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ON KROPINA CHANGE OF m-TH ROOT FINSLER METRICS

ПРО ЗАМІНУ КРОПІНОЇ ДЛЯ m-КОРЕНЕВИХ ФІНСЛЕРОВИХ МЕТРИК

We study the Kropina change for m-th root Finsler metrics. We find necessary and sufficient condition under which the Kropina change of an mth root Finsler metric is locally dually flat. Then we prove that the Kropina change of an mth root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

Розглянуто заміну Кропіної для m-кореневих фінслерових метрик. Встановлено необхідні та достатні умови того, що заміна Кропіної для m-кореневої метрики Фінслера є локально дуально плоскою. Також доведено, що заміна Кропіної для m-кореневої метрики Фінслера є локально проективно плоскою тоді і тільки тоді, коли вона є локально мінковською.

1. Introduction. Let M be an n-dimensional C^∞ manifold, TM its tangent bundle. Let $F = \sqrt[m]{A}$ be a Finsler metric on M, where A is given by $A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ with $a_{i_1 \ldots i_m}$ symmetric in all its indices [3, 8 – 11]. Then F is called an m-th root Finsler metric. Suppose that A_{ij} define a positive definite tensor and A^{ij} denotes its inverse. For an m-th root metric F, put

$$A_i = \frac{\partial A}{\partial y^i}, \qquad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \qquad A_{x^i} = \frac{\partial A}{\partial x^i}, \qquad A_0 = A_{x^i} y^i.$$

Then the following hold:

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2} \big[mAA_{ij} + (2-m) A_i A_j \big],$$

$$y^i A_i = mA, \qquad y^i A_{ij} = (m-1) A_j, \qquad y_i = \frac{1}{m} A^{\frac{2}{m}-1} A_i,$$

$$A^{ij} A_{jk} = \delta^i_k, \qquad A^{ij} A_i = \frac{1}{m-1} y^j, \qquad A_i A_j A^{ij} = \frac{m}{m-1} A.$$

Let (M, F) be a Finsler manifold. For a 1-form $\beta(x,y) = b_i(x) y^i$ on M, we have a change of Finsler which is defined by following:

$$F(x,y) \to \bar{F}(x,y) = f(F, \beta),$$

where $f(F,\beta)$ is a positively homogeneous function of F. This is called a β-change of metric. It is easy to see that, if $\|\beta\|_F := \sup_{F(x,y)=1} \left| b_i(x) y^i \right| < 1$, then $\bar{F}$ is again a Finsler metric [7].

In this paper, we consider a special case of β-change, namely

$$\bar{F}(x,y) = \frac{F^2(x,y)}{\beta(x,y)} \tag{1}$$

which is called the Kropina change. If F reduces to a Riemannian metric α, then $\bar{F}$ reduces to a Kropina metric $F = \dfrac{\alpha^2}{\beta}$. Due to this reason, the transformation (1) has been called the Kropina change of Finsler metrics. It is remarkable that, the Kropina metrics are closely related to physical theories. These metrics, was introduced by Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [5].

In [2], Amari, Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [6]. A Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies $(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}$.

In this paper, we find necessary and sufficient condition under which a Kropina change of an m-th root metric be locally dually flat.

Theorem 1.1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = \dfrac{F^2}{\beta}$ be Kropina change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following hold*:

$$\beta_{0l}\beta - 3\beta_l\beta_0 = 2\beta\beta_{x^l}, \tag{2}$$

$$A_{x^l} = \frac{1}{3m}[mA\theta_l + 4\theta A_l], \tag{3}$$

$$\beta_0 A_l = -\beta_l A_0, \tag{4}$$

where $\beta_{0l} = \beta_{x^k y^l} y^k$, $\beta_{x^l} = (b_i)_{x^l} y^i$, $\beta_0 = \beta_{x^l} y^i$ and $\beta_{0l} = (b_l)_0$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x,y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G^i = Py^i$, where $P(x, \lambda y) = \lambda P(x,y)$, $\lambda > 0$ [4].

In this paper, we prove that the Kropina change of an m-th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

Theorem 1.2. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = \dfrac{F^2}{\beta}$ be Kropina change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.*

2. Proof of Theorem 1.1. A Finsler metric $F = F(x,y)$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM such that $L = F^2(x,y)$ satisfies

$$L_{x^k y^l} y^k = 2L_{x^l}.$$

In this case, the coordinate (x^i) is called an adapted local coordinate system. It is easy to see that every locally Minkowskian metric satisfies in the above equation, hence is locally dually flat.

In this section, we are going to prove the Theorem 1.1. To prove it, we need the following lemma.

Lemma 2.1. *Suppose that the equation* $\Phi A^2 + \Psi A + \Theta = 0$ *holds, where* Φ, Ψ, Θ *are polynomials in* y *and* $m > 2$. *Then* $\Phi = \Psi = \Theta = 0$.

***Proof of Theorem* 1.1.** Let $\bar{F}$ be a locally dually flat metric. We have

$$\bar{F}^2 = \frac{A^{\frac{4}{m}}}{\beta^2}, \qquad (\bar{F}^2)_{x^k} = \frac{1}{\beta^2}\frac{4}{m}A^{\frac{4}{m}-1}A_{x^k} - \frac{2}{\beta^3}A^{\frac{4}{m}}\beta_{x^k},$$

$$(\bar{F}^2)_{x^k y^l}y^k = \frac{1}{\beta^2}\left[\frac{4}{m}A^{\frac{4}{m}-1}A_{0l} + \left(\frac{4}{m}\right)\left(\frac{4}{m}-1\right)A^{\frac{4}{m}-2}A_0A_l\right] -$$

$$-\frac{2}{\beta^3}\left[\frac{4}{m}A^{\frac{4}{m}-1}A_l\beta_0 + \frac{4}{m}A^{\frac{4}{m}-1}A_0\beta_l + A^{\frac{4}{m}}\beta_{0l} + \frac{6}{\beta^4}A^{\frac{4}{m}}\beta_0\beta_l\right].$$

Thus, we get

$$\frac{A^{\frac{4}{m}-2}}{\beta^4}\left[\frac{4}{m}\beta^2\left(\left(\frac{4}{m}-1\right)A_0A_l + AA_{0l} - 2AA_{x^l}\right) - \frac{8}{m}A\beta(A_l\beta_0 + A_0\beta_l) + \right.$$

$$\left. +2A^2\Big(3\beta_0\beta_l + 2\beta\beta_{x^l} - \beta\beta_{0l}\Big)\right] = 0.$$

By Lemma 2.1, we obtain

$$\left(\frac{4}{m}-1\right)A_lA_0 + AA_{0l} = 2AA_{x^l}, \tag{5}$$

$$\beta_0A_l = -A_0\beta_l, \tag{6}$$

$$\beta_{0l}\beta - 3\beta_l\beta_0 = 2\beta_{x^l}\beta. \tag{7}$$

One can rewrite (5) as follows:

$$A(2A_{x^l} - A_{0l}) = \left(\frac{4}{m}-1\right)A_lA_0. \tag{8}$$

Irreducibility of A and $\deg(A_l) = m-1$ imply that there exists a 1-form $\theta = \theta_l y^l$ on U such that

$$A_0 = \theta A. \tag{9}$$

Plugging (9) into (8), yields

$$A_{0l} = A\theta_l + \theta A_l - A_{x^l}. \tag{10}$$

Substituting (9) and (10) into (8) yields (3). The converse is a direct computation.

Theorem 1.1 is proved.

3. Proof of Theorem 1.2. A Finsler metric $F(x,y)$ on an open domain $U \subset \mathbb{R}^n$ is said to be locally projectively flat if its geodesic coefficients G^i are in the form $G^i(x,y) = P(x,y)y^i$, where $P\colon TU = U \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x,\lambda y) = \lambda P(x,y)$, $\lambda > 0$. We call $P(x,y)$ the projective factor of F.

In this section, we are going to prove the Theorem 1.2. To prove it, we need the following proposition.

Proposition 3.1. *Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, $n \geq 3$, where A is irreducible. Suppose that $\bar{F} = \dfrac{F^2}{\beta}$ be Kropina change of F where $\beta = b_i(x)y^i$. If $\bar{F}$ is projectively flat metric then it reduces to a Berwald metric.*

Proof. Let $\bar{F}$ be projectively flat metric. We have

$$\bar{F}_{x^k} = \frac{2}{m\beta}A^{\frac{2}{m}-1}A_{x^k} - \frac{1}{\beta^2}A^{\frac{2}{m}}\beta_{x^k},$$

$$\bar{F}_{x^k y^l}y^k = \frac{1}{\beta}\left[\frac{2}{m}A^{\frac{2}{m}-1}A_{0l} + \left(\frac{2}{m}\right)\left(\frac{2}{m}-1\right)A^{\frac{2}{m}-2}A_0A_l\right] -$$

$$-\frac{1}{\beta^2}\left[\frac{2}{m}A^{\frac{2}{m}-1}A_l\beta_0 + \frac{2}{m}A^{\frac{2}{m}-1}A_0\beta_l + A^{\frac{2}{m}}\beta_{0l} + \frac{2}{\beta^3}A^{\frac{2}{m}}\beta_0\beta_l\right].$$

Thus, we get

$$\frac{A^{\frac{2}{m}-2}}{\beta^3}\left[\frac{2}{m}\beta^2\left(\left(\frac{2}{m}-1\right)A_0A_l + AA_{0l} - AA_{x^l}\right) -\right.$$

$$\left.-\frac{2}{m}A\beta\big(A_l\beta_0 + A_0\beta_l\big) + A^2\big(2\beta_0\beta_l + \beta\beta_{x^l} - \beta\beta_{0l}\big)\right] = 0.$$

By Lemma 2.1, we obtain

$$mA(A_{0l} - A_{x^l}) = (m-2)A_0A_l.$$

Then irreducibility of A and $\deg(A_l) = m-1 < \deg(A)$ implies that A_0 is divisible by A. This means that, there is a 1-form $\theta = \theta_l y^l$ on U such that the following holds $A_0 = 2mA\theta$. Then $G^i = Py^i$, where $P = \theta$. Then F is a Berwald metric.

Proposition 3.1 is proved.

***Proof of Theorem* 1.2.** By Proposition 3.1, if F is projectively flat then it reduces to a Berwald metric. Now, if $n \geq 3$ then by Numata's theorem every Berwald metric of non-zero scalar flag curvature $\mathbf{K}$ must be Riemaniann. This is contradicts with our assumption. Then $\mathbf{K} = 0$, and in this case F reduces to a locally Minkowskian metric.

Theorem 1.2 is proved.

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