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FINITE GROUPS WITH X-QUASIPERMUTABLE SYLOW SUBGROUPS

СКІНЧЕННІ ГРУПИ З X-КВАЗІПЕРЕСТАВНИМИ СИЛОВСЬКИМИ ПІДГРУПАМИ

Let $H \le E$ and X be subgroups of a finite group G. Then we say that H is X-*quasipermutable* (X_S-*quasipermutable*, respectively) in E provided that G has a subgroup B such that $E = N_E(H)B$ and H X-permutes with B and with all subgroups (with all Sylow subgroups, respectively) V of B such that $(|H|, |V|) = 1$. We analyze the influence of X-quasipermutable and X_S-quasipermutable subgroups on the structure of G. In particular, it is proved that if every Sylow subgroup P of G is $F(G)$-quasipermutable in its normal closure P^G in G, then G is supersoluble.

Нехай $H \le E$ і X — підгрупи скінченної групи G. Тоді говорять, що H є X-*квазіпереставною* (X_S-*квазіпереставною*, відповідно) в E, якщо G містить таку підгрупу B, що $E = N_E(H)B$ і H є X-переставною з B і з усіма підгрупами (з усіма силовськими підгрупами, відповідно) V з B такими, що $(|H|, |V|) = 1$. У даній роботі проаналізовано вплив X-квазіпереставних і X_S-квазіпереставних підгруп на будову G. Зокрема, доведено, що якщо кожна силовська підгрупа P із G є $F(G)$-квазіпереставною в її нормальному замиканні P^G в G, то G є надрозв'язною.

1. Introduction. Throughout this paper, all groups are finite and G always denotes a finite group. For any prime p we use C_p to denote a group of order p.

If $AB = BA$, then A is said to *permute* with B; if $G = AB$, then B is called a *supplement* of A to G; if $AB^x = B^xA$, for at least one element $x \in X \subseteq G$, then A is said to X-*permute* with B [1].

A large number of researches are connected with the study of subgroups H of G such that H permutes with some subgroups of H's supplement B in G. If, for example, H X-permutes with all subgroups of B, then H is called X-*semipermutable* in G [2]; if H permutes with all Sylow subgroups of B, then H is called SS-*quasinormal* in G [3]. Subgroups with a condition of such kind have been useful in the analysis of many aspects of the theory of finite groups.

In this paper, we introduce and analyze some applications of the following concept that cover the conditions of X-semipermutability and SS-quasinormality.

Definition 1.1. *Let $H \le E$ and X be subgroups of G. Then we say that H is X-quasipermutable* (X_S-*quasipermutable*, *respectively*) *in E provided G has a subgroup B such that $E = N_E(H)B$ and H X-permutes with B and with all subgroups* (*with all Sylow subgroups*, *respectively*) *V of B such that* $(|H|, |V|) = 1$.

***Example* 1.1.** Let p, q and r be different primes such that q divides $p-1$. Let $A = C_p \rtimes C_q$ be a non-Abelian group of order pq and R a simple $\mathbb{F}_rA$-module which is faithful for A. Let $G = R \rtimes A$. Then C_p, clearly, is R_S-quasipermutable in G. On the other hand, $|R| > r$ when $p > r$ and so C_p is not R-quasipermutable in G.

It is clear that every X-semipermutable subgroup and every SS-quasinormal subgroup of G are X_S-quasipermutable in G for any $X \subseteq G$. We shall show that the inverse statements are not true in general.

***Example* 1.2.** Let p, q and r be different primes such that qr divides $p-1$.

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(i) Let $G = (C_p \rtimes C_q) \times P$, where $C_p \rtimes C_q$ is a non-Abelian group of order pq and $C_p = \langle a\rangle$ and $P = \langle b\rangle$ are groups of order p. Then C_q is clearly quasipermutable in G, and for every $x \in G$, $\langle ab\rangle^x C_q \neq C_q\langle ab\rangle^x$. Thus C_q is not G-semipermutable in G.

(ii) Let $G = C_p \rtimes (C_q \times C_r)$, where $C_q \times C_r \leq \mathrm{Aut}(C_p)$. Then C_q is 1-quasipermutable in G. Assume that C_q is SS-quasinormal in G. For any supplement B of C_q to G we have $C_p \leq B$, so for every $1 \neq x \in C_p$ we have $C_qC_r^x = C_r^xC_q$, which implies that $G = C_r^G \leq N_G(C_q)$, so $C_q \leq C_G(C_p)$. This contradiction shows that C_q is not SS-quasinormal in G.

Our main goal here is to prove the following results.

Theorem A. *Let $X = F(G)$ be the Fitting subgroup of G and H a Hall X-quasipermutable subgroup of G. If $p > q$ for all primes p and q such that p divides $|H|$ and q divides $|G\colon H|$, then H is normal in G.*

***Corollary* 1.1** (see [1], Theorem 5.4)**.** *Let $X = F(G)$ be the Fitting subgroup of G and H a Hall X-semipermutable subgroup of G. If $p > q$ for all primes p and q such that p divides $|H|$ and q divides $|G : H|$, then H is normal in G.*

***Corollary* 1.2** (see [4], Theorem 3)**.** *If a Sylow p-subgroup P of G, where p is the largest prime dividing $|G|$, is 1-semipermutable in G, then P is normal in G.*

Theorem B. *Let $X = F(G)$ be the Fitting subgroup of G. If every Sylow subgroup P of G is X-quasipermutable in its normal closure P^G in G, then G is supersoluble.*

***Corollary* 1.3.** *If every Sylow subgroup P of G is 1-semipermutable in its normal closure P^G in G, then G is supersoluble.*

Note that if a subgroup H of G is 1-semipermutable in G, then H is 1-semipermutable in every subgroup of G containing H. Hence we get from Corollary 1.3 the following known result.

***Corollary* 1.4** (see [4], Theorem 5)**.** *If every Sylow subgroup of G is 1-semipermutable in G, then G is supersoluble.*

From Theorem B we also get the following result.

***Corollary* 1.5** (see [4], Theorem 1.11)**.** *If every Sylow subgroup of G is $F(G)$-quasipermutable in G, then G is supersoluble.*

We use $\mathcal{M}_\phi(G)$ to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathcal{M}_\phi(G)$.

Theorem C. *Let P be a Sylow p-subgroup of G and $X = O_{p',p(G)}$. Suppose that every number V of some fixed $\mathcal{M}_\phi(P)$ is X_S-quasipermutable in G.*

(i) *If $|P| > p$, then G is p-supersoluble.*

(ii) *If $(p-1, |G|) = 1$, then G is p-nilpotent.*

***Corollary* 1.6** (see [3], Theorem 1.1)**.** *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every number V of some fixed $\mathcal{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

***Corollary* 1.7.** *Let P be a Sylow p-subgroup of G and $X = F(G)$. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathcal{M}_\phi(P)$ is X_S-quasipermutable in G, then G is p-nilpotent.*

Proof. If $|P| = p$, then G is p-nilpotent by Burnside's theorem [6] (IV, 2.6). Otherwise, G is p-supersoluble by Theorem C. The hypothesis holds for $G/O_{p'}(G)$ (see Lemma 2.2 below) and so in the case when $O_{p'}(G) \neq 1$, $G/O_{p'}(G)$ is p-nilpotent by induction, which implies the p-nilpotency of G. Therefore we may assume that $O_{p'}(G) = 1$. But then, by Lemma 2.4(3) below, P is normal in G. Hence G is p-nilpotent by hypothesis.

From Corollary 1.7 we get the following corollary.

***Corollary* 1.8** (see [3], Theorem 1.2). *Let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathcal{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

2. Preliminaries. The first lemma is evident.

Lemma 2.1. *Let A, B and X be subgroups of G and N a normal subgroup of G. If A X-permutes with B, then AN/N (XN/N)-permutes with BN/N. Hence in the case when $X \leq N$, AN/N permutes with BN/N.*

Lemma 2.2. *Let H and X be subgroups of G and N a normal subgroup of G. Suppose that H is X-quasipermutable (X_S-quasipermutable, respectively) in G.*

(1) *If either H is a Hall subgroup of G or for every prime p dividing $|H|$ and for every Sylow p-subgroup H_p of H we have $H_p \nleq N$, then HN/N is (XN/N)-quasipermutable $\big((XN/N)_S$-quasipermutable, respectively$\big)$ in G/N.*

(2) *If H is 1_S-quasipermutable in G, then H permutes with some Sylow p-subgroup of G for all primes p such that $(|H|, p) = 1$.*

Proof. (1) By hypothesis there is a subgroup B of G such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (with all Sylow subgroups, respectively) L of B such that $(|H|, |L|) = 1$.

It is clear that $G/N = N_{G/N}(HN/N)(BN/N)$. Let K/N be any subgroup (any Sylow p-subgroup, respectively) of BN/N such that $(|HN/N|, |K/N|) = 1$. Then $K = (K \cap B)N$. Let B_0 be a minimal supplement of $K \cap B \cap N$ in $K \cap B$. Then $K/N = (K \cap B)N/N = B_0(K \cap B \cap N)N/N = B_0N/N$ and $K \cap B \cap N \cap B_0 \leq \Phi(B_0)$. Therefore $\pi(K/N) = \pi(B_0)$, so $(|HN/N|, |B_0|) = 1$. It follows that $(|H|, |B_0|) = 1$, so in the case when H is X-quasipermutable in G, H X-permutes with B_0 and hence HN/N (XN/N)-permutes with $K/N = B_0N/N$. Thus HN/N is (XN/N)-quasipermutable in G/N.

Finally, suppose that H is X_S-quasipermutable in G and K/N is a Sylow p-subgroup of BN/N. Then B_0 is a p-group, so $(|H|, p) = 1$ and for some Sylow p-subgroup B_p of B we have $B_0 \leq B_p$. Then $K/N = B_0N/N$ and hence HN/N (XN/N)-permutes with K/N. Thus HN/N is $(XN/N)_S$-quasipermutable in G/N.

(2) By [6] (VI, 4.6), there are Sylow p-subgroups P_1, P_2 and P of $N_G(H)$, B and G, respectively, such that $P = P_1P_2$. Hence H permutes with P.

Lemma 2.3. *Let A and B be subgroups of G such that $G = AB$. Then $G = AB^x$ for all $x \in G$.*

Proof. Let $x = ab$, where $a \in A$ and $b \in B$. Then $AB^x = AB^{ab} = AabBb^{-1}a^{-1} = ABa^{-1} = Ga^{-1} = G$.

We shall need in our proofs the following properties of p-supersoluble groups.

Lemma 2.4. (1) *If $G/\Phi(G)$ is p-supersoluble, then G is p-supersoluble* [6] (IV, 8.6).

(2) *Let N and R be distinct minimal normal subgroups of G. If G/N and G/R are p-supersoluble, then G is p-supersoluble.*

(3) *Let $A = G/O_{p'}(G)$. Then G is p-supersoluble if and only if $A/O_p(A)$ is an Abelian group of exponent dividing $p-1$, p is the largest prime dividing $|A|$ and $F(A) = O_p(A)$ is a normal Sylow subgroup of A.*

Proof. (2) This follows from the G-isomorphism $NR/N \simeq R$.

(3) Since G is p-supersoluble if and only if $G/O_{p'}(G)$ is p-supersoluble, we may assume without loss of generality that $O_{p'}(G) = 1$.

First assume that G is p-supersoluble. In this case $G/C_G(H/K)$ is an Abelian group of exponent dividing $p-1$ for any chief factor H/K of G of order divisible by p. On the other hand,

$$O_{p',p}(G) = O_p(G) = \cap\big\{C_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K)\big\}$$

by [8] (A, 13.2). Hence $G/O_p(G)$ is an Abelian group of exponent dividing $p-1$. Thus p is the largest prime dividing $|G|$ and $F(G) = O_p(G)$ is a normal Sylow p-subgroup of G.

Finally, if $G/O_p(G)$ is an Abelian group of exponent dividing $p-1$, then every chief factor H/K of G below $O_p(G)$ is cyclic by [8] (B, 9.8(d)). Hence G is supersoluble.

Lemma 2.5 [7]**.** *If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G\colon A_1|$, $|G\colon A_2|$, $|G\colon A_3|$ are pairwise coprime, then G is itself nilpotent.*

Lemma 2.6. *Let $G = P \rtimes E$, where P is the Sylow p-subgroup of G and E is a Sylow tower group. Suppose that for every Sylow subgroup Q of E there is a subgroup B of P such that $P = N_P(Q)B$ and Q permutes with all subgroups of B. Then G is p-supersoluble.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. It is clear that G is soluble and $|P| > p$. Let $p_1 > \ldots > p_t$ be the set of all prime divisors of $|E|$. Let P_i be a Sylow p_i-subgroup of E.

Let N be a normal subgroup of G. Then the hypothesis holds for G/N, so the choice of G and Lemma 2.4 imply that N is the only minimal normal subgroup of G and $N \nleq \Phi(G)$. Therefore $N = C_G(N) = F(G) = P$ by [8] (A, 15.2), so E is a maximal subgroup of G.

Assume that $|\pi(E)| > 2$. Then $t > 2$. Let E_i be a Hall p_i'-subgroup of E. Then the hypothesis holds for PE_i, so PE_i is p-supersoluble by the choice of G. Moreover, since $P = C_G(P)$ we have $O_{p'}(PE_i) = 1$. Therefore PE_i is supersoluble by Lemma 2.4(3), and $F(PE_i) = P$. Thus $PE_i/P \simeq E_i$ is an Abelian group of exponent dividing $p-1$. Therefore E has at least three Abelian subgroups E_i, E_j and E_k of exponent dividing $p-1$ whose indices $|E\colon E_i|$, $|E\colon E_j|$, $|E\colon E_k|$ are pairwise coprime. But then by Lemma 2.5, E is nilpotent, and every Sylow subgroup of E is an Abelian group of exponent dividing $p-1$. Hence E is an Abelian group of exponent dividing $p-1$, which implies that $|P| = p$. This contradiction shows that $|\pi(E)| = 2$.

Since E is a Sylow tower group, P_1 is normal in E and so $N_G(P_1) \cap P = 1$. Therefore P_1 permutes with all subgroups of P. If $P \leq N_G(P_2)$, then $PP_2 = P \times P_2$. Hence in this case $P_2 \leq C_G(P) = P$. This contradiction shows that $N_G(P_2) \cap P \neq P$, so there is a nonidentity subgroup $B < P$ such that $P_2B = BP_2$. Hence $BE = B(P_1P_2) = (P_1P_2)B = BE$ is a subgroup of G, which contradicts the maximality of $E = P_1P_2$.

Lemma 2.7 (see [9], Theorem E)**.** *Suppose that $G = AB$ and $P \leq O_p(A)$. Assume that every conjugate of P in A permutes with every Sylow q-subgroup of B for all primes $q \neq p$. Then P^G is soluble and the p-complements in P^G are nilpotent.*

Lemma 2.8 (see [10], Lemma 2.15)**.** *Let E be a normal nonidentity quasinilpotent subgroup of G. If $\Phi(G) \cap E = 1$, then E is the direct product of some minimal normal subgroups of G.*

Lemma 2.9. *Let H be a subnormal subgroup of G. If H is nilpotent, soluble, or a π-group, then H^G is nilpotent, soluble, or a π-group, respectively.*

Proof. See the proof of Theorem 2.2 in [11, Ch. 2].

3. Proofs of the results. ***Proof of Theorem A.*** Suppose that this theorem is false and let G be a counterexample of minimal order. Let π be the set of all prime divisors of H. By hypothesis, there is a subgroup B of G such that $G = N_G(H)B$ and H X-permutes with B and with every π'-subgroup of B. Let $x \in X$ such that $HB^x = B^xH$. Then $\langle H, B^x\rangle = HB^x$ and $G = N_G(H)B^x$ by Lemma 2.3. Therefore $H^G = H^{N_G(H)B^x} = H^{B^x} \leq HB^x$. Hence $H^G = H(H^G \cap B^x)$.

(1) *HN is normal in G for any nonidentity normal subgroup N of G. Hence $O_\pi(G) = 1$.*

It is clear that HN/N is a Hall π-subgroup of G/N and the hypothesis holds for $(G/N, HN/N)$ by Lemma 2.2. Hence HN/N is normal in G by the choice of G. Thus HN is normal in G. Since $O_\pi(G) \leq H$, it follows that in the case when $O_\pi(G) \neq 1$, $O_\pi(G)H = H$ is normal in G, contrary to the choice of G. Hence we have (1).

(2) *$F(G)$ is a π'-group.*

Since $O_\pi(F(G))$ is characteristic in $F(G)$, it is normal in G. Hence by (1), $O_\pi(F(G)) \leq$ $\leq O_\pi(G) = 1$.

(3) *$F(G) = O_p(G)$ for some prime $p \notin \pi$.*

Let p be a prime dividing $|F(G)|$ and P the Sylow p-subgroup of $F(G)$. Then by claim (2), $p \notin \pi$. Suppose that $P \neq F(G)$. Then $F(G) = P \times E$, where $E \neq 1$ is the Hall p'-subgroup of $F(G)$. Since P and E are characteristic in $F(G)$, both these subgroup are normal in G. But then HP and HE are normal in G by claim (1), so $H = HP \cap HE$ is normal in G. This contradiction shows that $F(G) = P$.

(4) *$F(G)$ is an elementary Abelian p-group.*

Assume that this is false. Then $\Phi(F(G)) \neq 1$. Since $\Phi(F(G))$ is characteristic in $F(G)$, it is normal in G. Hence by claim (1), $\Phi(F(G))H$ is normal in G. But $\Phi(F(G))H$ is π-soluble and so any two Hall π-subgroups of $\Phi(F(G))H$ are conjugate in $\Phi(F(G))H$. Therefore, by the Frattini argument, $G = (\Phi(F(G))H)N_G(H) = \Phi(F(G))N_G(H) = N_G(H)$ since $\Phi(F(G)) \leq \Phi(G)$, a contradiction. Hence we have (4).

(5) *$G \neq HB$.*

Suppose that $G = HB$. Without loss of generality we may assume that B is a minimal supplement of H in G. First assume that H permutes with all π'-subgroups of B. Then the hypothesis holds for every subgroup of G containing H. Therefore for every maximal subgroup V of B we have $V \leq N_G(H)$ by the choice of G, so V is the only maximal subgroup of B. Hence B is a cyclic group of order q^n for some prime q. It is clear that q is the smallest prime dividing $|G|$ and, in view of claim (1), $(H \cap B)^G = (H \cap B)^{HB} = (H \cap B)^H \leq H_G = 1$. Hence $H \cap B = 1$. Therefore $|G : HV| = q$, which implies that HV is normal in G. But then, since $V \leq N_G(H)$, H is normal in G. This contradiction shows that for some π'-subgroup A of B we have $HA \neq AH$. It follows that $F(G) \neq 1$. Moreover, since $G = HB$, $F(G) \leq B$ by claim (3). Hence by claim (4), the hypothesis holds for $(HF(G), H)$. Therefore, if $HF(G) \neq G$, then H is normal (and so characteristic) in $HF(G)$. Hence in this case H is normal in G by claim (1). Thus $HF(G) = G$ and so the minimality of B implies that $B = F(G)$. But then, by claim (4), $HA = AH$. This contradiction shows that we have (5).

(6) *H permutes with every subgroup of $B \cap O_p(G)$* (this directly follows from claim (4)).

(7) $O_p(G) = 1$.

Suppose that $F(G) = O_p(G) \neq 1$. Then:

(a) $O_p(G)N_G(H) = G$.

By claim (1), $HO_p(G)$ is normal in G. On the other hand, $HO_p(G)$ is p-soluble and so any two Hall π-subgroups of $HO_p(G)$ are conjugate in $HO_p(G)$. Therefore, by the Frattini argument, $G = (HO_p(G))N_G(H) = O_p(G)N_G(H)$.

(b) $H^G = H(H^G \cap O_p(G))$.

In view of (a) we have

$$H^G = H^{O_p(G)N_G(H)} = H^{O_p(G)} \leq HO_p(G),$$

so $H^G = H^G \cap HO_p(G) = H(H^G \cap O_p(G))$.

(c) $H^G \cap O_p(G)$ *is a subgroup of* B.

$H^G = H(H^G \cap B^x) = H(H^G \cap O_p(G))$ by (b). Hence $H^G \cap O_p(G) \leq B$ by claim (3).

Final contradiction for (7). In view of claims (6), (b) and (c), the hypothesis holds for H^G. Hence in the case when $H^G \neq G$, H is normal in H^G, which implies the normality H in G. Thus $H^G = G$. But then $G = H^G = H(H^G \cap B^x) = HB^x = HB$, which contradicts (5).

Final contradiction. Since $X = F(G) = O_p(G) = 1$ by claim (7), the hypothesis holds for $H^G = H(H^G \cap B) \leq G$. Hence $H^G = G$, which implies that $G = HB$, contrary to (5).

The theorem is proved.

***Proof of Theorem* B.** Suppose that this theorem is false and let G be a counterexample with $|G|$ minimal. Let R be a minimal normal subgroup of G. Then $X/X \cap R \simeq XR/R \leq F(G/R)$.

(1) *The hypothesis holds for* G/R. *Hence* G/R *is supersoluble.*

Let P be a Sylow p-subgroup of G and $D = P^G$. Suppose that $P \nleq R$. By hypothesis, $D = N_D(P)B$, where B is a subgroup of D such that P X-permutes with B and with all p'-subgroups of B. Then

$$(PR/R)^{G/R} = (PR)^G/R = P^G R/R =$$

$$= DR/R = (N_D(P)R/R)(BR/R) = N_{DR/R}(PR/R)(BR/R)$$

and PR/R (XR/R)-permutes with BR/R by Lemma 2.1.

Now, let $V/R \leq BR/R$, where $(p, |V/R|) = 1$. Let U be a minimal supplement to R in V. Then $U \cap R \leq \Phi(U)$, so $(p, |U|) = 1$. Then for some $x \in X$ we have $PU^x = U^x P$, so

$$(PR/R)(UR/R)^{xR} = (PR/R)(V/R)^{xR} = (V/R)^{xR}(PR/R),$$

where $xR \in XR/R \leq F(G/R)$. Therefore PR/R is $F(G/R)$-quasipermutable in $(PR/R)^{G/R}$, so the hypothesis holds for G/R. Thus G/R is supersoluble by the choice of G.

(2) G *is soluble.*

If $X \neq 1$, this follows from claim (1). Now assume that $X = 1$. Let p be the largest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then P is normal, and so, characteristic in P^G by Theorem A. Hence P is normal in G and so $P \leq X$, a contradiction.

(3) $R = X = C_G(R) = O_p(G)$ *for some prime* p, *and* $G = R \rtimes M$, *where* M *is a supersoluble maximal subgroup of* G.

Claim (1) and Lemma 2.4 imply that R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$, so $C_G(R) \leq R$. Thus we have (3) by claims (1), (2) and [8] (A, 17.2).

(4) p *is the largest prime dividing.*

Assume that this is false. Let q be the largest prime dividing $|G|$ and Q a Sylow q-subgroup of M. Then $D = Q^G = R \rtimes Q$ by claims (1) and (3). Moreover, $N_G(Q) = M$ by claim (3). Hence

$N_D(Q) = Q$. By hypothesis and claim (3), there is a subgroup B of D such that $D = QB$ and Q R-permutes with all p-subgroups of B. But, clearly, $R \leq B$. Hence Q is X-quasipermutable in D, so Q is normal in D by Theorem A. That implies that $Q \leq C_G(R)$, contrary to (3).

(5) *R is a Sylow p-subgroup of G* (this directly follows from claims (1), (3) and (4)).

Final contradiction. Let Q be any Sylow subgroup of M. Then Q is a Sylow subgroup of G and so, by hypothesis and claim (3), there is a subgroup B of G such that $Q^G = N_{Q^G}(Q)B$ and Q R-permutes with every p-subgroup of B. It is clear that $R = (R \cap N_{Q^G}(Q))(R \cap B) = N_R(Q)(R \cap B)$. Therefore G is p-supersoluble by Lemma 2.6, which implies that $|N| = p$. This contradiction completes the proof of the result.

***Proof of Theorem* C.** (i) Suppose that this assertion is false and let G be a counterexample of minimal order.

Let $V \in \mathcal{M}_\phi(P)$ and $D = V^G$. By hypothesis, there is a subgroup B of G such that $G = N_G(V)B$ and V is X-permutable with B and with all Sylow subgroups S of B such that $(p, |S|) = 1$.

(1) $O_{p'}(N) = 1$ *for every subnormal subgroup N of G. Hence $X \leq O_p(G)$.*

Indeed, suppose that for some subnormal subgroup N of G we have $O_{p'}(N) \neq 1$. Then $O_{p'}(G) \neq$ $\neq 1$ by Lemma 2.9, and the hypothesis holds for $G/O_{p'}(G)$ by Lemma 2.2. Hence $G/O_{p'}(N)$ is p-supersoluble by the choice of G. Thus G is p-supersoluble, a contradiction. Therefore $O_{p'}(N) = 1$. Therefore, since X is p-nilpotent, $X \leq O_p(G)$.

(2) *If L is a minimal normal subgroup of G, then $L \nleq \Phi(P)$.*

Indeed, in the case when $L \leq \Phi(P)$, we have $L \leq \Phi(G)$ and the hypothesis holds for G/L by Lemma 2.2. Hence G/L is p-supersoluble by the choice of L. Therefore G is p-supersoluble by Lemma 2.4(1), a contradiction.

(3) *D is soluble, so $O_p(G) \neq 1$.*

Assume that $O_p(G) = 1$. Then in view of claim (1), $X = 1$. Therefore V permutes with B and with all Sylow subgroups S of B such that $(p, |S|) = 1$. Therefore $D = V^G = V^{N_G(V)B} =$ $= V^B \leq VB$, so $D = V(D \cap B)$. Hence V^D is soluble by Lemma 2.7. But claim (1) implies that $O_{p'}(V^D) = 1$. Hence $O_p(V^D) \neq 1$, and $O_p(V^D) \leq O_p(G)$ by Lemma 2.9. Thus $O_p(G) \neq 1$, a contradiction.

(4) *P is not cyclic.*

Assume that P is cyclic. Claim (3) implies that for some minimal normal subgroup L of G we have $L \leq O_p(G) \leq P$. Then $|L| = p$, and since $L \nleq \Phi(P)$ by claim (2), we get $L = P$, contrary to the hypothesis.

(5) *Every normal p-soluble subgroup of G is supersoluble and p-closed* (see claim (5)(a) in the proof of Proposition in [12]).

(6) *G is not p-soluble* (this directly follows from claim (5)).

Final contradiction for (i). In view of claim (4), there is a subgroup $W \in \mathcal{M}_\phi(P)$ such that $V \neq W$. Then $P = VW$. In view of claims (3) and (6), $P \nleq D$. Hence V is a Sylow subgroup of D, so V is normal in D (and also in G) by claim (5). Similarly, W is normal in G. Hence P is normal in G, contrary to claim (6). This final contradiction completes the proof of assertion (i).

(ii) If $|P| = p$, then G is p-nilpotent by [6] (IV, 2.6). Let $|P| > p$ and H/K any chief factor of G of order divisible by p. Then $|H/K| = p$ by assertion (i), so $C_G(H/K) = G$ since $(p-1, |G|) = 1$. Hence G is p-nilpotent.

The theorem is proved.

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