{2.7}$$

Proof. Let us consider $F_{kj}(f;x,y)$, by changing of variables in the integral we get

$$F_{kj}(f;x,y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \varphi(x,y,u,v) G_k(u) H_j(v) du dv +$$

$$+ \frac{d_{xy}(f)}{\pi^2} \int_0^\pi \int_0^\pi G_k(u) H_j(v) du dv = A_1(k,j) + A_2(k,j). \tag{2.8}$$

By (2.5) for every $\varepsilon > 0$ we can choose δ such that

$$\Psi(x,y,\delta,\delta)/\delta^2 < \varepsilon. \tag{2.9}$$

According the definition of $m(k)$ and $n(j)$ we can choose k and j such that $1/m(k)$, $1/n(j) < \delta$. Therefore

$$A_1(k,j) = \frac{1}{\pi^2} \left(\int_0^{1/m(k)} + \int_{1/m(k)}^{\delta} + \int_\delta^\pi \right) \left(\int_0^{1/n(j)} + \int_{1/n(j)}^{\delta} + \int_\delta^\pi \right) \times$$

$$\times \varphi(x,y,u,v) G_k(u) H_j(v) du dv = \sum_{r,s=1}^{3} B_{rs}. \tag{2.10}$$

Throughout, we use C to stand for an absolute positive constant, which may have different values in different occurrences.

Note that B_{rs} and B_{sr}, $s, r \in \{1,2,3\}$, $r \neq s$, can be estimated similarly. By (2.1) and (2.9) we have

$$|B_{11}| \leq \frac{Cm(k)n(j)}{\pi^2} \int\limits_0^{1/m(k)} \int\limits_0^{1/n(j)} |\varphi(x,y,u,v)|dudv = O(1). \tag{2.11}$$

Using (2.2) and integration by parts with respect to v we obtain

$$|B_{12}| \leq Cm(k) \int\limits_0^{1/m(k)} \int\limits_{1/n(j)}^{\delta} \frac{|\varphi(x,y,u,v)|}{v} dudv \leq C\varepsilon + C\varepsilon \int\limits_{1/n(j)}^{\delta} \frac{dv}{v} = O(\ln n(j)). \tag{2.12}$$

By (2.1), (2.2), (2.6) and integration by parts with respect to v we get

$$|B_{13}| \leq Cm(k) \int\limits_0^{1/m(k)} \int\limits_{\delta}^{\pi} \frac{|\varphi(x,y,u,v)|}{v} dudv \leq$$

$$\leq Cm(k) \left(\int\limits_0^{\pi} \int\limits_0^{1/m(k)} |\varphi(x,y,u,t)|dudt + \int\limits_{\delta}^{\pi} \frac{1}{v^2} \int\limits_0^{\pi} \int\limits_0^{1/m(k)} |\varphi(x,y,u,t)|dudtdv \right) =$$

$$= Cm(k) \left(\frac{1}{\pi} \Psi\left(x, y, \frac{1}{m(k)}, \pi\right) + \Psi\left(x, y, \frac{1}{m(k)}, \pi\right) \int\limits_{\delta}^{\pi} \frac{dv}{v^2} \right) = O(1) \tag{2.13}$$

and

$$|B_{22}| \leq C \int\limits_{1/m(k)}^{\delta} \left(\int\limits_{1/n(j)}^{\delta} |\varphi(x,y,u,v)| \frac{dv}{v} \right) \frac{du}{u}. \tag{2.14}$$

Furthermore, integration by parts with respect to v gives

$$\int\limits_{1/n(j)}^{\delta} |\varphi(x,y,u,v)| \frac{dv}{v} \leq \frac{1}{\delta} \int\limits_0^{\delta} |\varphi(x,y,u,t)|dt + \int\limits_{1/n(j)}^{\delta} \frac{1}{v^2} \int\limits_0^{v} |\varphi(x,y,u,t)|dtdv.$$

Now by using (2.9) and integration by parts with respect to u we have

$$\int\limits_{1/m(k)}^{\delta} \left(\frac{1}{\delta} \int\limits_0^{\delta} |\varphi(x,y,u,t)|dt + \int\limits_{1/n(j)}^{\delta} \frac{1}{v^2} \int\limits_0^{v} |\varphi(x,y,u,t)|dtdv \right) \frac{du}{u} \leq$$

$$\leq \frac{1}{\delta^2} \int\limits_0^{\delta} \int\limits_0^{\delta} |\varphi(x,y,s,t)|dsdt + \frac{1}{\delta} \int\limits_0^{\delta} \int\limits_{1/n(j)}^{\delta} \frac{1}{v^2} \int\limits_0^{v} |\varphi(x,y,s,t)|dtdvds+$$

$$+\int\limits_{1/m(k)}^{\delta}\int\limits_{1/n(j)}^{\delta}\frac{1}{v^2u^2}\int\limits_0^v\int\limits_0^u|\varphi(x,y,s,t)|dsdtdudv\le$$

$$\le\varepsilon+\varepsilon\int\limits_{1/n(j)}^{\delta}\frac{dv}{v}+\varepsilon\int\limits_{1/m(k)}^{\delta}\int\limits_{1/n(j)}^{\delta}\frac{dudv}{vu}.$$

By (2.14) and the last estimations we get

$$|B_{22}|\le C\varepsilon\ln m(k)\ln n(j), \tag{2.15}$$

where C is a fixed positive constant.

Analogously, using once more integration by parts with respect to u and (2.6) we obtain

$$|B_{23}|\le C\int\limits_{1/m(k)}^{\delta}\int\limits_{\delta}^{\pi}\frac{|\varphi(x,y,u,v)|}{uv}dudv\le$$

$$\le\frac{C}{\delta}\int\limits_{1/m(k)}^{\delta}\left(\int\limits_0^{\pi}|\varphi(x,y,u,v)|dv\right)\frac{du}{u}\le$$

$$\le\frac{C}{\delta^2}\int\limits_0^{\delta}\int\limits_0^{\pi}|\varphi(x,y,s,v)|dsdv+\frac{C}{\delta}\int\limits_{1/m(k)}^{\delta}\frac{1}{u^2}\int\limits_0^{u}\int\limits_0^{\pi}|\varphi(x,y,s,v)|dsdvdu=$$

$$=C\frac{\Psi(x,y,\delta,\pi)}{\delta^2}+\frac{C}{\delta}\int\limits_{1/m(k)}^{\delta}\frac{\Psi(x,y,u,\pi)}{u^2}du=O(\ln m(k)) \tag{2.16}$$

and

$$|B_{33}|\le\frac{4}{\pi^4\delta^2}\int\limits_0^{\pi}\int\limits_0^{\pi}|\varphi(x,y,u,v)|dudv=O(1). \tag{2.17}$$

Finally by (2.10) – (2.17) we have

$$\lim_{k,j\to+\infty}A_1(k,j)/(\ln m(k)\ln n(k))=0. \tag{2.18}$$

Now consider $A_2(k,j)$. By (2.3) we get

$$I_2(k)=\frac{d_{xy}(f)}{\pi^2}\int\limits_0^{\pi}\int\limits_0^{\pi}G_k(u)H_j(v)dudv=\frac{d_{xy}(f)}{\pi^2}\int\limits_0^{\pi}G_k(u)du\int\limits_0^{\pi}H_j(v)dv\simeq$$

$$\simeq\frac{d_{xy}(f)}{\pi^2}\ln m(k)\ln n(j).$$

If we combine (2.18) and the last estimation it completes the proof of theorem.

3. Applications. The question arises naturally: which kind of kernel satisfies conditions (2.1) – (2.3)?

First of all let us note that the conjugate Dirichlet kernel satisfies above mentioned conditions. Indeed assume that $m(k) = k$ and $n(j) = j$. We know that $|\tilde{D}_k(t)| \leq k$ for all t, $|\tilde{D}_k(t)| \leq 2/t$, $0 < t \leq \pi$, (see [16], Chap. II, (5.11)) and

$$\int_0^{\pi} \tilde{D}_k(u)du \simeq \ln k, \quad k \to +\infty,$$

where $\tilde{D}_k(t)$ denotes the conjugate Dirichlet kernel. In this case get Moricz's [6] result.

Analogously generalized Cesáro mean of the conjugate Dirichlet kernel satisfies (2.1) – (2.3) conditions (see proof of Theorem 2.1 in [20] or proof of Theorem 2.2 in [21]). Thus we get the author's result [21].

In [21] we generalized Moricz's theorem in case of matrix summability. It is easy to see that in the case if 4th dimensional matrix can be represented as a product of two dimensional matrixes, then Theorem 2.1 generalizes our [21] result.

Even more we can make sure that generalized de la Vallée Poussin mean of the conjugate Dirichlet kernel satisfies conditions (2.1) – (2.3). In [15] by authors' was obtained Lukacs type theorem where kernel satisfies conditions (2.1) – (2.4) from [15]. Authors also have shown that generalized de la Vallée Poussin mean of the conjugate Dirichlet kernel satisfies (2.1) – (2.4) conditions in [15]. But (2.1) – (2.3) conditions are different from (2.1) – (2.4) and we introduced different, self-contained proof.

Let $\tilde{S}_{kj}(f; x, y)$ be a rectangular partial sum of the double conjugate trigonometric Fourier series and let

$$V_{m,n,k,s}(f; x, y) = \frac{1}{(k+1)(s+1)} \sum_{i=m-k}^{m} \sum_{j=n-s}^{n} \tilde{S}_{ij}(f; x, y),$$

be generalized de la Vallée Poussin mean of the rectangular partial sum of the double conjugate trigonometric Fourier series where $0 \leq k \leq m$ and $0 \leq s \leq n$.

It is easy to verify that

$$V_{m,n,0,0}(f; x, y) = \tilde{S}_{mn}(f; x, y), \qquad V_{m,n,m,n}(f; x, y) = \tilde{\sigma}_{mn}(f; x, y),$$

where $\tilde{\sigma}_{mn}(f; x, y)$ denotes (C,1,1) means of the rectangular partial sums of conjugate trigonometric Fourier series.

By definition of the generalized de la Vallée Poussin sums we get

$$V_{m,n,k,s}(f; x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v)\tilde{K}_m(u - x)\tilde{K}_n(v - y)dudv,$$

where

$$\tilde{K}_m(t) = \frac{1}{k+1} \sum_{i=m-k}^{m} \tilde{D}_i(t).$$

***Corollary* 3.1.** *Let* $f \in L(-\pi;\pi]^2$ *and suppose that for a point* $(x,y) \in (-\pi;\pi]^2$ (2.5) *and* (2.6) *hold. Then*

$$\lim_{m,n\to+\infty} \frac{V_{m,n,k,s}(f;x,y)}{\ln m \ln n} = \frac{d_{xy}(f)}{\pi^2}.$$

Proof. We will show that (2.1), (2.2) and (2.3) conditions are true for generalized de la Vallée Poussin mean of the conjugate Dirichlet kernel.

Suppose that $G_m(t) = \tilde{K}_m(t)$ and $H_n(t) = \tilde{K}_n(t)$ then by the definition of generalized de la Vallée Poussin mean of the conjugate Dirichlet kernel and the following estimation $|\tilde{D}_k(t)| \le k$ for all t (see [16], Chap. II, (5.11)) and by the formula for the sum of the terms of an arithmetic sequence we have

$$|\tilde{K}_m(t)| \le \frac{1}{k+1}\sum_{i=m-k}^{m} |\tilde{D}_i(t)| \le \frac{1}{k+1}\sum_{i=m-k}^{m} i =$$

$$= \frac{1}{k+1}\frac{m-k+m}{2}(m-(m-k)+1) = \frac{2m-k}{2} \le m,$$

for all $t \in [0;\pi]$, and condition (2.1) holds.

By the estimation $|\tilde{D}_i(t)| \le 2/t$, $t \in (0;\pi]$ (see [16], Chap. II, (5.11)) we obtain

$$|\tilde{K}_m(t)| \le \frac{1}{k+1}\sum_{i=m-k}^{m} |\tilde{D}_i(t)| \le \frac{1}{k+1}\sum_{i=m-k}^{m} \frac{2}{t} = \frac{2}{t}.$$

Thus condition (2.2) holds.

Now consider

$$\int_0^{\pi} \tilde{K}_m(u)du = \frac{1}{k+1}\sum_{i=m-k}^{m}\int_0^{\pi} \tilde{D}_i(u)du = \frac{1}{k+1}\sum_{i=m-k}^{m} U_i.$$

It is well known that for any $\varepsilon > 0$ there exists number $N = N(\varepsilon)$ such that, for all $i > N$ we get

$$1-\varepsilon < \frac{U_i}{\ln i} < 1+\varepsilon. \tag{3.1}$$

Consider following sum:

$$\frac{1}{k+1}\sum_{i=m-k}^{m} U_i = \frac{1}{k+1}\left(\sum_{i=m-k}^{N} U_i + \sum_{i=N+1}^{m} U_i\right) = I_1 + I_2.$$

It is easy to see that

$$I_1 \le \max_{i\le N} |U_i| \cdot N.$$

Also by the right-hand side inequality of (3.1) we have

$$I_2 \le \frac{1+\varepsilon}{k+1}\sum_{i=N+1}^{m} \ln i \le \frac{(1+\varepsilon)\ln m}{k+1}(m-N) \le (1+\varepsilon)\ln m.$$

Therefore we get

$$\overline{\lim_{m\to+\infty}} \frac{1}{\ln m} \int\limits_0^{\pi} \tilde{K}_m(u)du \le 1. \tag{3.2}$$

If for the same ε we choose M such that $2/M < \varepsilon$, then we get following lower estimation of I_2 :

$$I_2 = \frac{1}{k+1} \sum_{i=N+1}^{m} U_i = \frac{1}{k+1} \sum_{i=N+1}^{m/M} U_i + \frac{1}{k+1} \sum_{i=m/M+1}^{m} U_i = J_1 + J_2.$$

By the left-hand side inequality of (3.1) we obtain

$$J_1 \ge \frac{1-\varepsilon}{k+1} \sum_{i=N}^{m/M} \ln i \ge \frac{1-\varepsilon}{k+1} \ln N \sum_{i=N}^{m/M} \ge \frac{1-\varepsilon}{k+1} \ln N = O(1).$$

Reasoning analogously as in previous estimation we get

$$J_2 \ge \frac{1-\varepsilon}{k+1} \sum_{i=m/M+1}^{m} \ln i \ge \frac{1-\varepsilon}{k+1} (\ln m - \ln M) \left(\sum_{i=m-k}^{m} - \sum_{i=m-k}^{m/M} \right) = L_1 - L_2.$$

It is easy to see that

$$L_1 = (1-\varepsilon)(\ln m - \ln M).$$

On the other hand $L_2 = o(\ln m)$, indeed

$$L_2 \le \frac{1-\varepsilon}{k+1} (\ln m - \ln M) \cdot 2 \cdot \left(\frac{m}{M} - (m-k) \right) \le \frac{k}{k+1} (\ln m - \ln M) \frac{2}{M} <$$

$$< \varepsilon(\ln m - \ln M) = o(\ln m).$$

Thus we obtain

$$\underline{\lim_{m\to+\infty}} \frac{1}{\ln m} \int\limits_0^{\pi} \tilde{K}_m(u)du \ge 1.$$

From the (3.2) and last estimation we conclude that $\tilde{K}_m(u)$ satisfies condition (2.3).

Corollary 3.1 is proved.

In all above examined examples kernels G_m and H_m are the same type, and it's integrals have the same order (logarithmic). Advantage of above considered integral operator and general kernel is that we can consider kernels with different order.

Consider matrices (b_{ki}) and (c_{js}):

$$b_{ki} = \begin{cases} 1 & \text{if} \quad i = \left[k^{1/\ln\ln k}\right], \\ 0 & \text{if} \quad i \ne \left[k^{1/\ln\ln k}\right], \end{cases}$$

and

$$c_{js} = \begin{cases} 1 & \text{if} \quad s = j, \\ 0 & \text{if} \quad s \ne j. \end{cases}$$

We construct matrix (a_{kjis}) where $a_{kjis} = b_{ki} \cdot c_{js}$. Let cite an example.

Consider

$$\sigma_{kj}(f;x,y) = \sum_{i=0}^{+\infty}\sum_{s=0}^{+\infty} a_{kjis}\tilde{S}_{is}(f;x,y) =$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sum_{i=0}^{+\infty} b_{ki}\tilde{D}_i(u-x)\sum_{s=0}^{+\infty} c_{js}\tilde{D}_s(v-y)dudv =$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\tilde{D}_{[k^{1/\ln\ln k}]}(u-x)\tilde{D}_j(v-y)dudv.$$

It is easy to see that in this case $G_{m(k)}(t) = \tilde{D}_{[k^{1/\ln\ln k}]}$ and $m(k) = [k^{1/\ln\ln k}]$.

1. *Akhobadze T.* On generalized Cesáro summability of trigonometric Fourier series // Bull. Georg. Acad. Sci. – 2004. – **170**. – P. 23 – 24.
2. *Akhobadze T.* On the convergence of generalized Cesáro means of trigonometric Fourier series I // Acta math. hung. – 2007. – **115**. – P. 59 – 78.
3. *Akhobadze T.* On the convergence of generalized Cesáro means of trigonometric Fourier series II // Acta math. hung. – 2007. – **115**. – P. 79 – 100.
4. *Lukács F.* Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe // J. reine und angew. Math. – 1920. – **150**. – S. 107 – 112.
5. *Móricz F.* Determination of jumps in terms of Abel – Poisson means // Acta math. hung. – 2003. – **98**. – P. 259 – 262.
6. *Móricz F.* Ferenc Lukács type theorems in terms of the Abel – Poisson mean of conjugate series // Proc. Amer. Math. Soc. – 2003. – **131**. – P. 1234 – 1250.
7. *Móricz F.* Extension of a theorem of Ferenc Lukács from single to double conjugate series // J. Math. Anal. and Appl. – 2001. – **259**, № 2. – P. 582 – 595.
8. *Móricz F., Wade W. R.* An analogue of a theorem of Ferenc Lukács for double Walsh – Fourier series // Acta math. hung. – 2002. – **95**, № 4. – P. 323 – 336.
9. *Pinsky M. A.* Inverse problems in Fourier analysis and summability theory // Methods Appl. Anal. – 2004. – **11**. – P. 317 – 330.
10. *Riad R.* On the conjugate of the Walsh series // Ann. Univ. sci. budapest. Sect. math. – 1987. – **30**. – P. 69 – 76.
11. *Shi Q., Shi X.* Determination of jumps in terms of spectral data // Acta math. hung. – 2006. – **110**. – P. 193 – 206.
12. *Taberski R.* Convergence of some trigonometric sums // Demonstr. math. – 1973. – **5**. – P. 101 – 117.
13. *Taberski R.* On general Dirichlet's integrals // An. Soc. math. pol. Ser. I. Prace math. – 1974. – **17**. – P. 499 – 512.
14. *Yu D. S., Zhou P., Zhou S.* On determinatioin of jumps in terms of Abel – Poisson mean of Fourier series // J. Math. Anal. and Appl. – 2008. – **341**. – P. 12 – 23.
15. *Zhou P., Zhou S. P.* More on determination of jumps // Acta math. hung. – 2008. – **118**. – P. 41 – 52.
16. *Zigmund A.* Trigonometric series. – Cambridge Univ. Press, 1959. – Vol. 1.
17. *Zviadadze Sh.* On the statement of F. Lukács // Bull. Georg. Acad. Sci. – 2005. – **171**. – P. 411 – 412.
18. *Zviadadze Sh.* On a Lukács theorem for regular linear means of conjugate trigonometric Fourier series // Bull. Georg. Acad. Sci. – 2006. – **173**. – P. 435 – 438.
19. *Zviadadze Sh.* On generalizations of a theorem of Ferenc Lukács // Acta math. hung. – 2009. – **122**, № 1-2. – P. 105 – 120.
20. *Zviadadze Sh.* On some properties of conjugate trigonometric Fourier series // Georg. Math. J. – 2013. – **20**, № 2. – P. 397 – 413.
21. *Zviadadze Sh.* On some properties of double conjugate trigonometric Fourier series // Acta math. hung. – 2012. – **134**, № 4. – P. 452 – 417.

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