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T. Zhao (School Sci., Shandong Univ. Technology, China),
X. Zhang (School Math. Sci., Huaiyin Normal Univ., Jiangsu, China)

WEAKLY SS-QUASINORMAL MINIMAL SUBGROUPS AND THE NILPOTENCY OF A FINITE GROUP *

СЛАБКО SS-КВАЗІНОРМАЛЬНІ МІНІМАЛЬНІ ПІДГРУПИ ТА НІЛЬПОТЕНТНІСТЬ СКІНЧЕННОЇ ГРУПИ

A subgroup H is said to be an s-permutable subgroup of a finite group G provided that $HP = PH$ holds for every Sylow subgroup P of G, and H is said to be SS-quasinormal in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. We show that H is weakly SS-quasinormal in G if there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G. We investigate the influence of some weakly SS-quasinormal minimal subgroups on the nilpotency of a finite group G. Numerous results known from the literature are unified and generalized.

Підгрупа H називається s-переставною підгрупою скінченної групи G за умови, що $HP = PH$ виконується для кожної силовської підгрупи P групи G; H називається SS-квазінормальною в G, якщо існує доповнення B підгрупи H до G таке, що H можна переставити з кожною силовською підгрупою B. Показано, що H є слабко SS-квазінормальною в G, якщо існує нормальна підгрупа T групи G така, що HT є s-переставною, а $H \cap T$ є SS-квазінормальною в G. Досліджено вплив деяких слабко SS-квазінормальних мінімальних підгруп на нільпотентність скінченної групи G. Велику кількість відомих з літератури результатів упорядковано та узагальнено.

1. Introduction. All groups considered in this paper will be finite and we use conventional notions and notation, as in D. Gorenstein [7]. We use $\mathcal{F}$ to denote a formation, $\mathcal{N}$ and $\mathcal{N}_p$ denote the classes of all nilpotent groups and p-nilpotent groups, respectively. $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G, that is, $G^{\mathcal{F}} = \cap\{N \trianglelefteq G \big| G/N \in \mathcal{F}\}$. A normal subgroup N is said to be $\mathcal{F}$-hypercentral in G, provided that N has a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = N$ such that each N_{i+1}/N_i is an $\mathcal{F}$-central chief factor of G. The product of all $\mathcal{F}$-hypercentral subgroups of G is again an $\mathcal{F}$-hypercentral subgroup of G, it is denoted by $Z_{\mathcal{F}}(G)$ and called the $\mathcal{F}$-hypercenter of G. For the formation $\mathcal{N}$, we use the notation $Z_{\mathcal{N}}(G) = Z_{\infty}(G)$, which is the hypercenter of G.

In the study of group theory, from the generalized normalities of some primary subgroups to investigate the structures of a finite group is a common method. Recently, many new generalized normal subgroups were introduced successively. Following Kegel [12], a subgroup H is said to be s-permutable in G, if H is permutable with every Sylow subgroup P of G. As a development, in [13] the authors introduced that: a subgroup H is called an SS-quasinormal subgroup of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. Recently, in [8] Guo et al. introduced that: a subgroup H is said to be S-embedded in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained H. This concept integrated both the s-permutability and another related concept called c-normal subgroup, introduced by Wang in [18] and investigated extensively by many scholars. By assuming that some primary subgroups of G satisfying the s-permutability, SS-quasinormality or S-embedded properties, many interesting results have been derived (see, for example, [1, 8, 9, 13, 14, 16]).

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In order to unify and generalize the related results, in this paper, we introduce a new kind of generalized normal subgroup which can generalize both the SS-quasinormality and the S-embedded property (and so it contains the s-permutability and c-normality) properly.

Definition 1.1. *Let H be a subgroup of a finite group G, then H is said to be weakly SS-quasinormal in G, if there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G.*

***Remark* 1.1.** From the definition, it is easy to see that every S-embedded subgroup and SS-quasinormal subgroup of G is weakly SS-quasinormal in G. In general, a weakly SS-quasinormal subgroup of G need not to be S-embedded or SS-quasinormal in G. For instance, we let $G = S_5$ be the symmetric group of degree 5.

***Example* 1.1.** Let $H = S_4$ and $P \in \mathrm{Syl}_5(G)$. Since $HP = PH = G$, H is SS-quasinormal and thus weakly SS-quasinormal in G. Since the only nontrivial normal subgroups of G are A_5 and G, but neither H nor $H \cap A_5 = A_4$ is s-permutable in G, H is not S-embedded in G.

***Example* 1.2.** Let $K = \langle (12) \rangle$ and $T = A_5$. Since $T \trianglelefteq G$ is a complement of K, K is weakly SS-quasinormal in G. Since the only supplement of K to G are A_5 and G itself, but $K\langle(12345)\rangle \neq$ $\neq \langle(12345)\rangle K$, K is not SS-quasinormal in G.

From some minimal subgroup's normalities to characterize the structure of a finite group is an active topic in the group theory. A number of meaningful results have been obtained under the assumption that some minimal subgroups of G are well located. For example, Buckley [3] and Itô (see [11], III, 5.3) have got some well-known results about the supersolublity and nilpotency of a finite group, respectively. Since then, a series of papers have dealt with generalizations of the results of Itô and Buckley by using the theory of formations and some generalized normal subgroups (see, for example, [1, 2, 4, 10, 16]). In this paper, we investigate the influence of some weakly SS-quasinormal minimal subgroups on the structures of a finite group G. Some new results about the nilpotency of G are obtained, we also generalized some known ones.

2. Preliminaries. In this section, we list some basic results which will be useful in the sequel.

Lemma 2.1. *Let H be an s-permutable subgroup of G.*

(1) *If $K \leq G$, then $H \cap K$ is s-permutable in K.*

(2) *If $N \trianglelefteq G$, then HN/N is s-permutable in G/N.*

(3) *If H is a p-subgroup of G for some prime p, then $N_G(H) \geq O^p(G)$.*

Proof. The proof of the statements can be seen in [12] and [5].

Lemma 2.2 ([13], Lemma 2.1)**.** *Suppose that H is SS-quasinormal in a group G.*

(1) *If $H \leq K \leq G$, then H is SS-quasinormal in K.*

(2) *If $N \trianglelefteq G$, then HN/N is SS-quasinormal in G/N.*

Lemma 2.3 ([13], Lemma 2.2)**.** *Let P be a p-subgroup of G, p a prime. Then P is s-permutable in G if and only if $P \leq O_p(G)$ and P is SS-quasinormal in G.*

Now, we can prove that:

Lemma 2.4. *Suppose that H is weakly SS-quasinormal in a group G, $N \trianglelefteq G$.*

(1) *If $H \leq K \leq G$, then H is weakly SS-quasinormal in K.*

(2) *If $N \leq H$, then H/N is weakly SS-quasinormal in G/N.*

(3) *Let π be a set of primes, H a π-subgroup and N a normal π'-subgroup of G. Then HN/N is weakly SS-quasinormal in G/N.*

(4) *If $H \leq K \trianglelefteq G$, then G has a normal subgroup L contained in K such that HL is s-permutable and $H \cap L$ is SS-quasinormal in G.*

Proof. The statements (1), (2) and (4) can be deduced directly by Lemmas 2.1 and 2.2. Now we prove the statement (3). By hypotheses, there exists a normal subgroup T of G such that HT is s-permutable and $H \cap T$ is SS-quasinormal in G. It is easy to see that $TN/N \trianglelefteq G/N$, by Lemma 2.1(2) we know $(HN/N)(TN/N) = HTN/N$ is s-permutable in G/N. Since H is a π-group and N a π'-group,

$$|H \cap TN| = \frac{|H| \cdot |TN|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HT|_\pi} = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$. Hence $(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap \cap TN)N/N = (H \cap T)N/N$, which is SS-quasinormal in G/N by Lemma 2.2(2). Thus HN/N is weakly SS-quasinormal in G/N, as required.

Lemma 2.4 is proved.

The following results is well known, one can see [21] (Lemma 2.2) for example.

Lemma 2.5. *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then N lies in $Z(G)$.*

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*

(3) *If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.6. *Let $\mathcal{F}$ be a saturated formation containing the classes of all nilpotent groups $\mathcal{N}$, H a normal subgroup of G. If $G/H \in \mathcal{F}$ and $H \leq Z(G)$, then $G \in \mathcal{F}$.*

Proof. Let f and F be the canonical definitions of $\mathcal{N}$ and $\mathcal{F}$, respectively. Pick an chief factor M/N of G contained in H, then M/N is a p-group for some prime p. Since $M \leq H \leq Z(G)$, $M/N \leq Z(G/N)$. Thus $G/C_G(M/N) = 1 \in f(p)$. Since $\mathcal{N} \subseteq \mathcal{F}$, $f(p) \subseteq F(p)$ by [6] (IV, Proposition 3.11). It follows that $G/C_G(M/N) \in F(p)$. The arbitrary choice of M/N implies that there exists a normal chain of G contained in H such that every G-chief factor is $\mathcal{F}$-central. Since $G/H \in \mathcal{F}$, it follows that $G \in \mathcal{F}$.

Lemma 2.7 ([16], Lemma 2.8)**.** *Suppose that P is a normal p-subgroup of G contained in $Z_\infty(G)$, then $C_G(P) \geq O^p(G)$.*

Lemma 2.8 ([11], X. 13)**.** *Let $F^*(G)$ be the generalized Fitting subgroup of G.*

(1) *If M is a normal subgroup of G, then $F^*(M) \leq F^*(G)$.*

(2) *$F^*(G) \neq 1$, if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *If $K \leq Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

3. Main results.

Theorem 3.1. *Suppose that p is a prime divisor of a group G with $(|G|, p-1) = 1$, $P \in \mathrm{Syl}_p(G)$. If every cyclic subgroup of $P \cap G^{\mathcal{N}_p}$ with prime order or order 4 (if $p = 2$ and P is non-abelian) not having a p-nilpotent supplement in G is weakly SS-quasinormal in G, then G is a p-nilpotent group.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Then we have

(1) *Every proper subgroup of G is p-nilpotent, $G^{\mathcal{N}_p} = P$ is not a cyclic group.*

Let M be a proper subgroup of G. Since $M/(M \cap G^{\mathcal{N}_p}) \cong MG^{\mathcal{N}_p}/G^{\mathcal{N}_p} \leq G/G^{\mathcal{N}_p}$ is p-nilpotent, $M^{\mathcal{N}_p} \leq M \cap G^{\mathcal{N}_p}$. Now, let M_p be a Sylow p-subgroup of M. Without loss of generality, we may assume that $M_p \leq P$ and so $M_p \cap M^{\mathcal{N}_p} \leq P \cap G^{\mathcal{N}_p}$. By Lemma 2.4, we know every cyclic subgroup of $M_p \cap M^{\mathcal{N}_p}$ with prime order or order 4 (if $p = 2$ and M_p is non-abelian) not having a p-nilpotent supplement in M is weakly SS-quasinormal in M. Thus M satisfies the hypotheses of the theorem.

The minimal choice of G implies that M is p-nilpotent and so G is a minimal non-p-nilpotent group. By [11] (IV, Theorem 5.4), G has a normal Sylow p-subgroup P and a non-normal cyclic Sylow q-subgroup Q such that $G = PQ$; $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Moreover, P is of exponent p if $p > 2$ and exponent at most 4 if $p = 2$. On the other hand, the minimal choice of G implies that $G^{\mathcal{N}_p} = P$. By Lemma 2.5, we may also assume that P is not cyclic.

(2) *Some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.*

If every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then by [17] (Lemma 2.11) we know $P/\Phi(P)$ has a maximal subgroup which is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $|P/\Phi(P)| = p$ and so P is cyclic, this contradicts with (1). Thus there exists some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ such that $X/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.

(3) *$\langle x\rangle$ is weakly SS-quasinormal in G for any $x \in X\backslash\Phi(P)$.*

Let $x \in X\backslash\Phi(P)$, then by (1) we know $\langle x\rangle$ is a cyclic group of order p or 4. Let T be any supplement of $\langle x\rangle$ in G, then $G = \langle x\rangle T$ and $P = P\cap\langle x\rangle T = \langle x\rangle(P\cap T)$. Since $P/\Phi(P)$ is abelian, $(P\cap T)\Phi(P)/\Phi(P)\trianglelefteq G/\Phi(P)$ and hence $(P\cap T)\Phi(P)\trianglelefteq G$. Thus $P\cap T \le \Phi(P)$ or $P\cap T = P$, as $P/\Phi(P)$ is a chief factor of G. If $P\cap T \le \Phi(P)$ for some supplement T of $\langle x\rangle$ in G, then $P = \langle x\rangle$ is cyclic, this contradicts with (1). Now assume that $P\cap T = P$ for any supplement T. Then $T = G$ is the unique supplement of $\langle x\rangle$ in G. Since G is not p-nilpotent, $\langle x\rangle$ is weakly SS-quasinormal in G by the hypotheses.

(4) *The final contradiction.*

By (3) and Lemma 2.4(4), there exists a normal subgroup K of G contained in P such that $\langle x\rangle K$ is s-permutable and $\langle x\rangle\cap K$ is SS-quasinormal in G. Since $\langle x\rangle\cap K \le P = O_p(G)$, $\langle x\rangle\cap K$ is s-permutable in G by Lemma 2.3. Since $P/\Phi(P)$ is a chief factor of G, $K \le \Phi(P)$ or $K = P$. If $K \le \Phi(P)$, then $X/\Phi(P) = \langle x\rangle K\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction. If $K = P$, then $\langle x\rangle = \langle x\rangle\cap K$ is s-permutable in G and so $X/\Phi(P) = \langle x\rangle\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction too.

Theorem 3.1 is proved.

Next, by assuming that some minimal subgroups lie in the hypercenter of G and some cyclic subgroups of order 4 having the weakly SS-quasinormal properties, we give out some criteria about the nilpotency of a group G.

Theorem 3.2. *Let E be a normal subgroup of G such that G/E is nilpotent. If every minimal subgroup of E is contained in $Z_\infty(G)$ and every cyclic subgroup of E with order 4 is weakly SS-quasinormal in G or also lies in $Z_\infty(G)$, then G is nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Then we have

(1) *Every proper subgroup of G is nilpotent.*

Let K be an arbitrary proper subgroup of G. Since G/E is nilpotent, $K/K\cap E \cong KE/E$ is nilpotent. Let H be a minimal subgroup of $K\cap E$, then $H \le Z_\infty(G)\cap K \le Z_\infty(K)$. For any cyclic subgroup U of $K\cap E$ of order 4, by hypotheses U is weakly SS-quasinormal in G or lies in $Z_\infty(G)$. Then by Lemma 2.4, U is weakly SS-quasinormal in K or lies in $Z_\infty(G)\cap K \le Z_\infty(K)$. Thus $(K, K\cap E)$ satisfies the hypotheses of the theorem in any case. The minimal choice of G implies that K is nilpotent, thus G is a minimal non-nilpotent group. By [11] (II, Theorem 5.2), we can deduce that $G = PQ$, where P is a normal Sylow p-subgroup and Q a non-normal cyclic Sylow q-subgroup of G; $P/\Phi(P)$ is a chief factor of G; $\exp(P) = p$ or 4.

(2) $p = 2$, $\exp(P) = 4$ *and every cyclic subgroup of $P \le E$ with order 4 is weakly SS-quasinormal in G.*

Since G/E is nilpotent and $G/P \cap E \lesssim G/P \times G/E$, $G/P \cap E$ is nilpotent. If $P \nleq E$, then $P \cap E < P$ and $Q(P \cap E) < G$. Thus $Q(P \cap E)$ is nilpotent by (1), then $Q(P \cap E) = Q \times (P \cap E)$ and $Q \operatorname{char} Q(P \cap E)$. On the other hand, $G/P \cap E = P/P \cap E \times Q(P \cap E)/P \cap E$, it follows that $Q(P \cap E)/P \cap E \trianglelefteq G/P \cap E$ and $Q(P \cap E) \trianglelefteq G$. Therefore, $Q \trianglelefteq G$ and $G = P \times Q$, a contradiction. Thus we have $P \leq E$. Since P is a normal Sylow p-subgroup of G, all elements of order p or 4 (if $p = 2$) of G are contained in P and so contained in E. If $p > 2$ or $p = 2$ and every cyclic subgroup of P with order 4 lies in $Z_\infty(G)$, then by (1) and hypotheses, $P \leq Z_\infty(G)$. Therefore, Lemma 2.7 implies that $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus by hypotheses, we know that (2) holds.

(3) *Every $x \in P \backslash \Phi(P)$ is weakly SS-quasinormal in G.*

If there exists some $x \in P \backslash \Phi(P)$ such that $o(x) = 2$, we denote $M = \langle x \rangle^G \leq P$, then $M\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $M \nsubseteq \Phi(P)$, $P = M\Phi(P) = M \leq Z_\infty(G)$. Therefore, Lemma 2.7 implies that $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus every $x \in P \backslash \Phi(P)$ is of order 4. By (2), we know $\langle x \rangle$ is weakly SS-quasinormal in G.

(4) *Some minimal subgroup of $P/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.*

If every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$, then by [17] (Lemma 2.11) we know $P/\Phi(P)$ has a maximal subgroup which is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $|P/\Phi(P)| = p$. Since $\exp(P) = 4$, P is a cyclic group of order 4. Then Lemma 2.5 implies that $Q \trianglelefteq G$ and so G is nilpotent, a contradiction. Thus some minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ is not s-permutable in $G/\Phi(P)$.

(5) *The final contradiction.*

Let $x \in X \backslash \Phi(P)$, then by (3) we know that x is of order 4 and $\langle x \rangle$ is weakly SS-quasinormal in G. Thus there exists a normal subgroup K of G contained in P such that $\langle x \rangle K$ is s-permutable and $\langle x \rangle \cap K$ is SS-quasinormal in G. Since $\langle x \rangle \cap K \leq P = O_p(G)$, by Lemma 2.3 we know $\langle x \rangle \cap K$ is s-permutable in G. Since $P/\Phi(P)$ is a chief factor of G, $K \leq \Phi(P)$ or $K = P$. If $K \leq \Phi(P)$, then $X/\Phi(P) = \langle x \rangle K\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction. If $K = P$, then $\langle x \rangle = \langle x \rangle \cap K$ is s-permutable in G and so $X/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$, a contradiction too.

Theorem 3.2 is proved.

Now, we can prove that:

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$. If every minimal subgroup of $G^{\mathcal{F}}$ lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G, then $G \in \mathcal{F}$ if and only if every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is weakly SS-quasinormal in G.*

Proof. The *necessity* is obvious, we need to prove only the *sufficiency.*

Let $\langle x \rangle$ be a minimal subgroup of $G^{\mathcal{F}}$, then $\langle x \rangle \leq Z_{\mathcal{F}}(G) \cap G^{\mathcal{F}}$ which is contained in $Z(G^{\mathcal{F}})$ by [6] (IV, 6.10). From Lemma 2.4, we know that every cyclic subgroup of $G^{\mathcal{F}}$ with order 4 is weakly SS-quasinormal in $G^{\mathcal{F}}$. Theorem 3.2 implies that $G^{\mathcal{F}}$ is nilpotent and so it is soluble. If $G^{\mathcal{F}} \leq \Phi(G)$, then $G/\Phi(G) \in \mathcal{F}$, hence $G \in \mathcal{F}$. Thus we may assume that there exists a maximal subgroup M of G such that $G = MG^{\mathcal{F}} = MF(G)$. By [6] (IV, 1.17), we know $M^{\mathcal{F}} \leq G^{\mathcal{F}}$. Hence every minimal subgroup of $M^{\mathcal{F}}$ is contained in $Z_{\mathcal{F}}(G) \cap M \leq Z_{\mathcal{F}}(M)$. By Lemma 2.4, every cyclic subgroup of $M^{\mathcal{F}}$ with order 4 is SS-quasinormal in M. Therefore, M satisfies the hypotheses of the theorem. Then $M \in \mathcal{F}$ by induction. From [1] (Theorem 1 and Proposition 1), we know $G^{\mathcal{F}}$ is a p-group for some prime p ; $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$; $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.

If $\exp(G^{\mathcal{F}}) = p$, then $G^{\mathcal{F}} = \Omega_1(G^{\mathcal{F}}) \leq Z_{\mathcal{F}}(G)$ by the hypotheses, this would imply that $G \in \mathcal{F}$. Thus we may assume that $p = 2$ and $exp(G^{\mathcal{F}}) = 4$. If there exists some $x \in G^{\mathcal{F}} \backslash \Phi(G^{\mathcal{F}})$ such that $o(x) = 2$, denote $H = \langle x \rangle^G$, then $H \trianglelefteq G$ and $H \leq \Omega_1(G^{\mathcal{F}}) \leq Z_{\mathcal{F}}(G)$. On the other hand, $G^{\mathcal{F}} = H\Phi(G^{\mathcal{F}}) = H$ as $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathcal{F}})$. In this case, $G \in \mathcal{F}$. Next we assume that every $x \in G^{\mathcal{F}} \backslash \Phi(G^{\mathcal{F}})$ is of order 4, and so by hypotheses $\langle x \rangle$ is weakly SS-quasinormal in G. Let $X/\Phi(G^{\mathcal{F}})$ be an arbitrary minimal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ and $x \in X \backslash \Phi(G^{\mathcal{F}})$. Then there exists a normal subgroup K of G contained in $G^{\mathcal{F}}$ such that $\langle x \rangle K$ is s-permutable and $\langle x \rangle \cap K$ is SS-quasinormal in G. Since $\langle x \rangle \cap K \leq G^{\mathcal{F}} \leq O_2(G)$, $\langle x \rangle \cap K$ is s-permutable in G by Lemma 2.3. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $K \leq \Phi(G^{\mathcal{F}})$ or $K = G^{\mathcal{F}}$. If $K \leq \Phi(G^{\mathcal{F}})$, then $X/\Phi(G^{\mathcal{F}}) = \langle x \rangle K\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. If $K = G^{\mathcal{F}}$, then $\langle x \rangle = \langle x \rangle \cap K$ is s-permutable in G and we also deduce that $X/\Phi(G^{\mathcal{F}}) = \langle x \rangle \Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. This means that every minimal subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. Then by [17] (Lemma 2.11) we know $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$. Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = 2$. By Lemma 2.5, we know $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}}) \leq Z(G/\Phi(G^{\mathcal{F}}))$. Since $(G/\Phi(G^{\mathcal{F}}))/(G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})) \cong G/G^{\mathcal{F}} \in \mathcal{F}$, Lemma 2.6 implies that $G/\Phi(G^{\mathcal{F}}) \in \mathcal{F}$. Since $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$ and $\mathcal{F}$ is an saturated formation, $G \in \mathcal{F}$, as desired.

Theorem 3.3 is proved.

Theorem 3.4. *A group G is nilpotent if and only if every minimal subgroup of $F^*(G^{\mathcal{N}})$ lies in $Z_\infty(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{N}})$ with order 4 is weakly SS-quasinormal in G.*

Proof. The *necessity* is obvious, we need to prove only the *sufficiency*. Suppose that the result is false and let G be a counterexample of minimal order. Then

(1) *Every proper normal subgroup of G is nilpotent.*

Let M be a proper normal subgroup of G. Since $M/(M \cap G^{\mathcal{N}}) \cong MG^{\mathcal{N}}/G^{\mathcal{N}} \leq G/G^{\mathcal{N}}$ is nilpotent and $M^{\mathcal{N}} \trianglelefteq M \cap G^{\mathcal{N}} \trianglelefteq G^{\mathcal{N}}$, Lemma 2.8 implies that $F^*(M^{\mathcal{N}}) \leq F^*(M \cap G^{\mathcal{N}}) \leq F^*(G^{\mathcal{N}})$. Moreover, $M \cap Z_\infty(G) \leq Z_\infty(M)$. Now we can see easily that M satisfies the hypotheses of the theorem. The minimal choice of G implies that M is nilpotent.

(2) *$F(G)$ is the unique maximal normal subgroup of G.*

Let M be a maximal normal subgroup of G, then M is nilpotent by (1). Since the classes of all nilpotent groups formed a Fitting class, the nilpotency of M implies that $M = F(G)$ is the unique maximal normal subgroup of G.

(3) *$G^{\mathcal{N}} = G = G'$ and $F^*(G) = F(G) < G$.*

If $G^{\mathcal{N}} < G$, then $G^{\mathcal{N}}$ is nilpotent by (1). Thus, $F^*(G^{\mathcal{N}}) = G^{\mathcal{N}}$ by Lemma 2.8. Now Theorem 3.2 implies immediately that G is nilpotent, a contradiction. Hence, we must have $G^{\mathcal{N}} = G$. Since $G^{\mathcal{N}} \leq G'$, it follows that $G' = G$. Hence $G/F(G)$ cannot be cyclic of prime order. Thus $G/F(G)$ is a non-abelian simple group. If $F(G) < F^*(G)$, then $F^*(G^{\mathcal{N}}) = F^*(G) = G$ by (2). Again by Theorem 3.2, we can deduce that G is nilpotent, which is a contradiction.

(4) *The final contradiction.*

Since $F(G) = F^*(G) \neq 1$, we may choose the smallest prime divisor p of $|F(G)|$ such that $O_p(G) \neq 1$. Then for any Sylow q-subgroup Q of G ($q \neq p$), we consider the subgroup $G_0 = O_p(G)Q$. It is clear that $G_0^{\mathcal{N}} \leq O_p(G)$ and $G_0 \cap Z_\infty(G) \leq Z_\infty(G_0)$. Hence, every minimal subgroup of $G_0^{\mathcal{N}}$ lies in $Z_\infty(G_0)$ and every cyclic subgroup of $G_0^{\mathcal{N}}$ with order 4 is weakly SS-quasinormal in G_0. By Theorem 3.2, we know G_0 is nilpotent. Hence, $G_0 = O_p(G) \times Q$ and $Q \leq C_G(O_p(G))$. Consequently, $G/C_G(O_p(G))$ is a p-group. Thus we have $C_G(O_p(G)) = G$ by (3), namely $O_p(G) \leq Z(G)$. Now we consider the factor group $\overline{G} = G/O_p(G)$. First we have $F^*(\overline{G}) =$

$= F^*(G)/O_p(G)$ by Lemma 2.8(4). Besides that, for any element $\overline{x}$ of odd prime order in $F^*(\overline{G})$, since $O_p(G)$ is the Sylow p-subgroup of $F^*(G)$, $\overline{x}$ can be viewed as the image of an element x of odd prime order in $F^*(G)$. It follows that x lies in $Z_\infty(G)$ and $\overline{x}$ lies in $Z_\infty(\overline{G})$, as $Z_\infty(G/O_p(G)) = = Z_\infty(G)/O_p(G)$. This shows that $\overline{G}$ satisfies the hypotheses of the theorem. By the minimal choice of G, we can conclude that $\overline{G}$ is nilpotent and so is G.

Theorem 3.4 is proved.

Now, we can get a more precise result:

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$. Then $G \in \mathcal{F}$ if and only if every minimal subgroup of $F^*(G^{\mathcal{F}})$ lies in $Z_{\mathcal{F}}(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is weakly SS-quasinormal in G.*

Proof. Only the *sufficiency* needs to be verified. By [6] (IV, 6.10), $G^{\mathcal{F}} \cap Z_{\mathcal{F}}(G) \leq Z(G^{\mathcal{F}}) \leq \leq Z_\infty(G^{\mathcal{F}})$. Consequently, every minimal subgroup of $F^*(G^{\mathcal{F}})$ is contained in $Z_\infty(G^{\mathcal{F}})$. By the hypotheses and Lemma 2.4, every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is weakly SS-quasinormal in $G^{\mathcal{F}}$. By Theorem 3.4, we see that $G^{\mathcal{F}}$ is nilpotent and so $F^*(G^{\mathcal{F}}) = G^{\mathcal{F}}$. Now by Theorem 3.3, we can deduce that $G \in \mathcal{F}$, as required.

Theorem 3.5 is proved.

4. Applications. Since all normal, quasinormal, s-permutable, c-normal, SS-quasinormal, nearly s-normal [19] and S-embedded subgroups of G are weakly SS-quasinormal in G, our results have many meaningful corollaries. Here, we list some of them.

***Corollary* 4.1** (see [20])**.** *G is 2-nilpotent if every cyclic subgroup of G with order 2 or order 4 is c-normal in G.*

***Corollary* 4.2.** *Let p be a prime divisor of G with $(|G|, p-1) = 1$, $P \in \mathrm{Syl}_p(G)$. If every cyclic subgroup of $P \cap G^{\mathcal{N}_p}$ with prime order or order 4 (if $p = 2$ and P is non-abelian) not having a p-nilpotent supplement in G is SS-quasinormal (nearly s-normal, S-embedded) in G, then G is a p-nilpotent group.*

***Corollary* 4.3** (see [2])**.** *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$. Let G be a group such that every element of $G^{\mathcal{F}}$ of order 4 is c-normal in G. Then G belongs to $\mathcal{F}$ if and only if $\langle x \rangle$ lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G for every element $x \in G^{\mathcal{F}}$ of order 2.*

***Corollary* 4.4** (see [15])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if $G^{\mathcal{F}}$ is solvable and every element of order 4 of $F(G^{\mathcal{F}})$ is c-normal in G and x lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G for every element x of prime order of $F(G^{\mathcal{F}})$.*

***Corollary* 4.5** (see [16])**.** *Suppose N is a normal subgroup of a group G such that G/N is nilpotent. Suppose every minimal subgroup of N is contained in $Z_\infty(G)$, every cyclic subgroup of order 4 of N is s-permutable in G or lies also in $Z_\infty(G)$, then G is nilpotent.*

***Corollary* 4.6** (see [14])**.** *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$, and let G be a group. Every cyclic subgroup of order 4 of $G^{\mathcal{F}}$ (or $F^*(G^{\mathcal{F}})$) is SS-quasinormal in G. Then G belongs to $\mathcal{F}$ if and only if every subgroup of prime order of $G^{\mathcal{F}}$ (or $F^*(G^{\mathcal{F}})$) lies in the $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ of G.*

***Corollary* 4.7.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{N}$. Then $G \in \mathcal{F}$ if and only if every minimal subgroup of $F^*(G^{\mathcal{F}})$ lies in $Z_{\mathcal{F}}(G)$ and every cyclic subgroup of $F^*(G^{\mathcal{F}})$ with order 4 is nearly s-normal or S-embedded in G.*

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