UDC 517.9

H.-J. Sun (Nanjing Univ. Sci. and Technology, China)

INEQUALITIES FOR EIGENVALUES OF A SYSTEM OF HIGHER-ORDER DIFFERENTIAL EQUATIONS*

НЕРІВНОСТІ ДЛЯ ВЛАСНИХ ЗНАЧЕНЬ СИСТЕМИ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ВИЩОГО ПОРЯДКУ

We establish some sharper inequalities for eigenvalues of a system of higher-order differential equations. Moreover, we give some sharper estimates for the upper bound of the $(k+1)$th eigenvalue and the gaps of its consecutive eigenvalues.

Встановлено деякі більш точні оцінки для власних значень системи диференціальних рівнянь вищого порядку. Ми також наводимо деякі більш точні оцінки для оцінки зверху $(k+1)$-го власного значення і щілин між будь-якими двома послідовними власними значеннями.

1. Introduction. Let $[a,b] \subset \mathbb{R}$ and $l \geq 1$. We investigate the following Dirichlet eigenvalue problem of a system of higher-order differential equations:

$$\begin{gathered}(-1)^l D^l\big(a_{11}(x)D^l u_1(x) + \ldots + a_{1n}(x)D^l u_n(x)\big) = \lambda\rho(x)u_1(x),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (-1)^l D^l\big(a_{n1}(x)D^l u_1(x) + \ldots + a_{nn}(x)D^l u_n(x)\big) = \lambda\rho(x)u_n(x),\\ D^t u_q(a) = D^t u_q(b) = 0, \quad \text{for} \quad t = 0,1,\ldots,l-1 \quad \text{and} \quad q = 1,\ldots,n,\end{gathered} \tag{1.1}$$

where $D = \dfrac{d}{dx}$, $\rho(x)$ is a positive function on $[a,b]$ and $0 \leq a_{ij}(x) = a_{ji}(x) \in C^l[a,b]$ for $i,j = 1,2,\ldots,n$. Assume that the coefficients $a_{ij}(x)$, $i,j = 1,2,\ldots,n$, satisfy the uniform ellipticity assumption: for any column vector $\xi = \big(\xi_1,\ldots,\xi_n\big)^T$, there is a positive constant ς such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \varsigma|\xi|^2 \quad \forall x \in [a,b], \tag{1.2}$$

where $|\xi| = (\xi_1^2 + \ldots + \xi_n^2)^{1/2}$ denotes the Euclidean norm of ξ. Moreover, we assume that the coefficients $a_{ij}(x)$, $i,j = 1,2,\ldots,n$, satisfy

$$a_{ij}(x) \leq \zeta \quad \forall x \in [a,b], \tag{1.3}$$

where ζ is a positive constant. Suppose that the weight function satisfies:

$$\sigma^{-1} \leq \rho(x) \leq \tau^{-1} \quad \forall x \in [a,b], \tag{1.4}$$

where σ and τ are two positive constants. This problem is significant in physics, mechanics and engineering. Moreover, the weight function ρ denotes the density. Weighted estimates for eigenvalues have many applications (see [6, 7]).

* This paper was supported by the National Natural Science Foundation of China (Grant No.11001130).

There have been some results for some eigenvalues problems of systems of equations. Mokeichev [8] derive some expressions for eigenvalues of a system of equations in Hilbert space. Kostenko [5] investigated the spectrum of a system of second-order differential equations. In 2011, Jia, Huang and Liu [4] established the following inequalities for eigenvalues of problem (1.1):

$$\sum_{r=1}^{k} \frac{\lambda_r^{1/l}}{\lambda_{k+1}-\lambda_r} \geq \frac{\varsigma\tau^2 k^2}{4l(2l-1)\zeta\sigma^2}\left(\sum_{r=1}^{k}\lambda_r^{1-1/l}\right)^{-1} \tag{1.5}$$

and

$$\lambda_{k+1} \leq \left[1+\frac{4l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\right]\lambda_k. \tag{1.6}$$

The goal of this paper is to give some sharper estimates for eigenvalues of problem (1.1). We first establish the following inequality.

Theorem 1.1. *Let* λ_r *be the* r*th eigenvalue of problem* (1.1). *Suppose that the coefficients* $a_{ij}(x)$, $i,j=1,2,\ldots,n$, *and the weight function* $\rho(x)$ *satisfy* (1.2), (1.3) *and* (1.4). *Then we have*

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq \frac{4l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r. \tag{1.7}$$

Furthermore, making a modification in the proof of Theorem 1.1, we can obtain another inequality.

Theorem 1.2. *Under the same assumptions as Theorem* 1.1, *we have*

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq$$

$$\leq 2\left[\frac{l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\right]^{1/2}\left[\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r^{1/l}\right]^{1/2}\left[\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2\lambda_r^{1-1/l}\right]^{1/2}. \tag{1.8}$$

Remark **1.1.** When $l=1$, (1.7) and (1.8) all become

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq \frac{4\zeta\sigma^2}{\varsigma\tau^2}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r.$$

When $l\geq 2$, inequality (1.7) is better than (1.8). In fact, inequality (1.7) is always Yang-type (see [1, 9, 10]) for any l. Using Chebyshev's inequality, it is not difficult to get (1.5) from (1.8). That is to say, (1.8) and (1.7) is sharper than (1.5).

Using (1.7), we can derive some explicit estimates for upper bounds of eigenvalues and gaps of consecutive eigenvalues. In fact, noting that (1.7) is a quadratic inequality of λ_{k+1}, we can obtain the following two corollaries.

Corollary **1.1.** *Under the same assumptions of Theorem* 1.1, *we have*

$$\lambda_{k+1} \leq \left[1+\frac{2l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\right]\frac{1}{k}\sum_{r=1}^{k}\lambda_r +$$

$$+\left\{\left[\frac{2l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\frac{1}{k}\sum_{r=1}^{k}\lambda_r\right]^2-\left[1+\frac{4l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\right]\frac{1}{k}\sum_{s=1}^{k}(\lambda_s-\frac{1}{k}\sum_{r=1}^{k}\lambda_r)^2\right\}^{1/2}. \tag{1.9}$$

Using the Cauchy – Schwarz inequality, we can get the following inequality from (1.9).

***Corollary* 1.2.** *Under the same assumptions of Theorem* 1.1, *we have*

$$\lambda_{k+1}\le\left[1+\frac{4l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\right]\frac{1}{k}\sum_{r=1}^{k}\lambda_r. \tag{1.10}$$

***Remark* 1.2.** Inequalities (1.9) and (1.10) give some estimates for upper bounds of λ_{k+1} in terms of the first k eigenvalues of problem (1.1). Moreover, it is easy to find that (1.10) implies (1.6).

At the same time, an explicit estimate for the gaps of any two consecutive eigenvalues of problem (1.1) can be obtained from (1.9).

***Corollary* 1.3.** *Under the same assumptions of Theorem* 1.1, *we have*

$$\lambda_{k+1}-\lambda_k\le 2\left\{\left[\frac{2l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\frac{1}{k}\sum_{r=1}^{k}\lambda_r\right]^2-\right.$$

$$\left.-\left[1+\frac{4l(2l-1)\zeta\sigma^2}{\varsigma\tau^2}\right]\frac{1}{k}\sum_{s=1}^{k}(\lambda_s-\frac{1}{k}\sum_{r=1}^{k}\lambda_r)^2\right\}^{1/2}. \tag{1.11}$$

2. Proofs of the main results. Let

$$\mathbf{u}=\begin{pmatrix}u_1(x)\\ \vdots\\ u_n(x)\end{pmatrix},\quad D^l\mathbf{u}=\begin{pmatrix}D^lu_1(x)\\ \vdots\\ D^lu_n(x)\end{pmatrix},\quad A(x)=\begin{pmatrix}a_{11}(x) & \dots & a_{1n}(x)\\ \dots & \dots & \dots\\ a_{n1}(x) & \dots & a_{nn}(x)\end{pmatrix}.$$

Then we can rewrite problem (1.1) as the following simpler form:

$$\begin{gathered}(-1)^lD^l\big(A(x)D^l\mathbf{u}\big)=\lambda\rho\mathbf{u},\quad \text{on}\quad [a,b],\\ D^t\mathbf{u}(a)=D^t\mathbf{u}(b)=0,\quad \text{for}\quad t=0,1,\dots,l-1.\end{gathered} \tag{2.1}$$

In order to prove the main theorems, we first establish a general inequality for eigenvalues of problem (2.1).

Lemma 2.1. *Let* $\mathbf{u}_r=\big(u_{r1}(x),u_{r2}(x),\dots,u_{rn}(x)\big)^T$ *be the weighted orthonormal eigenvector corresponding to the* r*th eigenvalue* λ_r *of problem* (1.1) *for* $r=1,2,\dots,k$. *Then we have*

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2\int_a^b|\mathbf{u}_r|^2dx\le$$

$$\le\sum_{r=1}^{k}\delta_r(\lambda_{k+1}-\lambda_r)^2\omega_r+\sum_{r=1}^{k}\frac{1}{\delta_r}(\lambda_{k+1}-\lambda_r)\int_a^b\frac{1}{\rho}|D\mathbf{u}_r|^2dx, \tag{2.2}$$

where the positive constants δ_r, $r=1,\dots,k,\dots$, *construct a nonincreasing sequence and*

$$\omega_r = l^2 \int_a^b D^{l-1}\mathbf{u}_r^T A(x) D^{l-1}\mathbf{u}_r dx - l(l-1)\int_a^b D^l \mathbf{u}_r^T A(x) D^{l-2}\mathbf{u}_r dx.$$

Proof. According to (2.1) and the assumptions of Lemma 1, the weighted orthonormal eigenvector $\mathbf{u}_r$ satisfies

$$(-1)^l D^l\big(A(x)D^l\mathbf{u}_r\big) = \lambda_r \rho \mathbf{u}_r,$$

$$D^t\mathbf{u}_r(a) = D^t\mathbf{u}_r(b) = 0, \tag{2.3}$$

$$\int_a^b \rho \mathbf{u}_r^T \mathbf{u}_s dx = \delta_{rs},$$

for $r, s = 1, \ldots, k$ and $t = 0, 1, \ldots, l-1$. We define the trial vectors Φ_r by

$$\Phi_r = x\mathbf{u}_r - \sum_{s=1}^{k} b_{rs}\mathbf{u}_s, \tag{2.4}$$

where

$$b_{rs} = \int_a^b \rho x \mathbf{u}_r^T \mathbf{u}_s dx = b_{sr}. \tag{2.5}$$

Then we know that Φ_r is weighted orthogonal with $\mathbf{u}_s$. That is to say

$$\int_a^b \rho \Phi_r^T \mathbf{u}_s dx = 0, \quad \text{for} \quad r, s = 1, \ldots, k. \tag{2.6}$$

It yields

$$\int_a^b \rho \Phi_r^T x \mathbf{u}_r dx = \int_a^b \rho |\Phi_r|^2 dx. \tag{2.7}$$

Since

$$D^l\big[a_{ij}(x)D^l\big(xu_{rj}(x)\big)\big] =$$

$$= lD^l\big(a_{ij}(x)D^{l-1}u_{rj}(x)\big) + D^l\big(a_{ij}(x)xD^l u_{rj}(x)\big) =$$

$$= lD^l\big(a_{ij}(x)D^{l-1}u_{rj}(x)\big) + lD^{l-1}\big(a_{ij}(x)D^l u_{rj}(x)\big) + xD^l\big(a_{ij}(x)D^l u_{rj}(x)\big),$$

we can deduce

$$D^l\big(A(x)D^l\Phi_r\big) = D^l\big[A(x)D^l(x\mathbf{u}_r)\big] - \sum_{s=1}^{k} b_{rs} D^l\big(A(x)D^l\mathbf{u}_s\big) =$$

$$= lD^l\big(A(x)D^{l-1}\mathbf{u}_r\big) + lD^{l-1}\big(A(x)D^l\mathbf{u}_r\big) + xD^l\big(A(x)D^l\mathbf{u}_r\big) -$$

$$- \sum_{s=1}^{k} b_{rs} D^l\big(A(x)D^l\mathbf{u}_s\big) =$$

$$= lD^l\big(A(x)D^{l-1}\mathbf{u}_r\big) + lD^{l-1}\big(A(x)D^l\mathbf{u}_r\big) + (-1)^l\lambda_r\rho x\mathbf{u}_r - (-1)^l\sum_{s=1}^{k} b_{rs}\lambda_s\rho\mathbf{u}_s. \qquad (2.8)$$

Using (2.6), (2.7), and (2.8), we have

$$\int_a^b \Phi_r^T D^l\big(A(x)D^l\Phi_r\big)dx =$$

$$= \int_a^b \Phi_r^T\Big[D^l\big(A(x)D^{l-1}\mathbf{u}_r\big) + D^{l-1}\big(A(x)D^l\mathbf{u}_r\big)\Big]dx + (-1)^l\lambda_r\int_a^b \rho|\Phi_r|^2dx. \qquad (2.9)$$

Substituting (2.9) into the Rayleigh – Ritz inequality

$$\lambda_{k+1} \le \frac{(-1)^l\displaystyle\int_a^b \Phi_r^T D^l\big(A(x)D^l\Phi_r\big)dx}{\displaystyle\int_a^b \rho|\Phi_r|^2dx},$$

we obtain

$$(\lambda_{k+1} - \lambda_r)\int_a^b \rho\Phi_r^2dx \le$$

$$\le (-1)^l l\int_a^b \Phi_r^T\Big[D^l\big(A(x)D^{l-1}\mathbf{u}_r\big) + D^{l-1}\big(A(x)D^l\mathbf{u}_r\big)\Big]dx =$$

$$= (-1)^l l\int_a^b x\mathbf{u}_r^T\Big[D^l\big(A(x)D^{l-1}\mathbf{u}_r\big) + D^{l-1}\big(A(x)D^l\mathbf{u}_r\big)\Big]dx + \sum_{s=1}^{k} b_{rs}c_{rs}, \qquad (2.10)$$

where

$$c_{rs} = (-1)^{l+1}l\int_a^b \mathbf{u}_s^T\Big[D^l\big(A(x)D^{l-1}\mathbf{u}_r\big) + D^{l-1}\big(A(x)D^l\mathbf{u}_r\big)\Big]dx.$$

It is easy to find that $c_{rs} = -c_{sr}$.

Using integration by parts and making use of (2.3), we have

$$\lambda_r b_{rs} = \int_a^b x\mathbf{u}_s^T\Big[(-1)^l D^l\big(A(x)D^l\mathbf{u}_r\big)\Big]dx = (-1)^l\int_a^b \mathbf{u}_r^T D^l\Big[A(x)D^l\big(x\mathbf{u}_s\big)\Big]dx =$$

$$= (-1)^l l \int_a^b \mathbf{u}_r^T \Big[D^l \big(A(x) D^{l-1} \mathbf{u}_s \big) + D^{l-1} \big(A(x) D^l \mathbf{u}_s \big) \Big] dx +$$

$$+(-1)^l \int_a^b x \mathbf{u}_r^T D^l \big(A(x) D^l \mathbf{u}_s \big) dx = \lambda_s b_{rs} - c_{sr}.$$

It yields

$$c_{rs} = (\lambda_r - \lambda_s) b_{rs}. \tag{2.11}$$

At the same time, we get

$$(-1)^l l \int_a^b x \mathbf{u}_r^T \Big[D^l \big(A(x) D^{l-1} \mathbf{u}_r \big) + D^{l-1} \big(A(x) D^l \mathbf{u}_r \big) \Big] dx =$$

$$= l \int_a^b D^{l-1} \mathbf{u}_r^T A(x) D^l \big(x \mathbf{u}_r \big) dx - l \int_a^b D^l \mathbf{u}_r^T A(x) D^{l-1} \big(x \mathbf{u}_r \big) dx =$$

$$= l^2 \int_a^b D^{l-1} \mathbf{u}_r^T A(x) D^{l-1} \mathbf{u}_r dx + l \int_a^b x D^{l-1} \mathbf{u}_r^T A(x) D^l \mathbf{u}_r dx -$$

$$-l(l-1) \int_a^b D^l \mathbf{u}_r^T A(x) D^{l-2} \mathbf{u}_r dx - l \int_a^b x D^l \mathbf{u}_r^T A(x) D^{l-1} \mathbf{u}_r dx = \omega_r, \tag{2.12}$$

where

$$\omega_r = l^2 \int_a^b D^{l-1} \mathbf{u}_r^T A(x) D^{l-1} \mathbf{u}_r dx - l(l-1) \int_a^b D^l \mathbf{u}_r^T A(x) D^{l-2} \mathbf{u}_r dx.$$

Substituting (2.11) and (2.12) into (2.10), we derive

$$(\lambda_{k+1} - \lambda_r) \int_a^b \rho |\Phi_r|^2 dx \le \omega_r + \sum_{s=1}^k (\lambda_r - \lambda_s) b_{rs}^2. \tag{2.13}$$

It is not hard to find

$$\int_a^b x \mathbf{u}_r^T D \mathbf{u}_r dx = -\frac{1}{2} \int_a^b |\mathbf{u}_r|^2 dx. \tag{2.14}$$

Hence, using integration by parts again, we obtain

$$-2\int\limits_a^b \Phi_r^T D\mathbf{u}_r dx = -2\int\limits_a^b x\mathbf{u}_r^T D\mathbf{u}_r dx + 2\sum_{s=1}^k b_{rs}d_{rs} = \int\limits_a^b |\mathbf{u}_r|^2 dx + 2\sum_{s=1}^k b_{rs}d_{rs}, \qquad (2.15)$$

where

$$d_{rs} = \int\limits_a^b \mathbf{u}_s^T D\mathbf{u}_r dx = -d_{sr}.$$

On the other hand, we have

$$-2(\lambda_{k+1}-\lambda_r)^2 \int\limits_a^b \Phi_r^T D\mathbf{u}_r dx =$$

$$= -2(\lambda_{k+1}-\lambda_r)^2 \int\limits_a^b \sqrt{\rho}\Phi_r^T \left(\frac{1}{\sqrt{\rho}}D\mathbf{u}_r - \sqrt{\rho}\sum_{s=1}^k d_{rs}\mathbf{u}_s\right) dx \le$$

$$\le \delta_r(\lambda_{k+1}-\lambda_r)^3 \int\limits_a^b \rho|\Phi_r|^2 dx + \frac{\lambda_{k+1}-\lambda_r}{\delta_r}\int\limits_a^b \left(\frac{1}{\sqrt{\rho}}D\mathbf{u}_r - \sqrt{\rho}\sum_{s=1}^k d_{rs}\mathbf{u}_s\right)^2 dx =$$

$$= \frac{\lambda_{k+1}-\lambda_r}{\delta_r}\left[\int\limits_a^b \frac{1}{\rho}|D\mathbf{u}_r|^2 dx - 2\sum_{s=1}^k d_{rs}\int\limits_a^b \mathbf{u}_s^T D\mathbf{u}_r dx + \sum_{s,q=1}^k d_{rs}d_{rq}\int\limits_a^b \rho\mathbf{u}_s^T\mathbf{u}_q dx\right] +$$

$$+\delta_r(\lambda_{k+1}-\lambda_r)^3 \int\limits_a^b \rho|\Phi_r|^2 dx =$$

$$= \delta_r(\lambda_{k+1}-\lambda_r)^3 \int\limits_a^b \rho|\Phi_r|^2 dx + \frac{\lambda_{k+1}-\lambda_r}{\delta_r}\left[\int\limits_a^b \frac{1}{\rho}|D\mathbf{u}_r|^2 dx - \sum_{s=1}^k d_{rs}^2\right]. \qquad (2.16)$$

Substituting (2.13) into (2.16), we obtain

$$-2(\lambda_{k+1}-\lambda_r)^2 \int\limits_a^b \Phi_r^T D\mathbf{u}_r dx \le$$

$$\le \delta_r(\lambda_{k+1}-\lambda_r)^2 \left[\omega_r + \sum_{s=1}^k (\lambda_r-\lambda_s)b_{rs}^2\right] + \frac{\lambda_{k+1}-\lambda_r}{\delta_r}\left[\int\limits_a^b \frac{1}{\rho}|D\mathbf{u}_r|^2 dx - \sum_{s=1}^k d_{rs}^2\right]. \qquad (2.17)$$

Combining (2.15) and (2.17), and taking sum on r from 1 to k, we derive

$$\sum_{r=1}^k (\lambda_{k+1}-\lambda_r)^2 \int\limits_a^b |\mathbf{u}_r|^2 dx + 2\sum_{r,s=1}^k (\lambda_{k+1}-\lambda_r)^2 b_{rs}d_{rs} \le$$

$$\leq \sum_{r=1}^{k} \delta_r(\lambda_{k+1}-\lambda_r)^2\omega_r + \sum_{r,s=1}^{k} \delta_r(\lambda_{k+1}-\lambda_r)^2(\lambda_r-\lambda_s)b_{rs}^2 +$$

$$+\sum_{r=1}^{k} \frac{1}{\delta_r}(\lambda_{k+1}-\lambda_r)\int_a^b \frac{1}{\rho}|D\mathbf{u}_r|^2dx - \sum_{r,s=1}^{k} \frac{1}{\delta_r}(\lambda_{k+1}-\lambda_r)d_{rs}^2. \tag{2.18}$$

Since the sequence $\{\delta_r\}$ is nonincreasing, one can get

$$\sum_{r,s=1}^{k} \delta_r(\lambda_{k+1}-\lambda_r)^2(\lambda_r-\lambda_s)b_{rs}^2 \leq -\sum_{r,s=1}^{k} \delta_r(\lambda_{k+1}-\lambda_r)(\lambda_r-\lambda_s)^2b_{rs}^2. \tag{2.19}$$

Then we can eliminate the unwanted terms in both sides of (2.18) by using (2.19) and

$$\sum_{r,s=1}^{k} (\lambda_{k+1}-\lambda_r)^2 b_{rs}d_{rs} = -\sum_{r,s=1}^{k} (\lambda_{k+1}-\lambda_r)(\lambda_r-\lambda_s)b_{rs}d_{rs}.$$

Thereby, (2.2) is true.

Lemma 2.1 is proved.

Now we give the proof of Theorem 1.1.

***Proof of Thereom* 1.1.** It is not hard to find

$$0<\tau\leq \int_a^b |\mathbf{u}_r|^2dx = \int_a^b \frac{1}{\rho}\rho|\mathbf{u}_r|^2dx \leq \sigma. \tag{2.20}$$

By virtue of (1.2), we have

$$\lambda_r = \int_a^b \mathbf{u}_r^T\Big[(-1)^l D^l\big(A(x)D^l\mathbf{u}_r\big)\Big]dx = \int_a^b D^l\mathbf{u}_r^T A(x)D^l\mathbf{u}_r dx =$$

$$= \int_a^b \sum_{i,j=1}^{n} a_{ij}(x)D^l u_{ri}(x)D^l u_{rj}(x)dx \geq \varsigma \int_a^b |D^l\mathbf{u}_r|^2dx.$$

It yields

$$\int_a^b |D^l\mathbf{u}_r|^2dx \leq \frac{1}{\varsigma}\lambda_r. \tag{2.21}$$

Using (2.21) and the following inequality (see Lemma 2.1 of [4])

$$\int_a^b |D^t\mathbf{u}_r|^2dx \leq \sigma^{1-t/l}\left(\frac{\lambda_r}{\varsigma}\right)^{t/l},$$

we can deduce

$$\int\limits_a^b \frac{1}{\rho}|D\mathbf{u}_r|^2 dx \le \sigma^{2-1/l}\left(\frac{\lambda_r}{\varsigma}\right)^{1/l}, \tag{2.22}$$

$$\int\limits_a^b D^{l-1}\mathbf{u}_r^T A(x) D^{l-1}\mathbf{u}_r dx \le \zeta \int\limits_a^b |D^{l-1}\mathbf{u}_r|^2 dx \le \zeta\sigma^{1/l}\left(\frac{\lambda_r}{\varsigma}\right)^{1-1/l} \tag{2.23}$$

and

$$-\int\limits_a^b D^l\mathbf{u}_r^T A(x) D^{l-2}\mathbf{u}_r dx \le \zeta\left(\int\limits_a^b |D^{l-2}\mathbf{u}_r|^2 dx\right)^{1/2}\left(\int\limits_a^b |D^l\mathbf{u}_r|^2 dx\right)^{1/2} \le$$

$$\le \zeta\sigma^{1/l}\left(\frac{\lambda_r}{\varsigma}\right)^{1-1/l}. \tag{2.24}$$

Using (2.23) and (2.24), we obtain

$$\omega_r = l^2\int\limits_a^b D^{l-1}\mathbf{u}_r^T A(x) D^{l-1}\mathbf{u}_r dx - l(l-1)\int\limits_a^b D^l\mathbf{u}_r^T A(x) D^{l-2}\mathbf{u}_r dx \le$$

$$\le l(2l-1)\zeta\sigma^{1/l}\left(\frac{\lambda_r}{\varsigma}\right)^{1-1/l}. \tag{2.25}$$

Therefore, substituting (2.20), (2.22), and (2.25) into (2.2), we get

$$\tau\sum_{r=1}^k(\lambda_{k+1}-\lambda_r)^2 \le$$

$$\le l(2l-1)\zeta\sigma^{1/l}\sum_{r=1}^k \delta_r(\lambda_{k+1}-\lambda_r)^2\left(\frac{\lambda_r}{\varsigma}\right)^{1-1/l} + \sigma^{2-1/l}\sum_{r=1}^k\frac{1}{\delta_r}(\lambda_{k+1}-\lambda_r)\left(\frac{\lambda_r}{\varsigma}\right)^{1/l}. \tag{2.26}$$

Putting

$$\delta_r = \frac{\delta}{l(2l-1)\zeta\sigma^{1/l}\lambda_r^{1-1/l}\varsigma^{-1+1/l}}$$

in (2.26), we have

$$\tau\sum_{r=1}^k(\lambda_{k+1}-\lambda_r)^2 \le \delta\sum_{r=1}^k(\lambda_{k+1}-\lambda_r)^2 + \frac{1}{\delta}l(2l-1)\zeta\sigma^2\varsigma^{-1}\sum_{r=1}^k(\lambda_{k+1}-\lambda_r)\lambda_r. \tag{2.27}$$

Then, putting

$$\delta = \left[l(2l-1)\zeta\right]^{1/2}\sigma\varsigma^{-1/2}\left[\sum_{r=1}^k(\lambda_{k+1}-\lambda_r)\lambda_r\right]^{1/2}\left[\sum_{r=1}^k(\lambda_{k+1}-\lambda_r)^2\right]^{-1/2}$$

in (2.27), we obtain

$$\tau\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq 2\big[l(2l-1)\zeta\big]^{1/2}\sigma\varsigma^{-1/2}\left[\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r\right]^{1/2}. \quad (2.28)$$

This yields (1.7).

Theorem 1.1 is proved.

***Proof of Thereom* 1.2.** Since the sequence $\{\delta_r\}$ is nonincreasing, we can obtain

$$\tau\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq$$

$$\leq l(2l-1)\zeta\sigma^{1/l}\delta\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2\left(\frac{\lambda_r}{\varsigma}\right)^{1-1/l}+\sigma^{2-1/l}\frac{1}{\delta}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\left(\frac{\lambda_r}{\varsigma}\right)^{1/l} \quad (2.29)$$

by taking $\delta_r=\delta$ in (2.26) for $r=1,\ldots,k$. Putting

$$\delta=\big[l(2l-1)\zeta\big]^{-1/2}\sigma^{1-1/l}\varsigma^{1/2-1/l}\left[\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r^{1/l}\right]^{1/2}\left[\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2\lambda_r^{1-1/l}\right]^{-1/2}$$

in (2.29), we can derive (1.8).

Theorem 1.2 is proved.

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Received 22.11.11