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FUNCTIONS AND VECTOR FIELDS ON $C(\mathbb{C}P^n)$-SINGULAR MANIFOLDS
ФУНКЦІЇ І ВЕКТОРНІ ПОЛЯ НА $C(\mathbb{C}P^n)$-СИНГУЛЯРНИХ МНОГОВИДАХ

Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold. We study functions and vector fields with isolated singularities on M^{2n+1}. A $C(\mathbb{C}P^n)$-singular manifold is obtained from a smooth manifold M^{2n+1} with boundary which is a disjoint union of complex projective spaces $\mathbb{C}P^n \cup \mathbb{C}P^n \cup \ldots \cup \mathbb{C}P^n$ and subsequent capture of the cone over each component of the boundary. Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points. The Euler characteristic of M^{2n+1} is equal to $\chi(M^{2n+1}) = \dfrac{k(1-n)}{2}$. Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Suppose that, on M^{2n+1}, there exists an almost smooth vector field $V(x)$ with finite number of zeros $m_1, \ldots, m_k, x_1, \ldots, x_l$. Then $\chi(M^{2n+1}) = \sum_{i=1}^{l} \operatorname{ind}(x_i) + \sum_{i=1}^{k} \operatorname{ind}(m_i)$.

Нехай $M^{2n+1} - C(\mathbb{C}P^n)$-сингулярний многовид. Ми вивчаємо функції і векторні поля з ізольованими сингулярностями на M^{2n+1}. $C(\mathbb{C}P^n)$-сингулярний многовид виникає з гладкого многовиду M^{2n+1} з краєм, який є незв'язним об'єднанням комплексного проективного простору $\mathbb{C}P^n \cup \mathbb{C}P^n \cup \ldots \cup \mathbb{C}P^n$ і послідовності конусів над кожною компонентою краю. Нехай M^{2n+1} — компактний $C(\mathbb{C}P^n)$-сингулярний многовид із k сингулярними точками. Ейлерова характеристика M^{2n+1} дорівнює $\chi(M^{2n+1}) = \dfrac{k(1-n)}{2}$. Нехай $M^{2n+1} - C(\mathbb{C}P^n)$-сингулярний многовид із сингулярними точками $m_1, \ldots, m_k$. Припустимо, що на M^{2n+1} існує майже гладке векторне поле $V(x)$ із скінченним числом нулів $m_1, \ldots, m_k, x_1, \ldots, x_l$. Тоді $\chi(M^{2n+1}) = \sum_{i=1}^{l} \operatorname{ind}(x_i) + \sum_{i=1}^{k} \operatorname{ind}(m_i)$.

1. The functions on $C(\mathbb{C}P^n)$-manifolds. Let M^{2n+2} be a closed smooth manifold with semifree S^1-action

$$\theta : S^1 \times M^{2n+2} \to M^{2n+2}$$

which has only isolated fixed points. It is known that every isolated fixed point m of a semifree S^1-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n+2}$. More precisely, for every isolated fixed point m there exist an open invariant neighborhood U of m and a diffeomorphism h from U to an open unit disk D^{2n+2} in $\mathbb{C}^{n+1}$ centered at origin such that h is conjugate to the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1, \ldots, 1)$. We will use both complex, $(z_1, \ldots, z_{n+1})$, and real coordinates $(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$ on $\mathbb{C}^n = \mathbb{R}^{2n+2}$ with $z_j = x_j + \sqrt{-1} y_j$. The pair (U, h) will be called a **standard chart** at the point m.

Let M^{2n+2} be a manifold with finite many fixed points $m_1, \ldots, m_{2k}$. Denote by

$$\pi \colon M^{2n+2} \to M^{2n+2}/S^1$$

the canonical map. The set of orbits $N^{2n+1} = M^{2n+2}/S^1$ is a manifold with singular points $\pi(m_1), \ldots, \pi(m_{2k})$. It is clear that a neighborhood of any singular point is a cone over $\mathbb{C}P^n$.

In general, a $C(\mathbb{C}P^n)$-singular manifold is obtained from a smooth manifold M^{2n+1} with boundary which is a disjoint union of complex projective spaces $\mathbb{C}P^n \cup \mathbb{C}P^n \cup \ldots \cup \mathbb{C}P^n$ and subsequent capture of the cone over each component of the boundary. For this type of $C(\mathbb{C}P^n)$-singular manifold parity of the number of singular points depends on parity of the number n.

Lemma 1.1. *Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points. The Euler characteristic of M^{2n+1} is equal $\chi(M^{2n+1}) = \dfrac{k(1-n)}{2}$.*

Remark **1.1.** For n even, the complex projective space $\mathbb{C}P^n$ can not be the boundary of a smooth compact manifold X^{2n+1}.

Lemma 1.2. *Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold with k singular points. If n is an odd number then the number k of singular points can be any number. If n is an even number the number k of singular points is an even number.*

Since $C(\mathbb{C}P^n)$-singular manifolds are topological spaces we can consider continuous functions on them and because of the nature of $C(\mathbb{C}P^n)$-singular manifolds it is appropriate to consider continuous functions which are smooth on the complement of the set of singular points. Also it makes sense to study such functions on a $C(\mathbb{C}P^n)$-singular manifold whose singular points of the manifold are critical points of these functions. More precisely, this means the following.

Let M^{2n+1} be a compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1, \ldots, m_k$ and $U(m_1), \ldots, U(m_k)$ the respective closed neighborhood homeomorphic to the cone over $\mathbb{C}P^n$. For any neighborhood $U(m_i)$ there is a disc D_i^{2n+2} and a semifree action of the circle

$$\theta\colon D_i^{2n+2} \times S^1 \to D_i^{2n+2}$$

such that performed

$$D_i^{2n+2} \xrightarrow{\pi} D_i^{2n+2}/S^1 \approx U(m_i).$$

We introduce in the disc D_i^{2n+2} complex coordinates $z_1, \ldots, z_n$ and recall that the circle is the set of complex numbers of modulus one. We assume that the action of the circle on the disc is defined by the formula

$$\theta(z_1, \ldots, z_n) = e^{it} z_1, \ldots, e^{it} z_n.$$

Consider an arbitrary S^1-invariant smooth function $f\colon D_i^{2n+2} \to \mathbb{R}$ with a single critical point in the center of the disk. For example, let f be given by

$$f = -|z_1|^2 - \ldots - |z_{\lambda_i}|^2 + |z_{\lambda_i+1}|^2 + \ldots + |z_n|^2.$$

Notice that the index of the nondegenerate critical point $0 \in D_i^{2n+2}$ of such function f is always even.

Let $\pi_*(f)\colon U(m_i) \to \mathbb{R}$ be the continuous function induced on $U(m_i)$ by the natural map

$$\pi\colon D_i^{2n+2} \to D_i^{2n+2}/S^1 \approx U(m_i).$$

It is clear that the function $\pi_*(f)$ is smooth on the manifold $U(m_i) \setminus m_i$.

Definition 1.1. *The function $\pi_*(f)\colon U(m_i) \to \mathbb{R}$ is called almost smooth function on the neighborhood $U(m_i)$ with a singularity at the point m_i.*

If f is given by

$$f = -|z_1|^2 - \ldots - |z_{\lambda_i}|^2 + |z_{\lambda_i+1}|^2 + \ldots + |z_n|^2$$

then the function $\pi_*(f)\colon U(m_i) \to \mathbb{R}$ is called almost Morse function on the neighborhood $U(m_i)$.

Definition 1.2. *The function $f\colon M^{2n+1} \to \mathbb{R}$ is called almost Morse function on the $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} if f is an almost Morse function in the neighborhoods $U(m_i)$ of singular points m_i of M^{2n+1} and f is a Morse function on a smooth manifold $M^{2n+1} \setminus \bigcup_i m_i$.*

From Definition 1.1 follows that on any compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1, \ldots, m_k$ there exists an almost Morse function [1, 2].

The number of critical points of an almost Morse function is dependent of the structure of such function in the neighborhood of singular points of the $C(\mathbb{C}P^n)$-singular manifold. Let us examine this issue in more detail.

Definition 1.3. *Let f be an almost Morse function on the $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1, \ldots, m_k$. Denote by*

$$\pi_*(f_i)\colon U(m_i) \to \mathbb{R}$$

its almost Morse function in the neighborhood $U(m_i)$ of singular point m_i of M^{2n+1}. The state of the almost Morse function f is the collection of all almost Morse functions in the neighborhood $U(m_i)$ $\pi_*(f_1), \pi_*(f_2), \ldots, \pi_*(f_k)$, which we will be denoted by $St(f)$.

Consider the case where M^{2n+1}, $2n \geq 5$, is a compact simply connected $C(\mathbb{C}P^n)$-singular manifold.

Recall that for a simply connected smooth manifold we can calculate the Morse number via its homology groups. More precisely, if we consider a closed manifold N^n and Morse functions $f\colon N^n \to \mathbb{R}$ then to count the Morse number for the class of such functions we can use the homology group $H_j(N^n, \mathbb{Z})$.

If we consider a compact manifold N^n with boundary $\partial N^n = \partial_1 N^n \bigcup \partial_2 N^n$ and Morse functions $f\colon (N^n, \partial_1 N^n, \partial_2 N^n) \to \mathbb{R}$ such that $f^{-1}(0) = \partial_1 N^n$ and $f^{-1}(1) = \partial_2 N^n$ then to calculate the Morse numbers for this class of functions we use the group $H_j(N^n, \partial_1 N^n, \mathbb{Z})$ [6].

Let M^{2n+1}, $2n \geq 5$, be a compact simply connected $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Let σ be a permutation of $(1, 2, \ldots, k)$. We split the singular point $m_1, \ldots, m_k$ in two disjoint sets A and B consisting of s and $k - s$ points, respectively:

$$A = m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(s)}, \qquad B = m_{\sigma(s+1)}, m_{\sigma(s+2)}, \ldots, m_{\sigma(k)}.$$

The case when A or B is empty set is not excluded. Consider the homology groups

$$H_j(M^{2n+1} \setminus B, A, \mathbb{Z}).$$

***Remark* 1.2.** If τ is another permutation of $(1, 2, \ldots, k)$ and

$$\tilde{A} = m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(s)}, \qquad \tilde{B} = m_{\tau(s+1)}, m_{\tau(s+2)}, \ldots, m_{\tau(k)}$$

is another splitting of the singular points $m_1, \ldots, m_k$ in two disjoint sets $\tilde{A}$ and $\tilde{B}$ then, in general,

$$H_j(M^{2n+1} \setminus B, A, \mathbb{Z}) \neq H_j(M^{2n+1} \setminus \tilde{B}, \tilde{A}, \mathbb{Z}).$$

Theorem 1.1. *Let M^{2n+1}, $2n \geq 5$, be a compact simply connected $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Let σ be a permutation of $(1, 2, \ldots, k)$ and let A (with s points) and B (with $k - s$ points) be the split of the singular points $m_1, \ldots, m_k$ in the two disjoint sets*:

$$A = m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(s)}, \qquad B = m_{\sigma(s+1)}, m_{\sigma(s+2)}, \ldots, m_{\sigma(k)}.$$

We fix a collection of almost Morse functions

$$St = \underbrace{\pi_*(f_1), \pi_*(f_1), \ldots, \pi_*(f_1)}_{s}, \underbrace{\pi_*(f_2), \pi_*(f_2), \ldots, \pi_*(f_2)}_{k-s}$$

in the neighborhoods $U\big(m_{\sigma(1)}\big), U\big(m_{\sigma(2)}\big), \ldots, U\big(m_{\sigma(s)}\big), U\big(m_{\sigma(s+1)}\big), \ldots, U\big(m_{\sigma(k)}\big)$ respectively, where

$$f_1 = \sum_{i=1}^{2n} |z_i|^2, \qquad f_2 = 1 - \sum_{i=1}^{2n} |z_i|^2.$$

Then

$$\mathcal{M}_\lambda(M^{2n+1}, St) = \mu\left(H_\lambda(M^{2n+1} \setminus B, A, \mathbb{Z})\right) + \mu\left(\mathrm{Tors} H_{\lambda-1}(M^{2n+1} \setminus B, A, \mathbb{Z})\right),$$

where $\mu(H)$ is the minimal number of generators of the group H.

2. Vector fields on $C(\mathbb{C}P^n)$-manifolds. Let $V(x)$ be a smooth vector field on a smooth compact manifold N^{2n+1} with boundary with a finite number of points $C(\mathbb{C}P^n)$ in the interior of the manifold N^{2n+1} where $V(x)$ are zero. Suppose that the restriction of the field $V(x)$ on the boundary ∂N^{2n+1} of the manifold N^{2n+1} is outwardly directed to the manifold N^{2n+1}. Recall the definition of a index zero of vector field $V(x)$ on a smooth manifold N^{2n+1}.

Definition 2.1. *Let N^{2n+1} be a cone over $C(\mathbb{C}P^n)$ and let $V(x)$ be an almost smooth vector field on N^{2n+1} such that the singular point $n \in N^{2n+1}$ is an isolated zero point of $V(x)$, the field $V(x)$ has finite number of zeros $n_1, \ldots, n_l$ and such zeros points belong to $N^{2n+1} \setminus \partial N^{2n+1}$ and $V(x)$ on the boundary of the manifold N^{2n+1} is pointed out to the manifold N^{2n+1}. The index $ind(n)_{V(x)}$ of the vector field $V(x)$ at the point n is defined as the*

$$\mathrm{ind}(n)_{V(x)} = \chi(N^{2n+1}) - \sum_{i=1}^{l} \mathrm{ind}(n_i)_{V(x)}.$$

For definition of the index at a singular point of an arbitrary vector field on cone over $C(\mathbb{C}P^n)$ we will need to build additional. First we prove the following lemma.

Lemma 2.1. *Let N^{2n+1} be a cone over $C(\mathbb{C}P^n)$ and let $V(x)$ be an almost smooth vector field on N^{2n+1} such that the singular point $n \in N^{2n+1}$ is an isolated zero point of $V(x)$. Then on singular manifold N^{2n+1} there exists an almost smooth vector field $W(x)$ such that*:

1) *$W(x)$ coincides with the field $V(x)$ in some neighborhood of singular point $n \in N^{2n+1}$*;

2) *$W(x)$ has finite number of zero points and such zero points belong to $N^{2n+1} \setminus \partial N^{2n+1}$*;

3) *$W(x)$ on the boundary of the manifold N^{2n+1} is pointed out to the manifold N^{2n+1}.*

We must show that so defined index at the singular point $n \in N^{2n+1}$ does not depend on the choice of the vector field $W(x)$.

We prove the following lemma.

Lemma 2.2. *Let N^{2n+1} be a smooth closed manifold and let $V(x)$ and $W(x)$ be smooth vector fields on $N^{2n+1}\times[0,1)$ such that*:

1) *the vector field $W(x)$ coincides with the vector field $V(x)$ on $N^{2n+1}\times[1-\varepsilon,1)$, where* $0<\varepsilon<1$;

2) *vector fields $V(x)$ and $W(x)$ have finite number of zeros*;

3) *vector fields $V(x)$ and $W(x)$ are not zero on the boundary $N^{2n+1}\times 0$ of the manifold $N^{2n+1}\times[0,1)$ and pointed out to the manifold $N^{2n+1}\times[0,1)$.*

Let $x_1,\ldots,x_s$ and $y_1,\ldots,y_t$ be zeros of the vector fields $V(x)$ and $W(x)$ respectively. Then

$$\sum_{i=1}^{s}\operatorname{ind}(x_i)_{V(x)}=\sum_{i=1}^{t}\operatorname{ind}(y_i)_{W(x)}.$$

Proposition 2.1. *Let N^{2n+1} be a cone over $C(\mathbb{C}P^n)$ and $V(x)$ is an almost smooth vector field on N^{2n+1} such that the singular point $n\in N^{2n+1}$ is an isolated zero of $V(x)$. The index of the zero n of the vector field $V(x)$ in Definition* 2.1 *does not depend of the almost smooth vector field $W(x)$.*

Theorem 2.1. *Let M^{2n+1} be a $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1,\ldots,m_k$. Suppose that on M^{2n+1} there exists an almost smooth vector field $V(x)$ with finite number of zeros $m_1,\ldots,m_k,x_1,\ldots,x_l$. Then*

$$\chi(M^{2n+1})=\sum_{i=1}^{l}\operatorname{ind}(x_i)+\sum_{i=1}^{k}\operatorname{ind}(m_i).$$

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