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ULTRAFILTERS ON BALLEANS

УЛЬТРАФІЛЬТРИ НА БОЛЕАНАХ

A ballean (equivalently, a coarse structure) is an asymptotic counterpart of a uniform space. We introduce three ultrafilter satellites of a ballean (namely, corona, companion, and corona companion), evaluate the basic cardinal invariants of the corona and characterize the subsets of balleans in terms of companions.

Болеан (або груба структура) — це асимптотичний аналог рівномірного простору. За допомогою ультрафільтрів визначено три супутники болеанів (а саме, корону, компаньйон і коронний компаньйон), знайдено оцінки основних кардальних інваріантів корони та охарактеризовано підмножини болеанів за допомогою компаньйонів.

1. Introduction. A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets, $B\colon X \times P \to \mathcal{P}_X$, $x \in B(X, \alpha)$ for each $x \in X$ and $\alpha \in P$, $\mathcal{P}_X$ denotes the family of all subsets of X. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii* and $B(x, \alpha)$ is called a *ball of radius* α *around* x.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},\ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if
for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$

for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

A ballean $\mathcal{B}$ on X can also be defined in terms of entourages of the diagonal Δ_X of $X \times X$ (in this case it is called a *coarse structure* [1]), and can be considered as an asymptotic counterpart of a uniform space. For our goals, we prefer the ball language from [2, 3].

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be balleans. A mapping $f\colon X \to X'$ is called a $\prec$*-mapping* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that, for every $x \in X$, $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$. If there exists a bijection $f\colon X \to X'$ such that f and f^{-1} are $\prec$-mappings, $\mathcal{B}$ and $\mathcal{B}'$ are called *asymorphic* and f is called an *asymorphism*.

For a ballean $\mathcal{B} = (X, P, B)$, a subset $Y \subseteq X$ is called *large* if there is $\alpha \in P$ such that $X = B(Y, \alpha)$. A subset V of X is called *bounded* if $V \subseteq B(x, \alpha)$ for some $x \in X$ and $\alpha \in P$. Each nonempty subset $Y \subseteq X$ determines a *subballean* $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = Y \cap B(y, \alpha)$.

We say that $\mathcal{B}$ and $\mathcal{B}'$ are *coarsely equivalent* if there exist large subset $Y \subseteq X$ and $Y' \subseteq X'$ such that the subballeans $\mathcal{B}_Y$ and $\mathcal{B}'_{Y'}$ are asymorphic.

Given a ballean $\mathcal{B} = (X, P, B)$, $x, y \in X$ and $\alpha \in P$, we say that x and y are *α-path connected* if there exists a finite sequence $x_0, \dots, x_n$, $x_0 = x$, $x_n = y$ such that $x_{i+1} \in B(x_i, \alpha)$, $x_i \in B(x_{i+1}, \alpha)$

for each $i \in \{0, \dots, n-1\}$. For any $x \in X$ and $\alpha \in P$, we denote

$$B^{\square}(x, \alpha) = \{y \in X : x, y \text{ are } \alpha\text{-path connected}\}.$$

The ballean $\mathcal{B}^{\square} = (X, P, B^{\square})$ is called a *cellularization* of $\mathcal{B}$. A ballean $\mathcal{B}$ is called *cellular* if the identity mapping $id \colon X \to X$ is an asymorphism between $\mathcal{B}$ and $\mathcal{B}^{\square}$. By [3] (Theorem 3.1.3), $\mathcal{B}$ is cellular if and only if $\mathcal{B}$ is asymptotically zero-dimensional.

For a ballean $\mathcal{B} = (X, P, B)$, we use a natural preordering on P defined by the rule: $\alpha < \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for each $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if for every $\alpha \in P$, there is $\alpha' \in P'$ such that $\alpha < \alpha'$. The minimal cardinality $cf\mathcal{B}$ of cofinal subsets of P is called *cofinality* of $\mathcal{B}$.

A ballean $\mathcal{B}$ is called *ordinal* if there exists a cofinal subset of P well-ordered by $<$. Up to asymorphism, we can replace P with some segment $[0, \gamma)$ of ordinals and, moreover, we can assume that γ is a regular cardinal. It is easy to see that every ordinal ballean of uncountable cofinality is cellular. More on cellular balleans can be found in [3] (Chapter 3).

Let $\mathcal{B} = (X, P, B)$ be a ballean. We say that two subsets Y, Z of X are *asymptotically disjoint* if, for every $\alpha \in P$, there exists a bounded subset V_α of X such that $B(Y \setminus V_\alpha, \alpha) \cap B(Z \setminus V_\alpha, \alpha) = \varnothing$. The subsets Y, Z are called *asymptotically separated* if to each $\alpha \in P$ one can assign a bounded subset V_α of X such that

$$\left(\bigcup_{\alpha \in P} B(Y \setminus V_\alpha, \alpha) \right) \bigcap \left(\bigcup_{\alpha \in P} B(Z \setminus V_\alpha, \alpha) \right) = \varnothing.$$

A ballean $\mathcal{B}$ is called *normal* if any two asymptotically disjoint subsets of X are asymptotically separated. For normal balleans see [4] and [3] (Chapter 4). According to [3] (Chapter 4), every ordinal ballean is normal.

In Section 2 we give some examples of cellular and ordinal balleans. In Section 3 we introduce three ultrafilter satellites of a ballean: corona, ultracompanion and corona companion. In Section 4 we evaluate the basic cardinal invariants of coronas of ordinal balleans. In Section 5 we characterize the subsets of a ballean in terms of its ultracompanions and corona companions.

2. Examples.

***Example* 2.1.** Each metric space (X, d) defines a metric ballean $(X, \mathbb{R}^+, B_d)$, where $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$. By [3] (Theorem 2.1.1), for a ballean $\mathcal{B}$, the following conditions are equivalent:

$\mathcal{B}$ is asymorphic to some metric ballean;

$\mathcal{B}$ is coarsely equivalent to some metric ballean;

$cf\mathcal{B} \leq \aleph_0$.

Clearly, each metric ballean is ordinal. By [3] (Theorem 3.1.1), a metric ballean $\mathcal{B}$ is cellular if and only if $\mathcal{B}$ is asymorphic to a ballean of some ultrametric space.

***Example* 2.2.** Every infinite cardinal κ defines the cardinal ballean $\overleftrightarrow{\kappa} = (\kappa, \kappa, \overleftrightarrow{B})$, where

$$\overleftrightarrow{B}(x, \alpha) = \{y \in \kappa : x \leq y \leq x + \alpha \text{ or } y \leq x \leq y + \alpha\}.$$

For cardinal ballean see [5]. In particular [5] (Theorem 3), if $\kappa > \aleph_0$ then $\overleftrightarrow{\kappa}$ is cellular. Clearly, each cardinal ballean is ordinal.

Example **2.3.** Let γ be a limit ordinal, $\{\mu_\alpha : \alpha < \gamma\}$ be a family of cardinals. A direct product $\otimes_{\alpha<\gamma}\mu_\alpha$ is a set of all γ-sequences $x = (x_\alpha)_{\alpha<\gamma}$ such that $x_\alpha \in \mu_\alpha$ and $x_\alpha = 0$ for all but finitely many $\alpha < \gamma$. We consider a ballean

$$\mathcal{B} = (\otimes_{\alpha<\gamma}\mu_\alpha, [0, \gamma), B),$$

where $B(x, \beta) = \{y \in \otimes_{\alpha<\gamma}\mu_\alpha : y_\alpha = x_\alpha$ for every $\alpha \geq \beta\}$. Evidently, $\mathcal{B}$ is ordinal and cellular. For decompositions of balleans into direct products see [6] and [7].

Example **2.4.** Let G be a group. An ideal J in the Boolean algebra $\mathcal{P}_G$ of all subsets of G is called a *group ideal* if J contains all finite subsets of G and if $A, B \in J$ then $AB^{-1} \in J$.

Now let X be a transitive G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$, and let J be a group ideal in G. We define a ballean $\mathcal{B}(G, X, J)$ as a triple (X, J, B), where $B(x, A) = Ax \cup \{x\}$ for all $x \in X$, $A \in J$. By [8] (Theorem 1), every ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G, X, J)$ for some group G of permutations of X and some ideal J of G. By [8] (Theorem 3), every cellular ballean $\mathcal{B}$ with the support X is asymorphic to $\mathcal{B}(G, X, J)$ for some group G of permutations of X and some ideal J which has a base consisting of subgroups.

In the case $X = G$, and the left regular action of G on X, we write (G, J) instead $\mathcal{B}(G, X, J)$.

3. Ultrafilters. Let $\mathcal{B} = (X, P, B)$ be an unbounded ballean. We endow X with the discrete topology and consider the Stone – Čech compactification βX of X. We take the points of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters on X. For every subset $A \subseteq X$, we put $\overline{A} = \{q \in \beta X : A \in q\}$. The topology of βX can be defined by stating that the family $\{\overline{A} : A \subseteq X\}$ is a base for the open sets. Let Y be a compact Hausdorff space. For a mapping $f : X \to Y$, f^β denotes the Stone – Čech extension of f onto βX.

We denote by $X^\sharp_\mathcal{B}$ the set of all ultrafilters on X whose members are unbounded in $\mathcal{B}$, and note that $X^\sharp_\mathcal{B}$ is a closed subset of βX.

Given any $r, q \in X^\sharp_\mathcal{B}$, we say that r, q are *parallel* (and write $r||q$) if there exists $\alpha \in P$ such that $\mathcal{B}(R, \alpha) \in q$ for every $R \in r$. By [4] (Lemma 4.1), $||$ is an equivalence on $X^\sharp_\mathcal{B}$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\sharp_\mathcal{B} \times X^\sharp_\mathcal{B}$) equivalence on $X^\sharp$ such that $|| \subseteq \sim$. The quotient $X^\sharp_\mathcal{B}/\sim$ is a compact Hausdorff space. It is called the *corona* of $\mathcal{B}$ and is denoted by $\check{X}_\mathcal{B}$. Let (X, d) be a metric space such that each closed ball in X is compact, $\mathcal{B} = \mathcal{B}(X, d)$. Then $\check{X}_\mathcal{B}$ coincides with the Higson's corona of (X, d) (see [9, p. 154]).

For every $p \in X^\sharp_\mathcal{B}$, we denote by $\check{p}$ the class of the equivalence $\sim$, and say that two ultrafilters $p, q \in X^\sharp_\mathcal{B}$ are corona equivalent if $\check{p} = \check{q}$. To detect whether two ultrafilters $p, q \in X^\sharp_\mathcal{B}$ are corona equivalent we use the slowly oscillation functions.

A function $h : X \to [0, 1]$ is called $\mathcal{B}$-slowly oscillating if, for every $\varepsilon > 0$ and every $\alpha \in P$, there exists a bounded subset V of X such that

$$\operatorname{diam} h(B(x, \alpha)) < \varepsilon$$

for each $x \in X \setminus V$.

Proposition 3.1. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ballean, $q, r \in X^\sharp_\mathcal{B}$. Then $\check{q} = \check{r}$ if and only if $h^\beta(p) = h^\beta(q)$ for every $\mathcal{B}$-slowly oscillating function $h : X \to [0, 1]$.*

Proof. See [9] (Proposition 1).

Proposition 3.2. *Let $\mathcal{B} = (X, P, B)$ be an unbounded normal ballean, $q, r \in X^\sharp_\mathcal{B}$. Then $\check{q} = \check{r}$ if and only if for any $Q \in q$ and $R \in r$, there exists $\alpha \in P$ such that $B(Q, \alpha) \cap B(R, \alpha)$ is unbounded.*

Proof. See [4] (Lemma 4.2).

Proposition 3.3. *Let $\mathcal{B} = (X, P, B)$ be an unbounded normal ballean and let $q \in X^\sharp$. Then the family of subsets of the form*

$$\left\{ \check{r} \in \check{X}_{\mathcal{B}} : \bigcup_{\alpha \in Q} B(Q \setminus V_\alpha, \alpha) \in r \right\},$$

where $Q \in q$ and each subset V_α is bounded, is a base of the neighborhoods of the point $\check{q}$ in $\check{X}_{\mathcal{B}}$.

Proof. See [4, p. 15].

Proposition 3.4. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y \subseteq X$, $\alpha \in P$, $q \in X^\sharp$. If $B(Y, \alpha) \in q$, then there is $r \in Y^\sharp$ such that $q||r$.*

Proof. For each $Q \in q$, we denote $S_Q = B(Q, \alpha) \cap Y$ and note that the family $\{S_Q : Q \in q\}$ is contained in some ultrafilter $r \in Y^\sharp$. Clearly, $r||q$.

Proposition 3.4 is proved.

We note that, for a cellular ballean $\mathcal{B} = (X, P, B)$, corona $\check{X}_{\mathcal{B}}$ coincides with its binary corona (see [3], Chapter 8) and hence $\check{X}_{\mathcal{B}}$ is zero-dimensional.

Let $\mathcal{B} = (X, P, B)$ be a ballean, $A \subseteq X$, $p \in X^\sharp$ and $\bar{\bar{p}} = \{q \in X^\sharp : p||q\}$. A subset

$$\Delta_p(A) = \bar{\bar{p}} \cap A^\sharp$$

is called an *ultracompanion* of A. For ultracompanions of subsets of metric spaces, groups and G-spaces see [10 – 13].

Given a ballean $\mathcal{B} = (X, P, B)$ and a subset A of X, we say that the subset $\check{p} \cap A^\sharp$ is a corona companion of A.

4. Cardinal invariants. Given a ballean $\mathcal{B} = (X, P, B)$, a subset A of X is called

large if $X = B(A, \alpha)$ for some $\alpha \in P$;

small if $X \setminus B(A, \alpha)$ is large for every $\alpha \in P$;

thick if, for every $\alpha \in P$, there exists $a \in A$ such that $B(a, \alpha) \subseteq A$;

thin if, for every $\alpha \in P$, there exists a bounded subset V of X such that $B(a, \alpha) \cap B(a', \alpha) = \varnothing$ for all distinct $a, a' \in A \setminus V$.

We note that large, small, thick and thin subsets can be considered as asymptotic counterparts of dense, nowhere dense, open and discrete subsets of a uniform topological space. We use the following cardinal invariants of $\mathcal{B}$: *asymptotic density*, *thickness* and *spread* defined by

$\operatorname{asden} \mathcal{B} = \min\{|L| : L$ is a large subset of $X\}$,

$\operatorname{thick} \mathcal{B} = \sup\{|\mathcal{F}| : \mathcal{F}$ is a family of pairwise disjoint thick subsets of $X\}$,

$\operatorname{spread} \mathcal{B} = \sup\{|Y|_{\mathcal{B}} : Y$ is a thin subset of $X\}$, where $|Y|_{\mathcal{B}} = \min\{|Y \setminus V| : V$ is a bounded subset of $X\}$.

Theorem 4.1. *For every unbounded ordinal ballean $\mathcal{B}$ with the support X, we have*

$$\operatorname{asden} \mathcal{B} = \operatorname{thick} \mathcal{B} = \operatorname{spread} \mathcal{B},$$

and there exists a thin subset Y of X and a disjoint family $\mathcal{F}$ of thick subsets of X such that $|Y|_{\mathcal{B}} = |Y| = \operatorname{asden} \mathcal{B} = |\mathcal{F}|$.

Proof. See [14] (Theorem 3.1) and [15] (Theorem 2.3).

Theorem 4.2. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ordinal ballean and let $\kappa = asdens\,\mathcal{B}$. Then*

$$|\check{X}_{\mathcal{B}}| = 2^{2^{\kappa}}.$$

Proof. Let Z be a large subset of X such that $|Z| = \kappa$. By Proposition 3.4, $\check{p} \cap Z^{\sharp} \neq \varnothing$ for each $p \in X^{\sharp}$. Hence, $|X_{\mathcal{B}}| \leq |Z^{\sharp}| \leq \beta Z = 2^{2^{\kappa}}$.

To verify the inequality $|\check{X}_{\mathcal{B}}| \geq 2^{2^{\kappa}}$, we use Theorem 4.1 to find a thin subset Y of X such that $|Y|_{\mathcal{B}} = |Y| = \kappa$. Since Y is thin and $\mathcal{B}$ is normal, by Proposition 3.2, $\check{p} \neq \check{q}$ for any two distinct ultrafilters p, q from $Y^{\sharp}$. So it suffices to prove that $|Y^{\sharp}| = 2^{2^{\kappa}}$. We fix some $y_0 \in Y$ and consider two cases.

Case 1. There exists a cofinal subset $C = \{c_\alpha : \alpha < \lambda\}$ in P such that $|Y \cap (B(y_0, c_{\alpha+1}) \setminus B(y_0, c_\alpha))|$ and choose some ultrafilter p on C such that $\{c_\beta : \alpha < \beta < \lambda\} \in p$ for each $\alpha < \lambda$. If $q, q' \in \beta\kappa$ and $q \neq q'$ then

$$\text{p-lim} f_\alpha(q) \neq \text{p-lim} f_\alpha(q'),$$

so $|Y^{\sharp}| \geq \beta\kappa = 2^{2^{\kappa}}$.

Case 2. There exists $\beta \in P$ such that $|Y \cap (B(y_0, \alpha) \setminus B(y_0, \beta))| < \kappa$ for each $\alpha > \beta$. We put $Z = Y \setminus B(y_0, \beta)$ and note that $|Z| = \kappa$. By the choice of β, $Z^{\sharp}$ coincides with the set of all uniform ultrafilters on Z so $|Z^{\sharp}| = 2^{2^{\kappa}}$.

Theorem 4.2 is proved.

Let κ be an infinite cardinal, ϕ be a uniform ultrafilter on κ. We consider a ballean $\mathcal{B} = (\kappa, \phi, B)$, where, for any $F \in \phi$, $B(x, F) = \{x\}$ if $x \in F$ and $B(x, F) = X \setminus F$ if $x \notin F$. Clearly, $\mathcal{B}$ is normal but $\kappa^{\sharp} = \{\phi\}$ so $\check{\kappa}_{\mathcal{B}}$ is a singleton. On the other hand $asdens\,\mathcal{B} = \kappa$. So Theorem 4.2 does not hold for $\mathcal{B}$.

Recall that the *Souslin number* $s(X)$ of a topological space X is the supremum of cardinalities of disjoint families of open subsets of X.

Let κ be an infinite cardinal. A family $\mathcal{F}$ of subsets of κ is called *almost disjoint* if $|F| = \kappa$ for every $F \in \mathcal{F}$, and $|F' \cap F| < \kappa$ for all distinct $F, F' \in \mathcal{F}$. For $\kappa = \aleph_0$, there is an almost disjoint family of cardinality c. Baumgartner [16] proved that, for each κ, there is an almost disjoint family of cardinality κ^+, and it is independent of ZFC that if $\kappa = \aleph_1$ then there is no almost disjoint families of cardinality 2^{κ}.

Proposition 4.1. *Let κ be an infinite cardinal, $\mathcal{B} = (X, \kappa, P)$ be an unbounded ordinal ballean. Assume that there exists a subset $Y = \{y_\alpha : \alpha < \kappa\}$ of X such that*

$$B(B(y_\alpha, \alpha), \alpha) \cap B(B(y_\beta, \beta), \beta) = \varnothing$$

for all $\alpha < \beta < \kappa$. If $\mathcal{F}$ is an almost disjoint family of subsets of κ, then

$$s(\check{X}_{\mathcal{B}}) \geq |\mathcal{F}|.$$

Proof. For each $F \in \mathcal{F}$, we put

$$Y_F = \bigcup_{\alpha \in F} B(y_\alpha, \alpha),\ Z_F = \{\check{q} : Y_F \in q\}.$$

Applying Propositions 3.2 and 3.3 we conclude that $\{Z_F : F \in \mathcal{F}\}$ is a disjoint family of subsets of $\check{X}_{\mathcal{B}}$ and each Z_F has a nonempty interior.

Proposition 4.1 is proved.

Recall that the *density* $den\,X$ of a topological space X is the smallest cardinality of dense subset of X.

Proposition 4.2. *For every unbounded ballean $\mathcal{B} = (X, P, B)$, we have*

$$\operatorname{den}(\check{X}_{\mathcal{B}}) \leq 2^{\operatorname{asden} X}.$$

Proof. We take a large subset L of X of cardinality $asden\, X$ and denote by $\mathcal{F}$ the family of all unbounded subsets of L. For each $F \in \mathcal{F}$, we pick $q_F \in X^\sharp$ such that $F \in q_F$. Then $\{\check{q}_F : F \in \mathcal{F}\}$ is a dense subset of $\check{X}_{\mathcal{B}}$ and $|\{\check{q}_F : F \in \mathcal{F}\}| \leq 2^{|L|}$.

Proposition 4.2 is proved.

Theorem 4.3. *For every infinite cardinal κ, we have $asden \overleftrightarrow{\kappa} = \kappa$ and*

$$\kappa^+ \leq (\check{\kappa}) \leq \operatorname{den}(\check{\kappa}) \leq 2^\kappa,$$

where $\check{\kappa}$ is the corona of $\overleftrightarrow{\kappa}$.

Proof. By the definition of κ, each large subset of κ has cardinality κ. By Proposition 4.2, $\operatorname{den}(\check{\kappa}) \leq 2^\kappa$. In view of Proposition 4.1 and the Baumgartner theorem it suffices to construct corresponding subset Y.

We put $y_0 = 1$ and define a set $Y = \{y_\alpha : \alpha < \kappa\}$ recursively by $y_{\alpha+1} = y_\alpha + y_\alpha + y_\alpha + y_\alpha$ and $y_\beta = \sup\{y_\alpha : \alpha < \beta\}$ for every limit ordinal β.

Theorem 4.3 is proved.

For $\kappa = \aleph_0$, Proposition 4.1 gives more strong result

$$s(\check{\aleph}_0) = \operatorname{den}(\check{\aleph}_0) = \mathfrak{c}.$$

Recall that a character $\chi(x)$ of a topological space X at the point x is the minimal cardinality of bases of neighborhoods of x.

For a metric space X, $\check{X}$ denotes the corona of corresponding metric ballean. Under CH, if X is a countable ultrametric space then, by [17], $\check{X}$ is homeomorphic to $\omega^* = \beta\omega \setminus \omega$. By [18], this statement is independent of ZFC.

Theorem 4.4. *Let X be an unbounded metric space, $\kappa = \operatorname{asden} \check{X}$. Then $|\check{X}| = 2^{2^\kappa}$ and*

$$\mathfrak{c} \cdot \kappa \leq s(\check{X}) \leq \operatorname{den}(\check{X}) \leq 2^\kappa.$$

Proof. In view of Theorem 4.2 and Proposition 4.2, it suffices to verify only $\mathfrak{c} \cdot \kappa \leq s(\check{X})$. The inequality $\mathfrak{c} \leq s(\check{X})$ follows directly from Proposition 4.1. To prove $\kappa \leq s(\check{X})$, we use Theorem 3.1 and choose a disjoint family $\mathcal{F}$ of thick subsets of X such that $|\mathcal{F}| = \kappa$. For each $F \in \mathcal{F}$, use Proposition 3.3 to find a subset $X_F \subseteq F$ such that projections $p \mapsto \check{p}$ of $\{X_F^\sharp : F \in \mathcal{F}\}$ to corona are pairwise disjoint with nonempty interior.

Theorem 4.4 is proved.

***Corollary* 4.1.** *For an unbounded countable metric space X we have*

$$\operatorname{den}(\check{X}) = s(\check{X}) = \mathfrak{c}.$$

5. Companions.

Theorem 5.1. *Let $\mathcal{B}$ be a ballean with the support X. For a subset A of X, the following statements hold:*

(i) *A is large if and only if $\Delta_p(A) \neq \varnothing$ for each $p \in X^\sharp$;*

(ii) *A is thick if and only if $\bar{\bar{p}} = \Delta_p(A)$ for some $p \in X^\sharp$;*

(iii) *A is prethick if and only if there exist $p \in X^\sharp$ and $\alpha \in P$ such that $\bar{\bar{p}} = \Delta_p(B(A, \alpha))$;*

(iv) *A is thin if and only if $\Delta_p(A) \leq 1$ for each $p \in X^\sharp$.*

Proof. The theorem is proved in [10] (Theorems 4.1, 4.2, 4.3) for metric balleans, but the proof can be easily adopted to the general case.

Given $p \in X^\sharp$, we say [10] that a subset $S \subseteq \bar{\bar{p}}$ is *ultrabounded with respect to* p if there is $\alpha \in P$ such that, for each $q \in S$ and every $Q \in q$, we have $B(Q, \alpha) \in p$.

We say that a subset A of X is

sparse if $\Delta_p(A)$ is ultrabounded for each $p \in X^\sharp$;

scattered if, for every $Y \subseteq A$, there is $p \in Y^\sharp$ such that $\Delta_p(Y)$ is ultrabounded.

To prove Theorems 5.2 and 5.3, one can adopt the arguments from [10] and [12].

Theorem 5.2. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ordinal ballean. For a subset A of X, the following statements are equivalent:*

(i) *A is sparse;*

(ii) *for every unbounded subset Y of A, there exists $\beta \in P$ such that, for every $\alpha \in P$, we have*

$$\{y \in Y : B_A(y, \alpha) \setminus B_A(y, \beta) = \varnothing\} \neq \varnothing.$$

Theorem 5.3. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ordinal ballean. For a subset A of X, the following statements are equivalent*:

(i) *A is scattered;*

(ii) *for every unbounded subset Y of A, there exists $\beta \in P$ such that, for every $\alpha \in P$, we have*

$$\{y \in Y : B_Y(y, \alpha) \setminus B_Y(y, \beta) = \varnothing\} \neq \varnothing.$$

A ballean $\mathcal{B} = (X, P, B)$ is called *uniformly locally finite* if, for every $\alpha \in P$, there exists a natural number $n(\alpha)$ such that $|B(x, \alpha)| \leq n_\alpha$ for every $x \in X$. By [8] (Theorem 6), for every locally finite ballean $\mathcal{B} = (X, P, B)$, there exists a group G of permutations of X such that $\mathcal{B}$ is asymorphic to the ballean $\mathcal{B}(G, X, \mathfrak{F}_G)$ (see Example 2.4), where $\mathfrak{F}_G$ is the ideal of finite subsets of G.

The following statement is a part of Theorem 5.4 from [13].

Theorem 5.4. *Let $\mathcal{B} = (X, P, B)$ be a uniformly locally finite ballean with the support X. A subset A of X is scattered if and only if $\Delta_p(A)$ is discrete for each $p \in X^\sharp$.*

Now we discuss a possibility generalization of Theorem 5.4 to arbitrary balleans.

Let $\mathcal{B} = (X, P, B)$ be a ballean. For $p \in X^\sharp$ and $\alpha \in P$, we set

$$B(p, \alpha) = \{q \in X^\sharp : B(P', \alpha) \in q \text{ for each } P' \in p\}$$

and note that $\bar{\bar{p}} = \bigcup_{\alpha \in P} B(p, \alpha)$ and each subset $B(p, \alpha)$ is closed in $\bar{\bar{p}}$.

We say that a point $p \in X^\sharp$ is *ball isolated* if there exists $P' \in p$ and $\alpha \in P$ such that if $q \in \bar{\bar{p}}$ and $P' \in q$ then $q \in B(p, \alpha)$. Applying Proposition 3.4, it is easy to verify that if p is ball isolated then each point $q \in \bar{\bar{p}}$ is ball isolated. If $\mathcal{B}$ is uniformly locally finite then p is ball isolated if and only if p is an isolated point of the subset $\bar{\bar{p}}$ of $X^\sharp$.

Theorem 5.5. *Let $\mathcal{B} = (X, P, B)$ be a ballean, A be a subset of X. If each point $p \in A^\sharp$ is ball isolated, then A is scattered.*

Proof. We say that a subset $F \subseteq X^\sharp$ is *invariant* if $p \in F$ and $q \| p$ imply $q \in F$.

We take an arbitrary unbounded subset Y of A, denote by $\mathfrak{F}$ the family of all closed invariant subsets of $X^\sharp$ and put

$$\mathfrak{F}_Y = \{F \in Y^\sharp : F \in \mathfrak{F}\}.$$

By the Zorn lemma, there is a minimal by inclusion element $M \in \mathfrak{F}_Y$. We take an arbitrary $p \in M$ and show that $\Delta_p(Y)$ is ultrabounded. Assume the contrary and choose $q \in cl\bar{\bar{p}}$ such that $q \notin B(p, \alpha')$ for each $\alpha' \in P$. Since $q \in M$, by the minimality of M, $p \in cl(\bar{\bar{q}})$. By the assumption, p is ball isolated. We choose corresponding $P' \in p$ and $\alpha \in P$. We pick $q' \in \bar{\bar{q}}$ such that $P' \in q'$. Since $q' \| q$, there is $\beta \in P$ such that $q \in B(q', \beta)$ so $B(P', \beta) \in q$.

If $p' \in B(P', \beta)$ then, by Proposition 3.4, there is $p'' \in P'$ such that $p'' \in B(p', \beta)$. We choose $\gamma \in P$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma)$ for each $x \in X$. Then $q \in clB(p, \gamma)$ and $q \in B(p, \gamma)$, contradicting the choice of q.

Theorem 5.5 is proved.

***Question* 5.1.** *Let A be a scattered subset of X. Is every point $p \in A^\sharp$ ball isolated? By Theorem 5.4, this is so for each uniformly locally finite balleans but the question is open even for metric balleans.*

Recall that a topological space X is scattered if each nonempty subset Y of X has an isolated point in Y.

***Question* 5.2.** *Let $\mathcal{B}$ be a ballean with the support X, $A \subseteq X$. Assume that each subspace $\Delta_p(A)$, $p \in A^\sharp$ is sparse in $X^\sharp$. Is A a sparse subset of $\mathcal{B}$? By Theorem* 5.4 *and* [8] (*Theorem* 6), *this is so for every uniformly locally finite ballean because in this case each $\Delta_p(A)$ is discrete.*

Given a ballean $\mathcal{B} = (X, P, B)$ and a subset A of X, we remind that a subset $\check{p} \cap A^\sharp$ is a corona p-companion of A and characterize a size of A in terms of corona companions.

Theorem 5.6. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ordinal ballean, $A \subseteq X$. Then the following statements hold*:

(i) *A is large if and only if $\check{p} \cap A^\sharp \neq \varnothing$ for each $p \in X^\sharp$*;

(ii) *A is thick if and only if there exists $p \in X^\sharp$ such that $\check{p} \subseteq A^\sharp$*;

(iii) *A is thin if and only if $|\check{p} \cap A^\sharp| \leq 1$ for each $p \in X^\sharp$.*

Proof. (ii) Assume that A is thick. We may suppose that P is an infinite regular cardinal κ. We choose a κ-sequence $\{y_\alpha : \alpha < \kappa\}$ in A such that $B(y_\alpha, \alpha) \subseteq A$ and $B(y_\alpha, \alpha) \cap B(y_\beta, \beta) = \varnothing$ for each $\alpha < \beta < \kappa$. Then we pick an arbitrary ultrafilter $p \in X^\sharp$ such that $\{y_\alpha : \alpha < \kappa\} \in p$. By Proposition 3.2, we have $\check{p} \in A^\sharp$.

Suppose that $\check{p} \subseteq A^\sharp$ for some $p \in X^\sharp$. Given any $\alpha < \kappa$, there is $P \in p$ such that $B(P, \alpha) \in A$, because otherwise, by Proposition 3.4, we can find $q \in X^\sharp$ such that $p || q$ and $X \setminus A \in q$ so $\bar{\bar{p}} \nsubseteq A^\sharp$. Hence A is thick.

(i) It suffices to observe that A is large if and only if $X \setminus A$ is not thick and apply (ii).

(iii) Assume that there are two distinct ultrafilters $p, q \in X^\sharp$ such that $p \sim q$ and $A \in p$, $A \in q$. We choose $P \in p$, $Q \in q$ such that $P \subseteq A$, $Q \subseteq A$ and $P \cap Q = \varnothing$. By Proposition 3.2, there is $\alpha < \kappa$ such that $B(P, \alpha) \cap Q$ is unbounded. It follows that A is not thin.

If A is not thin, one can choose $\gamma < \kappa$ and two κ-sequences $\{x_\alpha : \alpha < \kappa\}$ and $\{y_\alpha : \alpha < \kappa\}$ such that $x_\alpha \neq y_\alpha$, $y_\alpha \in B(x_\alpha, \gamma)$ and $B(x_\alpha, \gamma) \cap B(x_\beta, \gamma) = \varnothing$ for all $\alpha < \beta < \kappa$. We take an ultrafilter $p \in X^\sharp$ such that $\{x_\alpha : \alpha < \kappa\} \in p$ and use Proposition 3.4 to find $q \in X^\sharp$ such that $q || p$ and $\{y_\alpha : \alpha < \kappa\} \in q$. Then $\{p, q\} \subseteq \check{p} \cap A^\sharp$.

Theorem 5.6 is proved.

Let G be an uncountable Abelian group, $\mathcal{B} = (G, [G]^{<\aleph_0})$. By [9] (Proposition 4), the corona $\check{G}_{\mathcal{B}}$ is a singleton so Theorem 4.2 does not hold for $\mathcal{B}$.

***Question* 5.3.** *Is Theorem* 5.6 *true for every unbounded normal ballean*?

1. *Roe J.* Lectures on coarse geometry. – Providence, R.I.: Amer. Math. Soc., 2003.
2. *Protasov I., Banakh T.* Ball structures and colorings of groups and graphs // Math. Stud. Monogr. Ser. – 2003. – **11**.
3. *Protasov I., Zarichnyi M.* General asymptology // Math. Stud. Monogr. Ser. – 2007. – Vol. 12.
4. *Protasov I. V.* Normal ball structures // Math. Stud. – 2003. – **20**. – P. 3 – 16.
5. *Petrenko O., Protasov I. V., Slobodianiuk S.* Asymptotic structures of cardinals // Appl. Gen. and Top. – 2014. – **15**. – P. 137 – 146.
6. *Protasov I. V., Tsvietkova A.* Decomposition of cellular balleans // Topology Proc. – 2010. – **36**. – P. 1 – 7.
7. *Banakh T., Protasov I., Repovs D., Slobodianiuk S.* Classifying homogeneous cellular ordinal balleans. – Preprint (http:axiv.org/abs/1409.3910).
8. *Petrenko O., Protasov I.V.* Balleans and G-spaces // Ukr. Math. J. – 2012. – **64**, № 3. – P. 344 – 350.
9. *Protasov I. V.* Coronas of balleans // Top. and Appl. – 2005. – **146**. – P. 149 – 161.
10. *Protasov I. V.* Ultrafilters on metric spaces // Top. and Appl. – 2014. – **164**. – P. 207 – 214.
11. *Protasov I., Slobodianiuk S.* Ultracompanions of subsets of groups // Comment. math. Univ. Carol. – 2014. – **55**. – P. 257 – 265.
12. *Banakh T., Protasov I., Slobodianiuk S.* Scattered subsets of groups // Ukr. Math. J. – 2015. – **67**, № 3. – P. 304 – 312.
13. *Protasov I., Slobodianiuk S.* On the subset combinatorics of G-spaces // Algebra and Discrete Math. – 2014. – **15**, № 1. – P. 98 – 109.
14. *Protasov I. V.* Cellularity and density of balleans // Appl. Gen. and Top. – 2007. – **8**. – P. 283 – 291.
15. *Filali M., Protasov I.V.* Spread of balleans // Appl. Gen. and Top. – 2008. – **9**. – P. 169 – 175.
16. *Baumgartner J.* Almost-disjoint sets, the dense set problem and the partition calculus // Ann. Math. Logic. – 1976. – **9**. – P. 401 – 439.
17. *Protasov I.V.* Coronas of ultrametric spaces // Comment. math. Univ. carol. – 2011. – **52**. – P. 303 – 307.
18. *Banakh T., Chervak O., Zdomsky L.* On character of points in the Higson corona of a metric space // Comment. math. Univ. carol. – 2013. – **50**. – P. 159 – 178.

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